SIAM J. MATH. ANAL.
Vol. 2, No. 1, February 1971

ASYMPTOTIC ANALYSIS OF A DIFFERENTIAL EQUATION OF TURRITTIN*

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0. Introduction. In [6] Turrittin considered the differential equation

$$\frac{d^n y}{dx^n} - x^v y = 0, \tag{0.1}$$

where n and v are positive integers, $n \geqq 2$. He gave a fundamental system of solutions $y_j(x), j = 0, \cdots, n-1$, in series of powers of x and derived the asymptotic expansions of these solutions as $x \to \infty$. To find these expansions he expressed $y_j(x)$ as a linear combination of another fundamental system of solutions of (0.1) which have a given simple asymptotic behavior as $x \to \infty$ on a certain fixed sector S in the complex x-plane. The coefficients in this linear combination are the so-called *Stokes multipliers.*

However, the latter solutions are not uniquely determined by their behavior as $x \to \infty$ in S, and so the Stokes multipliers are not uniquely determined. By requiring the given simple asymptotic behavior of solutions of (0.1) in a sector larger than S, a fundamental system of (0.1) can be determined uniquely. Turrittin determines this system in [6]. The Stokes multipliers of the $y_j(x)$ with respect to this system are uniquely determined, but appear to be rather complicated.

In [4] Heading considered the same differential equation (0.1) where now v is a rational number other than certain negative integers. He gives a fundamental set of solutions and considers the leading terms of their asymptotic expansions. In particular, the Stokes multipliers in the various sectors are investigated. He used Barnes integrals and the method of steepest descent to obtain the leading terms of the asymptotic expansions.

In [7] Turrittin considered (0.1) with $v = -1$. Here a fundamental system of solutions, which now may have a logarithmic singularity in $x = 0$, is given and their asymptotic behavior as $x \to \infty$ is deduced.

In this paper we consider (0.1) for $n \geqq 2$ and arbitrary complex values of v. In § 1 we give a fundamental set of solutions $y_{j,h}(x)$ which are characterized by their behavior near $x = 0$. As in [4] we define them by Barnes integrals. In § 2 we express these solutions as linear combinations of a special solution $\tilde{y}(x)$ of (0.1) and solutions which arise from $\tilde{y}(x)$ by rotating the argument. The asymptotic expansion of $\tilde{y}(x)$ has been given by Barnes [1] by means of a complicated method. In § 3 we give a new proof of this expansion using indirect Abelian asymptotics of the Laplace transform (cf. [3, Chap. 2]). In § 4 we derive the asymptotic expansions for the $y_{j,h}(x)$ from those for $\tilde{y}(x)$. We may characterize $\tilde{y}(x)$ uniquely by its asymptotic behavior in a given sector or on a ray. We obtain unique Stokes multipliers for the $y_{j,h}(x)$ which are simpler than those in [6].

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* Received by the editors February 11, 1970.

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The solutions defined by Barnes integrals mentioned above can be expressed as G-functions. Their asymptotic behavior has been derived by Meijer [5] from the result of Barnes mentioned above. However, these expansions do not hold uniformly in closed sectors covering the entire x-plane. Further, the explicit expressions of the Stokes multipliers for the special G-functions considered here can be derived directly in an easier way.

In § 5 we consider the fundamental system of solutions of (0.1) for general v which are uniquely characterized by their asymptotic behavior in the manner given by Turrittin [6, § 8]. Finally, we determine the Stokes multipliers with respect to this system in Theorem 4. These are more complicated than those with respect to the system of solutions $\tilde{y}$.

1. The fundamental system of solutions. We write the differential equation (0.1) in the form

$$x^n \frac{d^n y}{dx^n} - x^m y = 0. \tag{1.1}$$

If $m = 0$, this is an Euler equation, which has elementary solutions. Therefore we suppose that $m \neq 0$.

We try to find solutions of (1.1) in the form

$$y(x) = \int_C \varphi(s) x^{ms}\, ds, \tag{1.2}$$

where C is a contour in the complex s-plane from $s = \infty - ia$ to $s = \infty + ia$ with some positive constant a, and where $\varphi(s)$ is a meromorphic function with no singularities either on C or to the left of C. A formal calculation shows that $\varphi(s)$ has to satisfy

$$ms(ms - 1) \cdots (ms - n + 1)\varphi(s) = \varphi(s - 1). \tag{1.3}$$

The solutions of this equation can be written

$$\varphi(s) = \left\{ \prod_{k=0}^{n-1} \Gamma\left(1 + s - \frac{k}{m}\right) \right\}^{-1} m^{-ns} \varphi^*(s), \tag{1.4}$$

where $\varphi^*(s)$ is a function with period 1. By a suitable choice of $\varphi^*(s)$ we obtain the solutions

$$\varphi_j(s) = \left\{ \prod_{k=0}^{n-1} \Gamma\left(1 + s - \frac{k}{m}\right) \right\}^{-1} \left\{ \sin \pi\left(s - \frac{j}{m}\right) \right\}^{-1} m^{-ns}\, e^{-\pi i s}, \tag{1.5}$$

where $j = 0, 1, \cdots, n - 1$, and

$$\tilde{\varphi}(s) = \left\{ \prod_{k=0}^{n-1} \Gamma\left(\frac{k}{m} - s\right) \right\} m^{-ns}\, e^{-\pi i n s}. \tag{1.6}$$

The function $\varphi_j(s)$ has poles of the first order or removable singularities at the points $s = v + j/m$, $v = 0, 1, 2, \cdots$, and $\tilde{\varphi}(s)$ has poles at the points $s = k/m + v$, $k = 0, \cdots, n - 1$, $v = 0, 1, 2, \cdots$.

In (1.2) we now choose for C the contour from $s = \infty - ia$ to $s = w - ia$, then to $s = w + ia$ and finally to $s = \infty + ia$, where $a > (n - 1)|\mathrm{Im}\, 1/m|$ and $w < \mathrm{Re}\,(n - 1)/m$, $w < 0$.

With the choices (1.5) and (1.6) for $\varphi(s)$ in (1.2) it is easily verified that we obtain solutions of (1.1). Put

$$(1.7) \qquad y_j(x) = \frac{1}{2\pi i}\int_C \varphi_j(s)x^{ms}\,ds, \qquad \tilde{y}(x) = \frac{1}{2\pi i}\int_C \tilde{\varphi}(s)x^{ms}\,ds.$$

Here and in the following the powers x^{ms} are defined by $x^{ms} = \exp\{ms(\log|x| + i\arg x)\}$. They depend in general on the value of $\arg x$. By means of the calculus of residues we obtain

$$(1.8) \quad y_j(x) = -\frac{1}{\pi}e^{-\pi ij/m}m^{-nj/m}x^j\sum_{\nu=0}^{\infty}\left\{\prod_{k=0}^{n-1}\Gamma\left(1+\nu+\frac{j-k}{m}\right)\right\}^{-1}m^{-n\nu}x^{m\nu}.$$

These series are convergent for all values of $x^m \neq 0$. If $m \neq 0, \pm 1, \cdots, \pm(n-1)$, then the functions $y_0(x), \cdots, y_{n-1}(x)$ are linearly independent and form a fundamental system of solutions of (1.1). Equation (1.8) gives the behavior of these solutions for $x^m \to 0$.

Now suppose $m = \pm 1, \pm 2, \cdots, \pm(n-1)$. Then it is easy to see that the solutions $y_0(x), \cdots, y_{n-1}(x)$ are linearly dependent. In order to obtain n linearly independent solutions of (1.1) we select the following solutions of (1.3) (compare (1.4)):

$$(1.9) \quad \varphi_{j,h}(s) = \left\{\prod_{k=0}^{n-1}\Gamma\left(1+s-\frac{k}{m}\right)\right\}^{-1}\left\{\sin\pi\left(s-\frac{j}{m}\right)\right\}^{-h-1}m^{-ns}e^{-\pi i(h+1)s}.$$

Thus $\varphi_{j,0}(s) = \varphi_j(s)$. The integers j and h in (1.9) will be restricted as follows:

$$(1.10) \qquad 0 \leqq h|m| + j \leqq n-1, \quad 0 \leqq h \leqq n-1, \quad 0 \leqq j \leqq |m|-1.$$

Then we obtain n different functions $\varphi_{j,h}(s)$, whose only singularities are poles at the points $s = \nu + j/m$, where $\nu = 0, 1, 2, \cdots$. The poles are of order $h+1$ at most.

We now define

$$(1.11) \qquad y_{j,h}(x) = \frac{1}{2\pi i}\int_C \varphi_{j,h}(s)x^{ms}\,ds.$$

Then $y_{j,0}(x) = y_j(x)$. We show that the n solutions $y_{j,h}(x)$ of (1.1) with j and h satisfying (1.10) form a fundamental system of solutions of (1.1). To this end we calculate $y_{j,h}(x)$ by means of the calculus of residues.

We use the formula

$$(1.12) \qquad \frac{\pi^{h+1}e^{-\pi i(h+1)s}}{(\sin\pi s)^{h+1}} = s^{-h-1}\sum_{\mu=0}^{\infty}B_\mu^{(h+1)}\frac{(2\pi is)^\mu}{\mu!},$$

valid for $0 < |s| < 1$, where the $B_\mu^{(h+1)}$ are the Bernoulli numbers of order $h+1$. Hence the residue of $\varphi_{j,h}(s)x^{ms}$ at the point $s = \nu + j/m$, $\nu = 0, 1, \cdots$, is

$$(1.13) \qquad x^{j+m\nu}\sum_{l=0}^{h}a_{l,\nu}(j,h)\{\log(m^{-n}x^m)\}^{h-l},$$

where

$$a_{l,\nu}(j,h) = \frac{1}{(h-l)!}\pi^{-h-1}\, m^{-n\nu-nj/m}\, e^{-j(h+1)\pi i/m} \cdot \sum_{\mu=0}^{l} \frac{(2\pi i)^{\mu}}{\mu!(l-\mu)!} B_{\mu}^{(h+1)} f_j^{(l-\mu)}(\nu), \tag{1.14}$$

with

$$f_j(s) = \left\{\prod_{k=0}^{n-1} \Gamma\left(1+s+\frac{j-k}{m}\right)\right\}^{-1}. \tag{1.15}$$

Therefore,

$$y_{j,h}(x) = -\sum_{l=0}^{h} \{\log(m^{-n}x^m)\}^{h-l} \sum_{\nu=0}^{\infty} a_{l,\nu}(j,h)x^{m\nu+j}. \tag{1.16}$$

The series converges for $x^m \neq 0$, and we see that the n solutions $y_{j,h}(x)$ with (1.10) are linearly independent, and therefore form a fundamental system.

The behavior of the solutions of the fundamental system for $x^m \to \infty$ may be derived from the behavior of $\tilde{y}(x)$ as $x^m \to \infty$. The last function is a special case of a function whose asymptotic behavior has been derived by Barnes [1]. In the terminology of Meijer's G-function (cf. [5]) we have

$$\tilde{y}(x) = G_{0,n}^{n,0}\left(m^{-n}e^{-n\pi i}x^m \,\middle|\, 0, \frac{1}{m}, \cdots, \frac{n-1}{m}\right). \tag{1.17}$$

The functions $y_{j,h}(x)$ are also G-functions:

$$y_j(x) = -\frac{1}{\pi} G_{0,n}^{1,0}\left(m^{-n}e^{-\pi i}x^m \,\middle|\, \frac{j}{m}, 0, \frac{1}{m}, \overset{*}{\cdots}, \frac{n-1}{m}\right), \tag{1.18}$$

where the star denotes that in the sequence $0, 1/m, \cdots, (n-1)/m$ the term j/m is to be omitted, and

$$y_{j,h}(x) = (-1)^{(h+1)(h/2+1)}\pi^{-h-1} \cdot G_{0,n}^{h+1,0}\left(m^{-n}e^{-\pi i(h+1)}x^m \,\middle|\, \frac{j}{m}, \frac{j}{m} \pm 1, \cdots, \frac{j}{m} \pm h, 0, \frac{1}{m}, \overset{**}{\cdots}, \frac{n-1}{m}\right),$$

where the two stars denote that in the sequence $0, 1/m, \cdots, (n-1)/m$ the terms $j/m, j/m \pm 1, \cdots, j/m \pm h$ are to be omitted. The upper sign corresponds to $m > 0$, the lower sign to $m < 0$.

2. Reduction of the analysis to the asymptotic analysis of $\tilde{y}(x)$. We show first that it suffices to find the asymptotic expansion for $x^m \to \infty$ of the solutions of the fundamental system in the sector

$$0 \leqq \arg(m^{-n}x^m) \leqq 2\pi. \tag{2.1}$$

Next we show how the asymptotic expansions of $y_{j,h}(x)$ in this sector can be derived from those of $\tilde{y}(x)$ in the sector

$$0 \leqq \arg(m^{-n}x^m) \leqq 2n\pi. \tag{2.2}$$

These results are consequences of the fact that if $y(x)$ is a solution of (1.1), then also $y(xe^{2p\pi i/m})$ satisfies (1.1) for arbitrary choice of the integer p.

If $m \neq 0, \pm 1, \cdots, \pm(n-1)$, then we have by (1.8),

$$y_j(x) = e^{2pj\pi i/m} y_j(xe^{-2p\pi i/m}), \qquad p = 0, \pm 1, \pm 2, \cdots. \tag{2.3}$$

Hence the asymptotic behavior of $y_j(x)$ as $x^m \to \infty$ in

$$2p\pi \leqq \arg(m^{-n}x^m) \leqq 2(p+1)\pi \tag{2.4}$$

follows from the behavior of $y_j(x)$ as $x^m \to \infty$ on (2.1). Now suppose that $m = \pm 1, \pm 2, \cdots, \pm(n-1)$. Then from the binomial expansion we have

$$\begin{aligned} e^{2p\pi is} &= e^{2pj\pi i/m}\{(e^{2\pi i(s-j/m)} - 1) + 1\}^p \\ &= \sum_{q=0}^{h} \binom{p}{q} (2i)^q e^{(2p-q)j\pi i/m}\{e^{\pi is} \sin \pi(s - j/m)\}^q \\ &\quad + \{e^{\pi is} \sin \pi(s - j/m)\}^{h+1} g_0(s), \end{aligned} \tag{2.5}$$

where $g_0(s)$ is an analytic function of s. Moreover, $g_0(s)$ is bounded on C and also on the right of C. From this equation, (1.9) and (1.11) we deduce

$$y_{j,h}(x) = \sum_{q=0}^{h} \binom{p}{q} (2i)^q e^{(2p-q)j\pi i/m} y_{j,h-q}(xe^{-2p\pi i/m}). \tag{2.6}$$

Therefore the behavior of the functions $y_{j,h}(x)$ as $x^m \to \infty$ on (2.4) can be deduced from this behavior on (2.1).

In order to express the functions $y_{j,h}(x)$ in terms of the functions $\tilde{y}(x\,e^{2p\pi i/m})$ (p an integer) we first express $\varphi_{j,h}(s)$ in terms of $\tilde{\varphi}(s)$ (cf. (1.6) and (1.9)):

$$\begin{aligned} \varphi_{j,h}(s) &= \tilde{\varphi}(s)\pi^{-n} e^{(n-h-1)\pi is}\left\{\prod_{k=0}^{n-1} \sin \pi\left(\frac{k}{m} - s\right)\right\} \left\{\sin \pi\left(s - \frac{j}{m}\right)\right\}^{-h-1} \\ &= \tilde{\varphi}(s)\pi^{-n}(-1)^{(h+1)(h/2+1)} e^{(n-h-1)\pi is}\left\{\prod_{k=0}^{n-1}{}' \sin \pi\left(\frac{k}{m} - s\right)\right\}, \end{aligned} \tag{2.7}$$

where the prime here means that the factors with $k = j,\ k = j + |m|, \cdots, k = j + h|m|$ in the product are omitted. Next we expand this product in powers of $e^{\pi is}$:

$$\begin{aligned} \varphi_{j,h}(s) &= \tilde{\varphi}(s)(2\pi i)^{-n}(-2i)^{h+1} e^{\{n(n-1)/2 - j(h+1)\}\pi i/m} \\ &\quad \cdot \prod_{k=0}^{n-1}{}' \{1 - e^{2\pi i(s-k/m)}\}; \end{aligned} \tag{2.8}$$

so that

$$\varphi_{j,h}(s) = \sum_{k=0}^{n-h-1} c_k(j,h)\, e^{2k\pi is}\tilde{\varphi}(s), \tag{2.9}$$

where the constants $c_k(j, h)$ follow from (2.8). Thus the $c_k(j, h)$ are defined by

$$\sum_{k=0}^{n-h-1} c_k(j, h)x^k = c_0(j, h) \prod_{k=0}^{n-1}{}' (1 - xe^{-2k\pi i/m}), \tag{2.10}$$

$$\begin{aligned} c_0(j, h) &= (2\pi i)^{-n}(-2i)^{h+1}\, e^{\{n(n-1)/2 - j(h+1)\}\pi i/m}, \\ c_{n-h-1}(j, h) &= (-2\pi i)^{-n}(2i)^{h+1}\, e^{\{j(h+1) - n(n-1)/2\}\pi i/m}. \end{aligned} \tag{2.11}$$

From (2.9), (1.11) and (1.7) it follows that

$$y_{j,h}(x) = \sum_{k=0}^{n-h-1} c_k(j, h)\tilde{y}(xe^{2k\pi i/m}). \tag{2.12}$$

Hence the behavior of $y_{j,h}(x)$ on (2.1) can be deduced from the behavior of $\tilde{y}(x)$ on (2.2).

3. The asymptotic expansion of $\tilde{y}(x)$**.** The derivation of the asymptotic expansion of $\tilde{y}(x)$ is split into three lemmas. First we consider the product of gamma functions in (1.6).

LEMMA 1. *There exist constants* $b_0, b_1, b_2, \cdots$ *such that for any positive integer* N,

$$\prod_{k=0}^{n-1} \Gamma\left(\frac{k}{m} - s\right) = n^{ns}\left\{\sum_{j=0}^{N-1} b_j\Gamma(1 - ns - \alpha - j) + \rho_N(s)\Gamma(1 - ns - \alpha - N)\right\}, \tag{3.1}$$

where

$$\alpha = \frac{n+1}{2} - \frac{n(n-1)}{2m}, \qquad b_0 = n^{\alpha - 1/2}\,(2\pi)^{(n-1)/2}, \tag{3.2}$$

and $\rho_N(s)$ *is an analytic function of* s *for* $s \neq \nu + k/m, \nu = 0, 1, \cdots; k = 0, \cdots, n-1$, *with the property*

$$\rho_N(s) = O(1) \tag{3.3}$$

as $s \to \infty$ *uniformly in*

$$|\arg(-s)| \leqq \pi - \varepsilon. \tag{3.4}$$

Here ε *is an arbitrary constant with* $0 < \varepsilon < \pi$.

Further,

$$\rho_N(s) = r_N(s) + g_N(s), \tag{3.5}$$

where $r_N(s)$ *is analytic except for simple poles at*

$$s = -(\alpha + N + \nu)/n, \qquad \nu = 0, 1, 2, \cdots, \tag{3.6}$$

and $g_N(s)$ *is analytic for* $|\mathrm{Im}\, s| > (n-1)|\mathrm{Im}\, 1/m|$.

Finally,

$$r_N(s) = O(1) \tag{3.7}$$

as $s \to \infty$ *uniformly in* $|\arg s| \leqq \pi - \varepsilon$, *and there exists a constant* K *independent of* s *such that*

$$|g_N(s)| \leqq K\, e^{-2\pi|\mathrm{Im}\, s|}|s^N| \tag{3.8}$$

for $|\arg s| \leqq \pi - \varepsilon$, $|\operatorname{Im} s| \geqq a$.

Proof. The assertions up to (3.4) easily follow from Stirling's formula (cf. [2, § 3.3]). From (3.1) we deduce

$$
\begin{aligned}
\rho_N(s) = (-1)^N \pi^{n-1} n^{-ns} & \frac{\Gamma(ns + \alpha + N)}{\prod_{k=0}^{n-1} \Gamma\left(1 - \frac{k}{m} + s\right)} \frac{\sin \pi(ns + \alpha)}{\prod_{k=0}^{n-1} \sin \pi\left(\frac{k}{m} - s\right)} \\
& - \sum_{j=0}^{N-1} b_j(-1)^{N-j}(ns + \alpha + j) \cdots (ns + \alpha + N - 1).
\end{aligned} \tag{3.9}
$$

For the products of gamma functions in (3.9) we again have from Stirling's formula,

$$
\begin{aligned}
\frac{\Gamma(ns + \alpha + N)}{\prod_{k=0}^{n-1} \Gamma\left(1 - \frac{k}{m} + s\right)} = n^{ns} \Bigg\{ & \sum_{j=0}^{N-1} b_j^*(ns + \alpha + j) \cdots (ns + \alpha + N - 1) \\
& + (-1)^N (2\pi)^{1-n} r_N(s) \Bigg\},
\end{aligned} \tag{3.10}
$$

where $b_0^*, \cdots, b_{N-1}^*$ are constants and $r_N(s)$ satisfies the conditions of Lemma 1.

For the product of sines in (3.9) we find, by using Euler's formula for the sine function,

$$
\frac{\sin \pi(ns + \alpha)}{\prod_{k=0}^{n-1} \sin \pi\left(\frac{k}{m} - s\right)} = 2^{n-1} + g^*(s), \tag{3.11}
$$

where $g^*(s)$ is analytic for $|\operatorname{Im} s| > (n - 1)|\operatorname{Im} 1/m|$ and has the property

$$
|g^*(s)| \leqq K_1 e^{-2\pi|\operatorname{Im} s|}, \tag{3.12}
$$

for $|\operatorname{Im} s| \geqq a > (n - 1)|\operatorname{Im} 1/m|$. Here K_1 is a constant independent of s. Combining (3.9), (3.10) and (3.11), we obtain

$$
\begin{aligned}
\rho_N(s) = (-1)^N \pi^{n-1} \Bigg\{ & \sum_{j=0}^{N-1} b_j^*(ns + \alpha + j) \cdots (ns + \alpha + N - 1) \\
& + (-1)^N (2\pi)^{1-n} r_N(s) \Bigg\} \{s^{n-1} + g^*(s)\} \\
& - (-1)^N \sum_{j=0}^{N-1} (-1)^j b_j(ns + \alpha + j) \cdots (ns + \alpha + N - 1).
\end{aligned} \tag{3.13}
$$

By letting $s \to i\infty$ we find that

$$
(2\pi)^{n-1} b_j^* = (-1)^j b_j.
$$

From this equation and (3.13) and (3.12) we deduce (3.5) and the properties of $g_N(s)$ stated in Lemma 1.

Using Lemma 1 we deduce the following integral representation for $\tilde{y}(x)$.

LEMMA 2. *If*

$$
\left| \arg\left(\frac{1}{m} e^{-\pi i} x^{m/n}\right) \right| < \tfrac{1}{2}\pi, \tag{3.14}
$$

then

$$\tilde{y}(x) = \frac{1}{n}\sum_{j=0}^{N-1} b_j(n^n m^{-n} e^{-n\pi i} x^m)^{(1-\alpha-j)/n} \exp\left(\frac{n}{m} x^{m/n}\right) + \sigma_N(x), \tag{3.15}$$

where

$$\sigma_N(x) = (n^n m^{-n} e^{-n\pi i} x^m)^{(3-\alpha-N)/n} \int_0^\infty f(t) \exp\left(\frac{n}{m} x^{m/n} t\right) dt \tag{3.16}$$

with

$$f(t) = \frac{1}{2\pi i}\int_{w-i\infty}^{w+i\infty} \frac{\rho_N(s)}{(2-ns-\alpha-N)(1-ns-\alpha-N)} t^{2-ns-\alpha-N} ds \tag{3.17}$$

for $t \geqq 0$. *Here* w *is a real number such that*

$$w < (1 - \operatorname{Re}\alpha - N)/n. \tag{3.18}$$

Proof. We consider (1.7) with $\tilde{\varphi}(s)$ defined as in (1.6) and the contour C as defined after (1.6). Substituting (3.1) in (1.6) and using the calculus of residues we obtain (3.15) with

$$\sigma_N(x) = \frac{1}{2\pi i}\int_C \Gamma(1-ns-\alpha-N)\rho_N(s)(n^n m^{-n} e^{-n\pi i} x^m)^s\, ds. \tag{3.19}$$

Now

$$\Gamma(1-ns-\alpha-N) = \pi\{\Gamma(ns+\alpha+N)\sin\pi(ns+\alpha+N)\}^{-1}.$$

Applying Stirling's formula to the right-hand side we deduce that there exists a constant K_2 such that for $\operatorname{Re} s \geqq w$ and $|\operatorname{Im} s| \geqq a$ we have:

$$|\Gamma(1-ns-\alpha-N)| \leqq K_2\{\exp(-\tfrac{1}{2}n\pi|\operatorname{Im} s|)\}\cdot|s|^{-nw-\operatorname{Re}\alpha-N+1/2}.$$

From this inequality and the relations (3.5), (3.7) and (3.8) we deduce that we may replace the contour C in (3.19) by the straight line from $s = w - i\infty$ to $s = w + i\infty$ provided that (3.14) is satisfied. Then substituting the formula

$$\Gamma(3-ns-\alpha-N)z^s = z^{(3-\alpha-N)/n}\int_0^\infty t^{2-ns-\alpha-N}\exp(-z^{1/n}t)\,dt, \quad \operatorname{Re} z^{1/n} > 0,$$

in the new integral in (3.19) and changing the order of integration we obtain (3.15).

The function $f(t)$ of Lemma 2 has the following remarkable properties.

LEMMA 3. *If* $0 \leqq t \leqq 1$, *then* $f(t) = 0$. *The function* $f(t)$ *defined on* $t \geqq 1$, *can be continued analytically to the domain* D *where* $|\arg t| < 2\pi/n$ *or* $|t| > 1$, $|\arg t| < \pi$.

Further

$$f(t) = t^{2-nw-\alpha-N}O(1) \tag{3.20}$$

as $t \to \infty$, *uniformly on* D.

Proof. In view of Lemma 1 and (3.18) the integrand in (3.17) is analytic and $O(s^{-2})$ as $s \to \infty$ in $\operatorname{Re} s \leqq w$ and $0 \leqq t \leqq 1$. Hence $f(t) \equiv 0$ for $0 \leqq t \leqq 1$. For

$t \geqq 1$ we write $f(t) = R(t) + G(t)$, where

$$(3.21)\quad R(t) = \frac{1}{2\pi i}\int_{w-i\infty}^{w+i\infty} \frac{\rho_N(s)}{(2 - ns - \alpha - N)(1 - ns - \alpha - N)} t^{2-ns-\alpha-N}\, ds,$$

$$(3.22)\quad G(t) = \frac{1}{2\pi i}\int_{w-i\infty}^{w+i\infty} \frac{g_N(s)}{(2 - ns - \alpha - N)(1 - ns - \alpha - N)} t^{2-ns-\alpha-N}\, ds.$$

This is justified by (3.17), (3.5), (3.7) and (3.8). For $t \geqq 1$ the integrand in (3.21) is $O(s^{-2})$ as $s \to \infty$ in $\operatorname{Re} s \geqq w$, in view of (3.7). Therefore if $t \geqq 1$, then $R(t)$ is equal to minus the sum of the residues of this integrand in the poles to the right of $\operatorname{Re} s = w$. In view of Lemma 1 these poles are given by $s = (2 - \alpha - N - \nu)/n$, for a finite number of nonnegative integers ν. Hence $R(t)$ is a polynomial for $t \geqq 1$, and therefore by analytic continuation for all t. Further we easily deduce (3.20) with $f(t)$ replaced by $R(t)$ for $t \geqq 1$ and hence again for all t.

From (3.8) it follows that the integral in (3.22) exists and represents an analytic function of t for $|\arg t| < 2\pi/n$. Further, using the properties of $g_N(s)$ mentioned in Lemma 1 we see that for $t > 1$ we may replace the path of integration in (3.22) by C. The new integral converges and represents an analytic function of t for $|t| > 1$. Finally we see that this integral is $O(1)\, t^{2-nw-\alpha-N}$ as $t \to \infty$ uniformly in $|t| > 1$, $|\arg t| < \pi$. Combining these properties of $R(t)$ and $G(t)$ we obtain the result stated in Lemma 3.

Next we prove the following theorem.

THEOREM 1. *Let $E(x)$ be defined formally by the series*

$$(3.23)\qquad E(x) = \left\{\exp\left(\frac{n}{m} x^{m/n}\right)\right\} (m^{-n} e^{-n\pi i} x^m)^{(n-1)(1/m-1/n)/2} \cdot \left\{ n^{-1/2}(2\pi)^{(n-1)/2} + \sum_{\nu=1}^{\prime} d_\nu x^{-m\nu/n} \right\},$$

the constants $d_1, d_2, \cdots$ being chosen so that $E(x)$ satisfies (1.1). *Then*

$$(3.24)\qquad \tilde{y}(x) \sim E(x)$$

as $x^m \to \infty$ uniformly in

$$(3.25)\qquad -\pi + \varepsilon \leqq \arg (m^{-n} x^m) \leqq (2n + 1)\pi - \varepsilon,$$

where ε is an arbitrary constant satisfying $0 < \varepsilon < \pi$.

Proof. Lemmas 2 and 3 imply

$$(3.26)\qquad \sigma_N(x) = (n^n m^{-n} e^{-n\pi i} x^m)^{(3-\alpha-N)/n} \exp\left(\frac{n}{m} x^{m/n}\right) \cdot \int_0^\infty f(t + 1) \exp\left(\frac{n}{m} x^{m/n}\, t\right) dt,$$

provided that (3.14) holds. According to Lemma 3 the function $f(t + 1)$ is analytic for $|\arg t| < \pi(1/2 + 1/n)$, and satisfies (3.20) as $t \to \infty$ uniformly in this sector. Moreover, $f(1) = 0$, so $f(t + 1) = tO(1)$ as $t \to 0$. Now it is easy to see by rotating the path of integration in (3.26) that the integral in (3.26) can be continued analytically from the sector (3.14) to the sector (3.25) and that it is $x^{-2m/n}O(1)$ as $x^m \to \infty$ uniformly in (3.25), (cf. [3, p. 49]). Combining this result with (3.26), (3.15) and (3.2) we obtain (3.24) as $x^m \to \infty$ uniformly in (3.25). The constants d_ν are multiples of the b_ν of Lemma 1. A recurrence formula for them may be found by substituting (3.24) and (3.23) in (1.1).

Remark. The method used above also may be used to obtain asymptotic expansions of

$$\int_C \prod_{j=0}^{n-1} \Gamma(m_j s - a_j) z^s \, ds,$$

where $m_0, \cdots, m_{n-1}$ are positive constants and $a_0, \cdots, a_{n-1}$ are complex constants.

4. The asymptotic expansions for the fundamental system. From Theorem 1 and the formulas (2.12) and (2.6) we easily obtain the asymptotic expansions of the solutions $y_{j,h}(x)$ defined in § 1. We use the notation of § 1. If $m \neq 0, \pm 1, \cdots, \pm(n-1)$, then $h = 0$ and $j = 0, \cdots, n-1$; if $m = \pm 1, \pm 2, \cdots, \pm(n-1)$, then j and h are integers satisfying (1.10). Further we use $E(x)$ defined by (3.23). Then we derive the following result.

THEOREM 2. *Define*

$$c = -\pi^{-n}(2i)^{1-n} \exp\{\tfrac{1}{2}n(n-1)\pi i/m\}. \tag{4.1}$$

Let ε be a constant such that $0 < \varepsilon < \pi$. Then the following asymptotic expansions hold uniformly as $x^m \to \infty$ in the indicated sectors:

$$y_{j,h}(x) \sim c(-2i)^h e^{-j(h+1)\pi i/m} E(x) + c(2i)^h e^{j(h+1)\pi i/m} E(xe^{-2(h+1)\pi i/m}) \tag{4.2}$$

in

$$-\pi + \varepsilon \leqq \arg(m^{-n}x^m) \leqq (2h+3)\pi - \varepsilon, \tag{4.3}$$

except in the case $m = \pm 1$ and $j = 0$, $h = n-1$;

$$y_{j,h}(x) \sim c(-2i)^h e^{-j(h+1)\pi i/m} E(x) \tag{4.4}$$

in

$$-\pi + \varepsilon \leqq \arg(m^{-n}x^m) \leqq (h+1)\pi - \varepsilon; \tag{4.5}$$

relation (4.4) *also holds in* (4.3) *in the case $m = \pm 1, j = 0, h = n-1$;*

$$y_{j,h}(x) \sim c(2i)^h e^{j(h+1)\pi i/m} E(xe^{-2(h+1)\pi i/m}) \tag{4.6}$$

in

$$(h+1)\pi + \varepsilon \leqq \arg(m^{-n}x^m) \leqq (2h+3)\pi - \varepsilon; \tag{4.7}$$

$$\begin{aligned} y_{j,h}(x) \sim c(2i)^h e^{j(2p-h-1)\pi i/m}\Bigg[&\binom{p-1}{h} E(xe^{-2p\pi i/m}) \\ &+ e^{2j\pi i/m}\binom{p}{h} E(xe^{-2(p+1)\pi i/m})\Bigg] \end{aligned} \tag{4.8}$$

in

$$(2p-1)\pi + \varepsilon \leqq \arg(m^{-n}x^m) \leqq (2p+3)\pi - \varepsilon \tag{4.9}$$

for all integers p with $p \leqq -1$ *or* $p \geqq h+1$;

$$y_{j,h}(x) \sim c(2i)^h \binom{p-1}{h} e^{j(2p-h-1)\pi i/m} E(xe^{-2p\pi i/m}) \tag{4.10}$$

in

$$(2p-1)\pi + \varepsilon \leqq \arg(m^{-n}x^m) \leqq (2p+1)\pi - \varepsilon \tag{4.11}$$

for all integers p with $p < 0$ *or* $p > h+1$.

Proof. Suppose (4.3) holds. Then we apply Theorem 1 to the functions on the right-hand side of (2.12) and determine the dominant terms. For $k = 1, \cdots, n-h-2$ we have

$$E(xe^{2k\pi i/m}) = o(1)E(x) \quad \text{or} \quad o(1)E(xe^{2(n-h-1)\pi i/m}) \tag{4.12}$$

since

$$-\pi + \varepsilon \leqq \arg(m^{-n}x^m) \leqq \arg(m^{-n}x^m e^{2k\pi i}) - 2\pi,$$

$$\arg(m^{-n}x^m e^{2k\pi i}) + 2\pi \leqq \arg(m^{-n}x^m e^{2(n-h-1)\pi i}) \leqq (2n+1)\pi - \varepsilon.$$

From this result and (2.12) and (3.24) we deduce

$$y_{j,h}(x) \sim c_0(j,h)E(x) + c_{n-h-1}(j,h)E(xe^{2(n-h-1)\pi i/m}) \tag{4.13}$$

in (4.3) except in the case $m = \pm 1, j = 0, h = n-1$. Now (3.23) implies

$$E(xe^{2n\pi i/m}) = (-1)^{n-1} e^{n(n-1)\pi i/m} E(x). \tag{4.14}$$

From (4.13), (4.14), (2.11) and (4.1) we deduce (4.2) on (4.3) except in the case $m = \pm 1$, $h = n-1$, $j = 0$. In the exceptional case the right-hand side of (2.12) consists only of $c_0(0, n-1)E(x)$. Hence (4.4) holds in (4.3) in the case $m = \pm 1$, $j = 0$, $h = n-1$.

Now we exclude the case $m = \pm 1$, $j = 0$, $h = n-1$. Then $0 \leqq h \leqq n-2$ by (1.10). Consider the sector (4.5). There we have

$$(h+1)\pi + \varepsilon \leqq -\arg(m^{-n}x^m e^{-2(h+1)\pi i}) \leqq 2n\pi - \arg(m^{-n}x^m) - 2\pi \leqq (2n-1)\pi.$$

Hence by (3.23) the second term in the right-hand side of (4.2) may be discarded in (4.5), that is, (4.4) holds in (4.5).

In (4.7) we have

$$\begin{aligned} -\pi + \varepsilon &\leqq -\arg(m^{-n}x^m e^{-2(h+1)\pi i}) \\ &\leqq (h+1)\pi - \varepsilon \leqq 2\varepsilon + \arg(m^{-n}x^m) \\ &\leqq 2n\pi + \arg(m^{-n}x^m e^{-2(h+1)\pi i}) - 2(\pi - \varepsilon). \end{aligned}$$

From this we deduce that the first term in the right-hand side of (4.2) may be discarded in (4.7), that is, (4.6) is valid in (4.7).

Now suppose (4.9) holds and $p \geqq h+1$ or $p \leqq -1$. Then we use (2.6) and expand the terms $y_{j,k}(xe^{-2p\pi i/m})$ by means of (4.4) if $k \geqq 2$, and by means of (4.2) if $k = 0$ or 1. This gives

$$\begin{aligned} y_{j,h}(x) \sim c(2i)^h e^{(2p-h-1)j\pi i/m} \Bigg[&\binom{p-1}{h} E(xe^{-2p\pi i/m}) \\ &+ \binom{p}{h} e^{2j\pi i/m} E(xe^{-2(p+1)\pi i/m}) + \binom{p}{h-1} e^{4j\pi i/m} E(xe^{-2(p+2)\pi i/m}) \Bigg] \end{aligned} \tag{4.15}$$

except in the case $h = 0$ and in the case $n = 2$, $m = \pm 1$, $h = 1$, $j = 0$. In these cases we omit the last term in (4.15), that is, (4.8) holds. In (4.9) we have $|\arg (m^{-n}x^m e^{-2p\pi i})| \leqq \pi$ or $|\arg (m^{-n}x^m e^{-2(p+1)\pi i})| \leqq \pi$. If $n \geqq 3$, then in (4.9),

$$\pi + \varepsilon \leqq |\arg (m^{-n}x^m e^{-2(p+2)\pi i})| \leqq 5\pi - \varepsilon \leqq (2n - 1)\pi - \varepsilon.$$

Hence the last term in (4.15) may be omitted in (4.9) if $n \geqq 3$, that is, (4.8) holds. If $n = 2$, $m \neq \pm 1$, then $h = 0$ according to (1.10) and so (4.8) holds again.

It is easy to see that in (4.11) we may discard the last term in (4.8), that is, (4.10) holds.

Remark. Theorem 2 gives the asymptotic expansions of $y_{j,h}(x)$ in overlapping sectors with vertex $x^m = 0$. The expansions of any other solution of (1.1) can be found by expressing this solution as a linear combination of $\tilde{y}(x\,e^{-2j\pi i/m})$, $j = 0, 1, \cdots, n - 1$, and then applying Theorem 1.

The constants $c_k(j, h)$ in (2.12), defined by (2.10), are the Stokes multipliers for the $y_{j,h}(x)$ with respect to the fundamental system $\tilde{y}(x\,e^{-2j\pi i/m}), j = 0, \cdots, n - 1$. The last system is uniquely determined by the condition that the jth solution is asymptotically equal to $E(x\,e^{-2j\pi i/m})$ as $x^m \to \infty$ in

$$-\pi + \varepsilon \leqq \arg (m^{-n}x^m e^{-2j\pi i}) \leqq (2n + 1)\pi - \varepsilon$$

or in particular for $\arg (m^{-n}x^m) = (n + 2j)\pi$. This easily follows from Theorem 1. Thus the Stokes multipliers for this system are uniquely determined.

If a solution of (1.1) is expressed as a linear combination of the solutions $y_{j,h}(x)$, then its asymptotic expansion may be obtained by applying (2.6), (2.12) and Theorem 1.

5. Another fundamental system defined by asymptotic properties. In [6] Turrittin gave a fundamental system of solutions $A_\mu(x), \mu = -[(n - 1)/2], -[(n - 1)/2] + 1, \ldots, [n/2]$, of (1.1) such that

$$A_\mu(x) \sim E(xe^{2\mu\pi i/m}) \tag{5.1}$$

as $x^m \to \infty$ uniformly in

$$-([n/2] + 1)\pi \leqq \arg (m^{-n}x^m) \leqq ([(n - 1)/2] + 1/2)\pi.$$

Here $[q]$ denotes, as usual, the largest integer which does not exceed the given real number q. Further n and $m - n$ are assumed to be positive integers in [6].

Turrittin used this system to obtain unique Stokes multipliers for the $y_j(x)$. However, these multipliers appear to be more complicated than the unique multipliers $c_k(j, h)$, defined in (2.10), for the $y_j(x)$ with respect to the system $\tilde{y}(xe^{-2j\pi i/m})$.

Here we first determine these functions $A_\mu(x)$ in the more general case of arbitrary m, other than $m = n$ or 0.

THEOREM 3. *Let the constants $\gamma_0, \gamma_1, \cdots, \gamma_n$ be defined by*

$$\prod_{j=0}^{n-1} (1 - x\,e^{-2j\pi i/m}) = \sum_{j=0}^{n} \gamma_j x^j, \quad \gamma_0 = 1, \quad \gamma_n = (-1)^n e^{-n(n-1)\pi i/m}, \tag{5.2}$$

and define for integral values of μ:

(5.3) $$A_\mu(x) = \tilde{y}(xe^{2\mu\pi i/m}) \quad \textit{if} \quad \left[\frac{n}{4}\right] + 1 \leqq \mu \leqq \left[\frac{n}{2}\right],$$

(5.4) $$A_\mu(x) = -\gamma_n \tilde{y}(xe^{2(\mu+n)\pi i/m}) \quad \textit{if} \quad -\left[\frac{n-1}{2}\right] \leqq \mu \leqq -\left[\frac{n-1}{4}\right],$$

(5.5) $$A_\mu(x) = \sum_{j=0}^{[n/2]-2\mu} \gamma_j \tilde{y}(xe^{2(\mu+j)\pi i/m}) \quad \textit{if} \quad -\left[\frac{n-1}{4}\right] \leqq \mu \leqq \left[\frac{n}{4}\right].$$

Then (5.1) *holds as* $x^m \to \infty$ *uniformly in*

(5.6) $$-\left(\left[\frac{n}{2}\right] + 1\right)\pi + \varepsilon \leqq \arg(m^{-n}x^m) \leqq \left(\left[\frac{n-1}{2}\right] + 1\right)\pi - \varepsilon$$

for any constant ε *with* $0 < \varepsilon \leqq \pi/2$.

The functions $A_\mu(x)$, $-[(n-1)/2] \leqq \mu \leqq [n/2]$, *are the only solutions of* (1.1) *satisfying* (5.1) *in* (5.6).

Proof. First suppose that $[n/4] + 1 \leqq \mu \leqq [n/2]$. Consider values of x with $\arg(m^{-n}x^m) = (n - 2\mu)\pi$, because the right-hand side of (5.1) is then as small as possible. These values of x satisfy (5.6) since

$$0 \leqq n - 2\mu \leqq n - 2[n/4] - 2 \leqq [(n-1)/2].$$

We may write any solution $A_\mu(x)$ of (1.1) in the form

$$A_\mu(x) = \sum_{j=-[n/2]}^{[(n-1)/2]} \alpha_j \tilde{y}(xe^{2(\mu+j)\pi i/m}).$$

If $\arg(m^{-n}x^m) = (n - 2\mu)\pi$, then we may apply Theorem 1 to each term on the right-hand side, since $0 \leqq n - 2[n/2] \leqq n + 2[(n-1)/2] \leqq 2n$. We deduce from this theorem that (5.1) holds if, and only if, all α_j's vanish except $\alpha_0 = 1$. This gives (5.3), and $A_\mu(z)$ satisfies (5.1) in (5.6) by Theorem 1. Next suppose that $0 \leqq \mu \leqq [n/4]$. We write a solution $A_\mu(x)$ of (1.1) in the form

$$A_\mu(x) = \sum_{j=-\mu-[(n-1)/4]}^{n-\mu-[(n-1)/4]-1} \alpha_j \tilde{y}(xe^{2(\mu+j)\pi i/m}).$$

For $\arg(m^{-n}x^m) = [(n-1)/2]\pi$ we may apply Theorem 1 to each term on the right-hand side, because

$$\begin{aligned} 0 &\leqq ([(n-1)/2] - 2[(n-1)/4]\pi \leqq \arg(m^{-n}x^m\, e^{2(\mu+j)\pi i}) \\ &\leqq (2n - 2[(n-1)/4] - 2 + [(n-1)/2])\pi \leqq 2n\pi. \end{aligned}$$

We then deduce that (5.1) holds for these values of x if, and only if, $\alpha_0 = 1$, $\alpha_j = 0$ for $j < 0$ and $j > [n/2] - 2\mu$ because for these values of j either

$$0 \leqq \arg(m^{-n}x^m\, e^{2(\mu+j)\pi i}) \leqq \arg(m^{-n}x^m\, e^{2\mu\pi i}) - 2\pi$$

or

$$2n\pi \geqq \arg(m^{-n}x^m\, e^{2(\mu+j)\pi i}) \geqq 2n\pi - \arg(m^{-n}x^m\, e^{2\mu\pi i}).$$

Hence

$$(5.7) \qquad A_\mu(x) = \sum_{j=0}^{[n/2]-2\mu} \alpha_j \tilde{y}(x\, e^{2(\mu+j)\pi i/m}), \quad \alpha_0 = 1.$$

Further we deduce from Theorem 1 that (5.1) holds in

$$(5.8) \qquad -2\mu\pi \leqq \arg(m^{-n}x^m) \leqq ([(n-1)/2]+1)\pi - \varepsilon,$$

since in this sector

$$\arg(m^{-n}x^m\, e^{2([n/2]-\mu)\pi i}) \leqq 2n\pi - \arg(m^{-n}x^m\, e^{2\mu\pi i}) - 2\varepsilon.$$

From (5.2) we have

$$\prod_{j=0}^{n-1} \sin \pi\left(s - \frac{j}{m}\right) = (-2i)^{-n}\, e^{n(n-1)\pi i/2m} \sum_{j=0}^{n} \gamma_j\, e^{(2j-n)\pi i s}.$$

Using this equation and (1.7) and (1.6) we find that

$$(5.9) \qquad \begin{aligned} \sum_{j=0}^{n} \gamma_j \tilde{y}(xe^{2j\pi i/m}) &= (2\pi i)^{n-1}\, e^{-n(n-1)\pi i/2m} \\ &\quad \cdot \int_C \left\{ \prod_{j=0}^{n-1} \Gamma\left(1+s-\frac{j}{m}\right)\right\}^{-1} m^{-ns}x^{ms}\, ds = 0. \end{aligned}$$

This result is substituted in (5.7), using $\gamma_0 = 1$. We find

$$(5.10) \quad A_\mu(x) = \sum_{j=1}^{[n/2]-2\mu} (\alpha_j - \gamma_j)\tilde{y}(xe^{2(\mu+j)\pi i/m}) - \sum_{j=[n/2]-2\mu+1}^{n} \gamma_j \tilde{y}(xe^{2(\mu+j)\pi i/m}).$$

To the last term in this formula we apply Theorem 1 with the definition (5.2) and (4.14). Then

$$(5.11) \qquad -\gamma_n \tilde{y}(xe^{2(\mu+n)\pi i/m}) \sim E(xe^{2\mu\pi i/m})$$

on the part of (5.6) complementary to (5.8).

Now we consider (5.10) in

$$(5.12) \qquad \arg(m^{-n}x^m) = -(2\mu+k)\pi$$

for $k = 1, \cdots, [n/2]-2\mu$. In this event

$$\arg(m^{-n}x^m) + 2(\mu+k)\pi = k\pi, \quad \arg(m^{-n}x^m) + 2(\mu+n)\pi = (2n-k)\pi.$$

In the case $k = 1$ we see from Theorem 1 and (5.11) that (5.1) holds in (5.12) if and only if $\alpha_1 = \gamma_1$. Then (5.1) is valid in

$$(5.13) \qquad -\pi + \varepsilon \leqq \arg(m^{-n}x^m) + (2\mu+k)\pi \leqq \pi.$$

By applying the same reasoning for $k = 2, \cdots, [n/2]-2\mu$ successively we find that (5.1) holds on (5.12) and consequently in (5.13) if and only if $\alpha_k = \gamma_k$. The result is that $A_\mu(x)$ has to satisfy (5.5) and that then (5.1) holds in (5.6).

For $-[(n-1)/4] \leqq \mu < 0$ the proof is similar. We consider

$$A_\mu(x) = \sum_{j=[n/4]+1-\mu}^{n+[n/4]-\mu} \beta_j \tilde{y}(xe^{2(j+\mu)\pi i/m}).$$

The condition (5.1) for $\arg(m^{-n}x^m) = -[n/2]\pi$ is satisfied according to Theorem 1 if and only if $\beta_j = 0$ for $j > n$ and $j \leqq [n/2] - 2\mu$, $\beta_n = -\gamma_n$. In the same way as above using (5.9) we deduce the relation (5.5). Relation (5.4) follows in the same way as (5.3).

COROLLARY. *From* (5.5) *and* (5.9) *we deduce that*

$$A_\mu(x) = -\sum_{j=[n/2]-2\mu+1}^{n} \gamma_j \tilde{y}(xe^{2(\mu+j)\pi i/m}) \quad \text{if} \quad -\left[\frac{n-1}{4}\right] \leqq \mu \leqq \left[\frac{n}{4}\right]. \tag{5.14}$$

Using Theorem 3 and (2.11) we may express the solutions of the fundamental system $y_{j,h}(x)$ of § 1 in terms of the solutions $A_\mu(x)$. The result is contained in the following theorem.

THEOREM 4. *Let* $c_0(j, h)$ *and* $\gamma_0, \cdots, \gamma_n$ *be defined by* (2.11) *and* (5.2). *Then*

$$y_{j,h}(x) = c_0(j, h) \sum_{\mu=-[(n-1)/2]}^{[n/2]} \alpha_\mu(j, h) A_\mu(x), \tag{5.15}$$

where

$$\alpha_\mu(j, h) = 0 \quad \text{if} \quad -h \leqq \mu \leqq -1; \tag{5.16}$$

$$\alpha_\mu(j, h) = \frac{-1}{\gamma_n} \sum_{r=0}^{n+\mu} \gamma_r \binom{\mu + n - r + h}{h} \exp\{-2j(n + \mu - r)\pi i/m\}, \tag{5.17}$$

if $-[(n-1)/2] \leqq \mu \leqq \min\{-[(n-1)/2] + h, -h - 1\}$;

$$\alpha_\mu(j, h) = \frac{1}{\gamma_n} \sum_{r=2\mu+2n-[n/2]}^{n} \gamma_r \binom{\mu + n - r + h}{h} \exp\{-2j(n + \mu - r)\pi i/m\}, \tag{5.18}$$

if $-[(n-1)/2] + h + 1 \leqq \mu \leqq -[(n-1)/4] - 1$;

$$\alpha_\mu(j, h) = \binom{\mu + h}{h} \exp(-2j\mu\pi i/m), \tag{5.19}$$

if $-[(n-1)/4] \leqq \mu \leqq -h - 1$ *and if* $0 \leqq \mu \leqq [n/4]$;

$$\alpha_\mu(j, h) = \sum_{r=0}^{2\mu-[n/2]-1} \gamma_r \binom{\mu - r + h}{h} \exp\{-2j(\mu - r)\pi i/m\}, \tag{5.20}$$

if $[n/4] + 1 \leqq \mu \leqq [n/2]$.

Proof. From (2.10) and (5.2) we deduce

$$\begin{aligned} \sum_{k=0}^{n-h-1} c_k(j, h)x^k &= c_0(j, h)\left\{\prod_{k=0}^{n-1} (1 - xe^{-2k\pi i/m})\right\}(1 - xe^{-2j\pi i/m})^{-h-1} \\ &= c_0(j, h) \sum_{g=0}^{n} \gamma_g x^g \sum_{r=0}^{\infty} \binom{h + r}{h} e^{-2jr\pi i/m} x^r. \end{aligned} \tag{5.21}$$

Hence

$$c_k(j, h) = c_0(j, h) \sum_{r=0}^{k} \gamma_r \binom{h + k - r}{k - r} e^{-2j(k-r)\pi i/m} \tag{5.22}$$

if $0 \leqq k \leqq n - h - 1$.

By expanding the last factor in the second term of (5.21) in a series of descending powers of x we obtain, in the same way,

$$c_k(j,h) = -c_0(j,h) \sum_{r=h+1+k}^{n} \gamma_r \binom{h+k-r}{h} e^{-2j(k-r)\pi i/m} \tag{5.23}$$

when $0 \leqq k \leqq n-h-1$. Equations (5.22) and (5.23) remain valid for $k > n-h-1$ if we put $c_k(j,h) = 0$ for $k > n-h-1$.

Next we consider the expression

$$\frac{y_{j,h}(x)}{c_0(j,h)} - \left(\sum_{\mu=-[(n-1)/4]}^{-h-1} + \sum_{\mu=0}^{[n/4]}\right)\binom{\mu+h}{h} e^{-2\mu j\pi i/m} A_\mu(x). \tag{5.24}$$

According to Theorems 2 and 3 the parts of (5.24) on each side of the first minus sign have the same asymptotic behavior in $-2[n/4]\pi \leqq \arg(m^{-n}x^m) \leqq 2[(n-1)/4]\pi$. Therefore we may expect that the difference can be written as a combination of A_μ's with $\mu > [n/4]$ and $\mu < -[(n-1)/4]$. Using (5.14) and (5.4) we see that (5.24) equals

$$\frac{y_{j,h}(x)}{c_0(j,h)} + \left(\sum_{\mu=-[(n-1)/4]}^{-h-1} \sum_{k=[n/2]-\mu+1}^{n+\mu} - \sum_{\mu=0}^{[n/4]} \sum_{k=\mu}^{[n/2]-\mu}\right)\binom{\mu+h}{h}$$
$$\cdot e^{-2\mu j\pi i/m}\gamma_{k-\mu}\tilde{y}(xe^{2k\pi i/m}).$$

Changing the order of summation in these sums, and performing elementary calculations, using (2.12), (5.22) and (5.23), we obtain the result formulated in Theorem 4.

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SIAM J. MATH. ANAL.
Vol. 2, No. 1, February 1971

ON THE APPROXIMATION OF SOLUTIONS OF ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS IN TWO AND THREE DIMENSIONS*

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1. Introduction. In this paper we shall develop a constructive approach to solving boundary value problems for elliptic partial differential equations in two and three variables with analytic coefficients. We shall consider only bounded regions D which are Lyapunov and starlike with respect to the origin. Such regions we shall refer to as being *appropriate.*

For the case of two variables we consider, in particular, the equation

$$\mathbf{E}(u) = \Delta u + a(x, y)\frac{\partial u}{\partial x} + b(x, y)\frac{\partial u}{\partial y} + c(x, y)u = 0, \tag{1.1}$$

whose coefficients are real, analytic in $D + \partial D$; and in addition we take $c(x, y) \leqq 0$. Furthermore, we assume that these coefficients may be holomorphically continued to $(D + \partial D) \times (D^* + \partial D^*)$ in $\mathbb{C}^2$, which permits us to transform (1.1) to an associated complex-valued hyperbolic equation,

$$\begin{gathered} U_{ZZ^*} + AU_Z + BU_{Z^*} + CU = 0, \qquad B = \bar{A}, \\ U(Z, Z^*) \equiv u\left(\frac{Z + Z^*}{2}, \frac{Z - Z^*}{2i}\right), \end{gathered} \tag{1.2}$$

where $Z \equiv x + iy$, $Z^* \equiv x - iy$, $(Z, Z^*) \in (D + \partial D) \times (D^* + \partial D^*)$. See also the monograph by one of the authors [3, Chap. III]. (Note. Z and Z^* are conjugate if and only if x and y are real.) The function theoretic methods of Bergman [1] and Vekua [11] show how to represent a general solution of (1.2). In the next section we shall show how these representations can be used for a constructive approach for solving the Dirichlet problem for (1.1) with data

$$u^-(\mathbf{t}) = f(\mathbf{t}) \quad \text{with } \mathbf{t} \in \partial D, \tag{1.3}$$

and where f is a continuous (real) function of the point $\mathbf{t}$ on ∂D.

For the case of three variables, we consider differential equations of the form

$$\Delta u + F(x, y, z)u = 0, \qquad F(x, y, z) < 0, \tag{1.4}$$

where $F(x, y, z)$ is real for real x, y, z and in addition is an entire function of three complex variables. These conditions may be relaxed to F being holomorphic in a suitably large polycylinder as will be clear from our development. The solutions of (1.4) have a representation in an integral form, which resembles the Whittaker–Bergman operator for harmonic functions of three variables. This operator was first discovered by Tjong in her dissertation [10]. For the purposes of solving boundary value problems it is necessary to show that this operator is invertible;

* Received by the editors March 3, 1970. This research was supported in part by the United States Air Force Office of Scientific Research under Grant AFOSR 1206-67.

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however, this was not done in [10], since that work was primarily devoted to studying the analytic structure of solutions to (1.4). We are able to show in § 3, that Tjong's operator is invertible. By this we mean if D is an appropriate domain there exists a unique harmonic function $H(\mathbf{X})$ *in* D (see (3.10)) (and hence a unique $\mathbf{B}_3$-associate of $H(\mathbf{X})$ given by (3.13)), which corresponds to a regular solution of (1.4) in D. This is accomplished by using a new inversion of the Whittaker–Bergman operator [1] to obtain a Fredholm integral equation for the harmonic (3.40). Having shown that there exists a unique harmonic function for each solution of (1.4) *in* an appropriate domain, we are strongly motivated to use the Tjong operator to obtain a complete system of solutions of (1.4) with respect to uniform convergence.

Remark. If a solution of $u(\mathbf{x})$ is of class $\mathscr{C}^2(D + \partial D)$, then it may be shown that the inversion is possible on $D + \partial D$.

2. Two-dimensional boundary value problems. Both Bergman [1, pp. 10–17] and Vekua [11, pp. 54–58] give integral representations for (1.2) and hence for (1.1). Bergman's representation for solutions of (1.2) is of the form

$$U(Z, Z^*) = \exp\left[-\int_0^{Z^*} A(Z, S)\,dS\right]\left[g(Z) + \sum_{n \geqq 1} \frac{Q^{(n)}(Z, Z^*)}{2^{2n}B(n, n+1)} \cdot \int_0^Z \int_0^{Z_1} \cdots \int_0^{Z_n} g(Z_n)\,dZ_n \cdots dZ_1\right], \tag{2.1}$$

where $g(Z)$ is taken to be analytic in $D + \partial D$. The functions $Q^{(n)}(Z, Z^*)$ are defined by

$$Q^{(n)}(Z, Z^*) \equiv \int_0^{Z^*} P^{(2n)}(Z, S)\,dS, \tag{2.2}$$

where the $P^{(2n)}(Z, Z^*)$ are defined recursively by the system

$$P^{(2)} = -2F \equiv -2(A_Z - AB + C) \tag{2.3}$$

and

$$(2n+1)P^{(2n+2)} = -2\left[P_Z^{(2n)} + \left\{\Phi' - \int_0^{Z^*} A_Z\,dZ^* + B\right\}P^{(2n)} + F\int_0^{Z^*} P^{(2n)}\,dZ^*\right], \qquad n \geqq 1. \tag{2.4}$$

Here $\Phi = \Phi(Z)$ is an arbitrary analytic function of one complex variable in $D + \partial D$.

For the situation where $g(0) = 0$, Bergman shows in [1, pp. 15–17] that (2.1) may be rewritten as

$$U(Z, Z^*) = \exp\left[-\int_0^{Z^*} A(Z, \zeta)\,d\zeta\right]\left[g(Z) + \sum_{n \geqq 1} \frac{Q^{(n)}(Z, Z^*)}{2^{2n}B(n, n+1)}\int_0^Z (Z - \zeta)^{n-1} g(\zeta)\,d\zeta\right]. \tag{2.5}$$

It is not explicitly stated in [1] that when $g(0) \neq 0$, the representation (2.1) is also a solution of the differential equation (1.2). However, several examples are given there that imply this must be the case. In order to demonstrate that this is true we need only consider the special case with $g(Z) \equiv 1$ and $A(Z, Z^*) \equiv 0$. Here we have

$$U(Z, Z^*) = 1 + \sum_{n \geqq 1} \frac{(2n)!}{(n!)^2} \left(\frac{Z}{4}\right)^n Q^{(n)}(Z, Z^*);$$

and substituting this directly into (1.2) and using (2.2)–(2.4) yields the desired result.

The function $U(Z, Z^*)$ defined by (2.1)–(2.4) converges uniformly in any relatively compact subset of $(D + \partial D) \times (D^* + \partial D^*)$.

It is not difficult to show that the representation (2.5) of Bergman is actually identical to Vekua's representation of solutions of (1.2), namely,

$$\begin{aligned} U(Z, Z^*) = R(Z, 0; Z, Z^*)g(Z) + \int_0^Z \{ -R_1(t, 0; Z, Z^*) \\ + B(t, 0)R(t, 0; Z, Z^*)\} g(t)\, dt, \end{aligned} \tag{2.6}$$

where $R(\zeta, \zeta^*; Z, Z^*)$ is the *complex* Riemann function. For details of this fact see, for example, [3, Chap. III]. Consequently, the Vekua approach of singular integral equations *must* also work for the Bergman representation. In the next section we establish this fact directly by estimating terms in (2.5); as a by-product of this effort we arrive at a sequence of approximating singular integral equations for Vekua's integral equation. This approach, consequently, provides us with a constructive method for solving boundary value problems when the Riemann function is not known.

2.1. A singular integral equation for the Dirichlet problem. There exists a unique real function $\mu(t)$ of the point t on ∂D, continuous in the Hölder sense, such that $g(Z)$ may be expressed as

$$g(Z) = \int_{\partial D} \frac{t\mu(t)\, ds}{t - Z}, \qquad z \in \partial D, \tag{2.7}$$

where ds is an element of arc length on ∂D at t (see, for example, [8, p. 192]). By substituting the representation (2.7) into (2.5), and taking x and y real, we have

$$u(x, y) = \operatorname{Re} \left\{ \hat{H}_0(Z) \int_{\partial D} t\mu(t) \left[\frac{1}{t - Z} + \sum_{n=1}^{\infty} \frac{Q^{(n)}(Z, \bar{Z})}{2^{2n} B(n, n+1)} \int_0^Z \frac{(Z - \zeta)^{n-1}}{t - \zeta} d\zeta \right] ds \right\}, \tag{2.8}$$

where $\hat{H}_0(Z) \equiv \exp \left[-\int_0^{\bar{Z}} A(Z, \sigma)\, d\sigma \right]$.

Now, let $Z = x + iy \to t_0 \in \partial D$. Then, because of the limit properties of Cauchy integrals [8, p. 42], we have

$$(2.9)\quad \begin{aligned} f(t_0) = \operatorname{Re}\Bigg\{ \hat{H}_0(t_0)\Bigg[\pi i t_0 \mu(t_0)\bar{t}'_0 + \int_L t\mu(t)\Bigg(\frac{1}{t - t_0} \\ + \sum_{n=1}^{\infty} \frac{Q^{(n)}(t, t_0)}{2^{2n}B(n, n+1)} \int_0^{t_0} \frac{(t_0 - \zeta)^{n-1}}{t - \zeta}\, d\zeta\Bigg)\Bigg] ds\Bigg\}, \end{aligned}$$

where $t'(s) \equiv dt/ds$. Introducing the functions

$$A(t_0) \equiv \operatorname{Re}\,[\pi i t_0 \bar{t}'_0 \hat{H}_0(t_0)],$$

$$B(t_0) \equiv i\pi \operatorname{Re}\,[t_0 \bar{t}'_0 \hat{H}_0(t_0)],$$

we see that (2.9) can be written as

$$(2.10)\quad \begin{aligned} f(t_0) = A(t_0)\mu(t_0) + \frac{B(t_0)}{i\pi} \int_{\partial D} \frac{\mu(t)\, dt}{t - t_0} \\ + \int_{\partial D} F_1(t_0, t)\mu(t)\, ds + \int_{\partial D} F_2(t_0, t)\mu(t)\, ds, \end{aligned}$$

where

$$(2.11)\quad F_1(t_0, t) = \operatorname{Re}\left\{\frac{t\hat{H}_0(t_0)}{t - t_0}\right\} - \frac{t'B(t_0)}{i\pi(t - t_0)}$$

and

$$(2.12)\quad F_2(t_0, t) = \operatorname{Re}\left\{ t \sum_{n=1}^{\infty} \frac{\hat{H}_0(t_0)Q^{(n)}(t_0, \bar{t}_0)}{2^{2n}B(n, n+1)} \int_0^{t_0} \frac{(t_0 - \zeta)^{n-1}}{t - \zeta}\, d\zeta \right\}.$$

Equation (2.10) is a singular integral equation. According to general theory [8], a unique solution of (2.10) exists if we can show both $F_1(t_0, t)$ and $F_2(t_0, t)$ have the form

$$(2.13)\quad F_i(t, t_0) = \frac{\mathfrak{F}_i(t, t_0)}{|t - t_0|^{\alpha_i}}, \qquad i = 1, 2,$$

where $0 \leqq \alpha_i < 1$ and $\mathfrak{F}_i(t, t_0)$ is a function, continuous in the Hölder sense, of the two points t_0, t of the curve ∂D. In this case the second and third integrals of (2.10) correspond to compact integral operators [7, p. 32], and it is then easy to show that the index of (2.10) is zero [11, p. 125]. It follows from Vekua's theory that (2.10) may be reduced to a Fredholm integral equation [11, p. 129]. To this end, we obtain estimates for $F_1(t_0, t)$ and $F_2(t_0, t)$.

It is obvious from [11, p. 125] that $F_1(t, t_0)$ has the form (2.13).

For $F_2(t, t_0)$, we recall the hypothesis that D is starlike and use the estimate [1, p. 14]

$$(2.14)\quad |P_{2n}(Z, Z^*)| \leqq \frac{2^{n+1}(n + A - 1)(n + A - 2)\cdots(1 + A)C}{[1 - (|Z|/r)]^n r^{n-1} 1 \cdot 3 \cdots (2n - 1)},$$

where $A = 2Cr(1 + r)$, and we assume the coefficients of (1.1) may be continued to a bicylinder $\{|z| \leqq r\} \times \{|z^*| \leqq r\}$:

$$
\begin{aligned}
|F_2(t, t_0)| &\leqq |\hat{H}_0(t_0)| \sum_{n=1}^{\infty} \frac{2^{n+1} r^2 (n + A - 1)(n + A - 2) \cdots (1 + A)C}{2^{2n}(r - |t_0|^n) 1 \cdot 3 \cdots (2n - 1)} \frac{(2n)!}{(n-1)!n!} \\
&\quad \cdot |2t_0|^{n-1} \ln \left| \frac{t - |t_0|}{t} \right| \\
&\leqq r^2 C |\hat{H}_0(t_0)| \sum_{n=1}^{\infty} \frac{2^n |t_0|^{n-1} (n + A - 1)(n + A - 2) \cdots (1 + A)}{(r - |t_0|)^n (n-1)!} \\
&\quad \cdot \ln \left| \frac{t - t_0}{t} \right| \\
&\equiv Q(t_0) \ln \left| \frac{t - t_0}{t} \right|,
\end{aligned}
\tag{2.15}
$$

where $Q(t_0)$ converges uniformly for

$$\left| \frac{2t_0}{r - |t_0|} \right| \leqq \theta < 1.$$

It follows that $F_2(t, t_0)$ also has the form (2.13) for $t, t_0 \in \partial D$, if the radius r of the above bicylinder is sufficiently large.

2.2. Conclusion. An approximate singular integral equation for (2.10) may be obtained by replacing $F_2(t_0, t)$ in that equation by

$$F_2^{(N)}(t_0, t) \equiv \operatorname{Re} \left\{ \sum_{n=1}^{N} \frac{\hat{H}_0(t_0) Q^{(n)}(t_0, \bar{t}_0)}{2^{2n} B(n, n+1)} \int_0^{t_0} \frac{(t_0 - \zeta)^{n-1}}{t - \zeta} d\zeta \right\}. \tag{2.16}$$

This leads to an equivalent Fredholm equation whose kernel $K^{(N)}(t_0, t)$ approximates the corresponding Fredholm kernel $K(t_0, t)$ associated with (2.10) in the following way:

There exists an α, $0 \leqq \alpha < 1$, such that, as $N \to \infty$,

$$|K(t_0, t) - K^{(N)}(t_0, t)| \cdot |t - t_0|^{\alpha} \to 0,$$

uniformly on $\partial D \times \partial D$. Indeed, by applying the Poincaré–Bertrand formula [8, p. 57] to the result, one obtains, by operating on the *truncated* version of (2.10) with the operator (see [11, p. 129])

$$A(t_0)(\cdot) - \frac{B(t_0)}{\pi i} \int_{\partial D} \frac{(\cdot)\, dt}{t - t_0}, \tag{2.17}$$

the Fredholm equation

$$\mu(t_0) + \int_{\partial D} K^{(N)}(t_0, t) \mu(t)\, ds = F(t_0), \tag{2.18}$$

where

$$F(t_0) \equiv \frac{1}{\pi^2 |t_0|^2 |\hat{H}(t_0)|^2} \left[A(t_0) f(t_0) - \frac{B(t_0)}{\pi i} \int_{\partial D} \frac{f(t)}{t - t_0} \right] \tag{2.19}$$

and where the truncated Fredholm kernel is given by

$$
\begin{aligned}
K^{(N)}(t_0, t) \equiv \frac{1}{\pi^2 |t_0|^2 |\hat{H}_0(t_0)|^2} \Bigg[& A(t_0) \left(\operatorname{Re} \left\{ \frac{t\hat{H}_0(t_0)}{t - t_0} \right\} - \frac{t' B(t_0)}{i\pi(t - t_0)} \right) \\
& + A(t_0) F_2^{(N)}(t_0, t) - \frac{B(t_0)}{\pi i} \int_{\partial D} \operatorname{Re} \left\{ \frac{t\hat{H}_0(t)}{t - \tau} \right\} \frac{d\tau}{\tau - t_0} \\
& - \frac{B(t_0)}{\pi i} \int_{\partial D} \frac{F_2^{(N)}(\tau, t)\, d\tau}{\tau - t_0} \Bigg].
\end{aligned} \tag{2.20}
$$

3. Three-dimensional boundary value problems. In [10, p. 548] Tjong showed that it is possible to generate solutions of

$$
\Delta \varphi + F(x, y, z)\varphi = 0, \tag{3.1}
$$

where $F(x, y, z)$ is an entire function in $\mathbb{C}^3$. In order to present her result we first introduce the following notations:

$$
\begin{aligned}
X &= x, \\
Z &= \tfrac{1}{2}[y + iz], \\
Z^* &= \tfrac{1}{2}[-y + iz], \\
w &= (1 - t^2)u, \\
u &= X + \zeta Z + \zeta^{-1} Z^*, \\
\xi_1 &= X, \\
\xi_2 &= X + 2\zeta Z, \\
\xi_3 &= X + 2\zeta^{-1} Z^*.
\end{aligned} \tag{3.2}
$$

Then (3.1) may be written in the form

$$
\frac{\partial^2 \psi}{\partial X^2} - \frac{\partial^2 \psi}{\partial Z \partial Z^*} + \hat{F}(X, Z, Z^*)\psi = 0, \tag{3.3}
$$

and there exist solutions of (3.3) of the form [1]

$$
\psi(X, Z, Z^*) = \mathbf{T}f \equiv \frac{1}{2\pi i} \int_{|\zeta| = 1} \int_{\gamma} E(X, Z, Z^*, \zeta, t) f(w, \zeta) \frac{dt}{\sqrt{1 - t^2}} \frac{d\zeta}{\zeta}, \tag{3.4}
$$

where γ is a rectifiable curve joining $t = -1$ to $t = +1$, provided $\hat{E}(\xi_1, \xi_2, \xi_3, \zeta, t) \equiv E(X, Z, Z^*, \zeta, t)$ satisfies the partial differential equation

$$
\begin{aligned}
ut \left(\frac{\partial^2 \hat{E}}{\partial \xi_1^2} + \frac{\partial^2 \hat{E}}{\partial \xi_2^2} + \frac{\partial^2 \hat{E}}{\partial \xi_3^2} + 2\frac{\partial^2 \hat{E}}{\partial \xi_1 \partial \xi_2} + 2\frac{\partial^2 \hat{E}}{\partial \xi_1 \partial \xi_3} - 2\frac{\partial^2 \hat{E}}{\partial \xi_2 \partial \xi_3} + \hat{F}\hat{E} \right) \\
+ (1 - t^2) \frac{\partial^2 \hat{E}}{\partial \xi_1 \partial t} - \frac{1}{t} \frac{\partial \hat{E}}{\partial \xi_1} = 0.
\end{aligned} \tag{3.5}
$$

[1] It is assumed that $f(u, \zeta)$ is holomorphic in a product space $\{|u| \leqq a\} \times \{1 - \varepsilon \leqq |\zeta| \leqq 1 + \varepsilon\}$, $0 < \varepsilon < 1$.

She goes on to show that $\hat{E}$ has a series representation of the form [9]

$$\hat{E}(\xi_1, \xi_2, \xi_3, \zeta, t) = 1 + \sum_{n \geqq 1} t^{2n} u^n p^{(n)}(\xi_1, \xi_2, \xi_3, \zeta), \tag{3.6}$$

where the $p^{(n)}$ are defined by

$$p_1^{(n+1)} = -\frac{1}{2n+1}\{p_{11}^{(n)} + p_{22}^{(n)} + p_{33}^{(n)} + 2p_{12}^{(n)} + 2p_{13}^{(n)} - 2p_{23}^{(n)} + \hat{F}p^{(n)}\} \tag{3.7}$$

and

$$p^{(n+1)}(0, \xi_2, \xi_3, \zeta) = 0.$$

It is also shown [10, p. 549] that (3.4) has the alternate representation

$$\begin{aligned} \psi &= \mathbf{T}f \\ &\equiv \frac{1}{2\pi i}\int_{|\zeta|=1} g(u, \zeta)\frac{d\zeta}{\zeta} \\ &\quad + \sum_{n \geqq 1} \frac{1}{2\pi i B(n, \frac{1}{2})}\int_{|\zeta|=1} \left\{p^{(n)}(\xi_1, \xi_2, \xi_3, \zeta)\int_0^u (u-s)^{n-1} g(s, \zeta)\, ds\right\}\frac{d\zeta}{\zeta}, \end{aligned} \tag{3.8}$$

where

$$g(u, \zeta) = \int_\gamma f(w, \zeta)\frac{dt}{\sqrt{1-t^2}}. \tag{3.9}$$

The expression (3.8) is reminiscent of the series solution (2.5) which occurs in the two-dimensional case.

Let us proceed to rewrite (3.8) by replacing the integral of $g(u, \zeta)$ by the harmonic function it represents, i.e.,

$$H(X, Z, Z^*) = (\mathbf{B}_3 g)(X, Z, Z^*) \equiv \frac{1}{2\pi i}\int_{|\zeta|=1} g(u, \zeta)\frac{d\zeta}{\zeta}. \tag{3.10}$$

(We remark that $\mathbf{B}_3$ is the Whittaker–Bergman operator [1], [3].) To do this we wish to use an integral representation for $\mathbf{B}_3^{-1}$. Bergman [1] has given a representation for $\mathbf{B}_3^{-1}$ as an integral over the characteristic space $X^2 - ZZ^* = 0$; however, such an inversion formula is not of use to us in formulating a real boundary value problem. In [3] Gilbert gives a representation for $\mathbf{B}_3^{-1}$ as an integral over a sphere for the case where the domain of $\mathbf{B}_3^{-1}$ is restricted to harmonic functions regular at infinity. It is of the form

$$(\mathbf{B}_3^{-1}J)(u, \zeta) \equiv \frac{1}{4\pi}\int_0^\pi d\theta' \int_0^{2\pi} d\varphi' R \sin\theta' \frac{(u'+u)}{(u'-u)^2} J(\mathbf{X}'), \tag{3.11}$$

where $g(u, \zeta) = (\mathbf{B}_3^{-1}J)(u, \zeta)$, $J(\mathbf{X}) \equiv J(X, Z, Z^*)$, and the integration is taken over a sphere $|\mathbf{X}| = R$, whose surface is entirely inside the domain of regularity of $J(\mathbf{X})$.

If $H(\mathbf{X})$ is a harmonic function, regular about the origin, we may modify the representation (3.11) to handle this case also. If $H(\mathbf{X})$ is regular about the origin, then $J_a(\mathbf{X}) = aH(\mathbf{X}a^2/r^2)/r$ is by Kelvin's transformation a harmonic function regular at ∞.[2] Let $H(\mathbf{X})$ be defined, for $r > 0$ sufficiently small, by the series

$$H(\mathbf{X}) \equiv \sum_{n \geqq 0} \sum_{m=-n}^{+n} \frac{a_{nm} n!}{(n+m)!} i^m r^n P_n^m(\cos\theta) e^{im\varphi}, \tag{3.12}$$

where the $P_n^m(x)$ are associated Legendre functions. Then if $h(u, \zeta)$ is a holomorphic function defined by

$$h(u, \zeta) \equiv \sum_{n \geqq 0} \sum_{m=-n}^{+n} a_{nm} u^n \zeta^m, \tag{3.13}$$

and u is the auxiliary variable given by (3.2), we have [3, p. 49]

$$H(\mathbf{X}) = (\mathbf{B}_3 h)(\mathbf{X}). \tag{3.14}$$

Likewise, if $j_a(u, \zeta)$ is defined by

$$j_a(u, \zeta) \equiv \sum_{n \geqq 0} \sum_{m=-n}^{+n} \frac{(-1)^m (n!)^2 a_{nm}}{(n-m)!(n+m)!} \left(\frac{u}{a}\right)^{-n-1} \zeta^m, \tag{3.15}$$

then [3, p. 53]

$$\begin{aligned} J_a(\mathbf{X}) &= (\mathbf{B}_3 j)(\mathbf{X}) \\ &= \sum_{n \geqq 0} \sum_{m=-n}^{+n} \frac{a_{nm} n!}{(n+m)!} i^m \left(\frac{r}{a}\right)^{-n-1} P_n^m(\cos\theta) e^{im\varphi}. \end{aligned} \tag{3.16}$$

The operator represented by (3.11) has $J_a(\mathbf{X})$ (as defined by (3.16)) in its domain; and the image of $J_a(\mathbf{X})$ is the $j_a(u, \zeta)$ given by (3.15). We may indicate this by

$$j_a(u, \zeta) = \mathbf{B}_3^{-1} \frac{1}{r} H(\mathbf{X}a^2/r^2). \tag{3.17}$$

We wish to exploit (3.17) to obtain a representation of $\mathbf{B}_3^{-1}$ for an $H(\mathbf{X})$ given by (3.12). To this end we note that

$$\begin{aligned} \frac{\pi}{n+1} & \frac{\Gamma(n-m+1)\Gamma(n+m+1)}{\Gamma(n+1)^2} \\ &= 2^{2n+1} B(n-m+1, n+m+1) \cdot B(n+\tfrac{3}{2}, \tfrac{1}{2}), \end{aligned} \tag{3.18}$$

where $B(x, y)$ is the beta function. We conclude, using formal arguments and an integral identity for the beta function, that

$$\begin{aligned} h(s, \zeta) = \sum_{n \geqq 0} \sum_{m=-n}^{+n} & \left\{ \frac{(n!)^2 a_{nm} s^n \zeta^m (-1)^m}{(n-m)!(n+m)!} \int_0^1 \alpha^{n-m} (1-\alpha)^{n+m} \, d\alpha \right. \\ & \left. \cdot (-1)^m \frac{(n+1)}{\pi} \cdot 2^{2n+1} \cdot \int_0^1 \beta^{n+1/2} (1-\beta)^{-1/2} \, d\beta \right\} \\ = \frac{1}{2\pi} \frac{\partial}{\partial s} \int_0^1 & \left\{ \int_0^1 j\left(\frac{a}{4\alpha\beta(1-\alpha)s}, \zeta\left(1 - \frac{1}{\alpha}\right) \right) \frac{d\beta}{\sqrt{(1-\beta)\beta}} \right\} \frac{da}{\alpha(1-\alpha)}. \end{aligned} \tag{3.19}$$

[2] We choose $a > 0$ sufficiently large so that the region D is contained in the sphere $S_a \equiv \{|\mathbf{X}| \leqq a\}$.

Consequently, if $H(\mathbf{X})$ is given by (3.12), we may formally represent $\mathbf{B}_3^{-1}$ operating on $H(\mathbf{X})$ by

$$(\mathbf{B}_3^{-1}H)(s,\zeta) = \frac{-a}{2\pi^2}\int_0^1 d\alpha \int_0^1 \frac{d\beta\sqrt{\beta}}{\sqrt{1-\beta}}\left\{\int_{\substack{0\\|\mathbf{X}'|=R}}^{2\pi} d\varphi' \int_0^\pi \sin\theta' \cdot H(\mathbf{X}'a^2/R^2) \cdot \left(\frac{12s\alpha\beta(1-\alpha)\hat{u}+a}{[4s\alpha\beta(1-\alpha)\hat{u}-a]^3}\right) d\theta'\right\}, \tag{3.20}$$

where

$$\hat{u} = X' + \zeta\left(1-\frac{1}{\alpha}\right)Z' + \zeta^{-1}\left(1-\frac{1}{\alpha}\right)^{-1} Z^{*\prime},$$

and $\{\mathbf{X}' \mid |\mathbf{X}'| \leqq 1/R\} \Subset D$. An alternate, and perhaps preferable, approach would be to introduce the "beta-function operation" on $H(\mathbf{X})$ first and then use $\mathbf{B}_3$ on the reflection of this harmonic function through the unit sphere. However, since (3.20) is suitable for our purposes, we shall refer to this as the representation for $\mathbf{B}_3^{-1}$ when $H(\mathbf{X})$ is of the form (3.12).

Returning to (3.8), we assume temporarily that this is a general representation for solutions in an appropriate domain D, and we take $|\mathbf{X}| \leqq R^{-1}$ as a solid sphere entirely within D. Then we may rewrite this as

$$\psi(\mathbf{X}) = H(\mathbf{X}) + \sum_{n\geqq 1}\frac{1}{2\pi i B(n,\frac{1}{2})}\int_{|\zeta|=1}\left\{p^{(n)}(\xi_1,\xi_2,\xi_3,\zeta) \cdot \int_0^u (u-s)^{n-1}\cdot\frac{-a}{2\pi^2}\int_0^1 d\alpha\int_0^1\frac{d\beta\sqrt{\beta}}{\sqrt{1-\beta}}\left[\int_{\substack{0\\|\mathbf{X}'|=R}}^{2\pi} d\varphi'\int_0^\pi \sin\theta' H(\mathbf{X}'a^2/R^2) \cdot \left(\frac{12s\alpha\beta(1-\alpha)\hat{u}+a}{[4s\alpha\beta(1-\alpha)\hat{u}-a]^3}\right) d\theta'\right] ds\right\}\frac{d\zeta}{\zeta}. \tag{3.21}$$

Interchanging orders of integration by Fubini's theorem yields

$$\psi(\mathbf{X}) = (\hat{\mathbf{T}}H)(\mathbf{X}) \equiv H(\mathbf{X}) + \frac{-a}{4\pi^3 i}\sum_{n\geqq 1}\frac{1}{B(n,\frac{1}{2})}\int_0^{2\pi} d\varphi' \int_0^\pi d\theta' \sin\theta' \cdot \left\{H(\mathbf{X}'a^2/R^2)\int_{|\zeta|=1} p^{(n)}(\xi_1,\xi_2,\xi_3,\zeta) D(u,\hat{u};n)\frac{d\zeta}{\zeta}\right\}, \tag{3.22}$$

where the coefficients $D(u,\hat{u};n)$ are independent of either $H(\mathbf{X})$ or $F(\mathbf{X})$ and are defined as

$$D(u,\hat{u};n) \equiv \int_0^u (u-s)^{n-1} A(s,\hat{u})\,\partial s, \tag{3.23}$$

with

$$A(s,\hat{u}) \equiv \int_0^1 d\alpha \int_0^1 \frac{d\beta\sqrt{\beta}}{\sqrt{1-\beta}}\left[\frac{12s\alpha\beta(1-\alpha)\hat{u}+a}{[4s\alpha\beta(1-\alpha)\hat{u}-\alpha]^3}\right]. \tag{3.24}$$

We now turn to the Dirichlet problem associated with (3.1). It will be shown in the corollary to Theorem 2 that if the Dirichlet data is sufficiently smooth then there exists a unique harmonic function $H(\mathbf{X})$, which is mapped by $\hat{\mathbf{T}}$ onto the unique solution of the Dirichlet problem for (3.1). Using this result we may give several schemes by which solutions to the Dirichlet problem may be constructed and approximated. One method depends on the construction of a complete system of solutions for (3.1). This can be accomplished by using the corollary to Theorem 2, and a recent result of du Plessis [9] which showed that the harmonic polynomials are complete (with respect to the uniform norm) for simply connected domains in $\mathscr{R}^n$. Indeed, one obtains in this way the following theorem.

THEOREM 1. *Let* $h_{n,m}(\mathbf{X})$, $m = 0, \pm 1, \pm 2, \cdots, \pm n$; $n = 0, 1, 2, \cdots$, *represent the tesseral harmonics; then the functions*

$$\psi_{n,m}(\mathbf{X}) \equiv (\hat{\mathbf{T}} h_{n,m})(\mathbf{X}), \tag{3.25}$$

$m = 0, \pm 1, \cdots, \pm n$; $n = 0, 1, 2, \cdots$, *form a complete system of solutions for* (3.1) *with respect to uniform convergence in D.*

Proof. Let $\mathscr{H}[D]$ and $\mathscr{E}[D]$ be the space of functions in $\mathscr{C}[D + \partial D]$ which satisfy respectively $\Delta\varphi = 0$, and $\Delta\psi + F\psi = 0$ for $\mathbf{X} \in D$. Let $\hat{\psi}(\mathbf{X}) \in \mathscr{E}[D]$ not lie in the space spanned by the $\psi_{n,m}(\mathbf{X})$ as given by (3.25). If D_0 is an arbitrary domain such that $\bar{D}_0 \subset D$, then it is well known that $\hat{\psi}(\mathbf{X}) \in \mathscr{C}^{\infty}[D_0]$. Hence, if D_0 is also appropriate, using the corollary to Theorem 2 we conclude that there exists a unique $\hat{H}(\mathbf{X}) \in \mathscr{H}[D_0]$, such that $\hat{\psi}(\mathbf{X}) = (\mathbf{T}\hat{H})(\mathbf{X})$. Since D_0 is fixed $\hat{H}(\mathbf{X})$ may be uniformly approximated in D_0 by harmonic polynomials. Hence $\hat{\psi}(\mathbf{X})$ may be uniformly approximated in D_0 by the functions $\psi_{n,m}(\mathbf{X})$, and since D_0 is arbitrary we have a contradiction.

Given a complete system of solutions $\{\psi_j(\mathbf{X})\}$ there are various procedures for approximating solutions of the Dirichlet problem. One such method is to approximate the boundary data as a linear combination

$$f(\mathbf{X}) \sim \sum_{j=1}^{N} c_j \psi_j(\mathbf{X}), \qquad \mathbf{X} \in \partial D, \tag{3.26}$$

such that

$$\max_{\mathbf{X} \in \delta D} |f(\mathbf{X}) - \sum_{j=1}^{N} c_j \psi_j(\mathbf{X})| < \varepsilon \tag{3.27}$$

for $\varepsilon > 0$ chosen sufficiently small. Then by the Hopf maximum principle the approximate solution

$$\Phi_N(\mathbf{X}) = \sum_{j=1}^{N} c_j \psi_j(\mathbf{X})$$

is within an ε-approximation of the actual solution in $D + \partial D$.

An alternate procedure is to introduce the Dirichlet inner product for (3.1), namely,

$$(\psi, \varphi) \equiv \int_D [\nabla\psi \cdot \nabla\varphi - F\psi\varphi] \, d\mathbf{X}, \tag{3.28}$$

and construct by the Gram–Schmidt process an orthonormal system $\{\Phi_j(\mathbf{X})\}$. The data may then be approximated in terms of a truncated Fourier series

$$f(\mathbf{X}) \sim \sum_{j=1}^{N} a_j \Phi_j(\mathbf{X}), \qquad \mathbf{X} \in \partial D,$$

where the Fourier coefficients are given by[3]

$$a_j = (f, \Phi_j) = -\int_{\partial D} f \frac{\partial \Phi_j}{\partial \nu} d\omega. \tag{3.29}$$

The approximate solutions

$$\Phi_N(\mathbf{X}) = \sum_{j=1}^{N} a_j \Phi_j(\mathbf{X}), \qquad N = 1, 2, \cdots,$$

may be seen to converge uniformly in $D_0 \subset D$ to the actual solution. An estimate on the error is given in terms of the kernel function

$$K(\mathbf{X}, \mathbf{Y}) = \sum_{j=1}^{\infty} \Phi_j(\mathbf{X})\Phi_j(\mathbf{Y}),$$

namely,

$$|\psi(\mathbf{X}) - \psi_N(\mathbf{X})|^2 \leqq \|\psi - \psi_N\|^2 \cdot K(\mathbf{X}, \mathbf{X}). \tag{3.30}$$

We next turn to the problem of showing that Tjong's operator is invertible. Let $\psi(\mathbf{X})$ be a solution of (3.1) of the class $\mathscr{C}^0(D + \partial D)$, where D is appropriate. Then, for any $D_0 \Subset D$, $\psi(\mathbf{X}) \in \mathscr{C}^2(D_0 + \partial D_0)$. Let D_0 also be appropriate, and let us consider the class of harmonic functions in $\mathscr{C}^1(D_0 + \partial D_0)$. Such a harmonic function $H(\mathbf{X})$ may be represented as a single layer potential,

$$H(\mathbf{X}) = \frac{1}{2\pi} \int_{\partial D_0} \rho(\mathbf{Y}) \frac{d\omega_y}{|\mathbf{X} - \mathbf{Y}|}. \tag{3.31}$$

If we substitute in the Whittaker representation for the fundamental solution, namely,

$$\frac{1}{|\mathbf{X} - \mathbf{Y}|} = \frac{1}{2\pi i} \int_{|\zeta| = 1} \frac{1}{\mathbf{N} \cdot (\mathbf{X} - \mathbf{Y})} \frac{d\zeta}{\zeta} \tag{3.32}$$

(where $\mathbf{N}(\zeta)$ is the isotropic vector introduced earlier in $u = \mathbf{N} \cdot \mathbf{X}$), we obtain the following expression upon interchanging orders of integration:

$$H(\mathbf{X}) = \frac{1}{2\pi i} \int_{|\zeta| = 1} \frac{d\zeta}{\zeta} \left\{ \frac{1}{2\pi} \int_{\partial D_0} \rho(\mathbf{Y}) \frac{1}{\mathbf{N} \cdot (\mathbf{X} - \mathbf{Y})} d\omega_y \right\}. \tag{3.33}$$

The change of orders of integration follows by Fubini's theorem. The representation (3.33) suggests that an alternate representation for $\mathbf{B}_3^{-1}$ might be given as

$$(\mathbf{B}_3^{-1} H)(\mathbf{X}) \equiv \frac{1}{2\pi} \int_{\partial D_0} \rho(\mathbf{Y}) \frac{1}{\mathbf{N} \cdot (\mathbf{X} - \mathbf{Y})} d\omega_y. \tag{3.34}$$

[3] The differentiation $\partial/\partial\nu$ is in the inner normal direction.

Returning to (3.8)–(3.9) and using $H(\mathbf{X}) = (\mathbf{B}_3 g)(\mathbf{X})$ yields

$$\psi(\mathbf{X}) = (\mathbf{T}f)(\mathbf{X})$$

$$= H(\mathbf{X}) + \sum_{n \geqq 1} \frac{1}{2\pi i B(n, \frac{1}{2})} \int_{|\zeta|=1} P^{(n)}(\boldsymbol{\xi}; \zeta) \cdot \left\{ \frac{1}{2\pi} \int_{\partial D_0} \Phi(u; \mathbf{N} \cdot \mathbf{Y}) \rho(\mathbf{Y})\, d\omega_y \right\} \frac{d\zeta}{\zeta}, \tag{3.35}$$

where

$$\Phi_n(u; \mathbf{N} \cdot \mathbf{Y}) \equiv \int_0^u \frac{(u-s)^{n-1}\, ds}{s - \mathbf{N} \cdot \mathbf{Y}}$$

$$= [(\mathbf{N} \cdot \mathbf{Y} - u)^{n-1} - (\mathbf{N} \cdot \mathbf{Y})^{n-1}] \log (u - \mathbf{N} \cdot \mathbf{Y}) - \sum_{\nu=1}^{n-1} \binom{n-1}{\nu} \frac{(\mathbf{N} \cdot \mathbf{Y})^{n-\nu-1}}{\nu}, \tag{3.36}$$

or upon interchanging orders of integration,

$$\psi(\mathbf{X}) = H(\mathbf{X}) + \int_{\partial D_0} \rho(\mathbf{Y}) \left\{ \frac{1}{2\pi i} \int_{|\zeta|=1} P(\mathbf{X}, \mathbf{Y}; \zeta) \frac{d\zeta}{\zeta} \right\} d\omega_y, \tag{3.37}$$

where

$$P(\mathbf{X}, \mathbf{Y}; \zeta) \equiv \sum_{n \geqq 1} \frac{1}{B(n, \frac{1}{2})} p^{(n)}(\boldsymbol{\xi}; \zeta) \Phi_n(u; \mathbf{N} \cdot \mathbf{Y}). \tag{3.38}$$

The function $\Phi_n(u; \mathbf{N} \cdot \mathbf{Y})$ is a universal function, whereas $P(\mathbf{X}, \mathbf{Y}; \zeta)$ depends merely on the coefficient $F(\mathbf{X})$ of (3.1).

Let us consider the Neumann problem for (3.1), $\partial\psi/\partial v_x = f(\mathbf{X})$ for $\mathbf{X} \in \partial D_0$. If we define the kernel

$$\mathscr{K}(\mathbf{X}, \mathbf{Y}) \equiv \frac{1}{2\pi i} \int_{|\zeta|=1} P(\mathbf{X}, \mathbf{Y}; \zeta) \frac{d\zeta}{\zeta}, \tag{3.39}$$

then it is tempting to reformulate this Neumann problem as the Fredholm integral equation

$$f(\mathbf{X}) = -\rho(\mathbf{X}) + \frac{1}{2\pi} \int_{\partial D_0} \rho(\mathbf{Y}) \frac{\partial}{\partial v_x} \left(\frac{1}{|\mathbf{X} - \mathbf{Y}|} \right) d\omega_y + \int_{\partial D_0} \rho(\mathbf{Y}) \frac{\partial}{\partial v_x} \mathscr{K}(\mathbf{X}, \mathbf{Y})\, d\omega_y, \tag{3.40}$$

where we have replaced $H(\mathbf{X})$ by (3.31) and computed the residue as $\mathbf{X} \to \partial D_0$. To verify this operation we must investigate the convergence properties of $P(\mathbf{X}, \mathbf{Y}; \zeta)$ more closely. To this end we recall that on page 549 of [10] Tjong gave the following estimates for $p^{(n)}(\boldsymbol{\xi}; \zeta)$, where

$$\boldsymbol{\xi} \in \{\boldsymbol{\xi} \,\|\, \xi_i| \leqq R_0 < \rho; i = 1, 2, 3\} \quad \text{and} \quad |\zeta| = 1:$$

$$|p^{(n)}(\boldsymbol{\xi}; \zeta)| \leqq \text{const.}\, r^n \quad \text{with } r = \left\{ \frac{4(9+\delta)\alpha^5}{(\alpha-1)^6 R_0} \right\}, \quad \alpha = \frac{\rho}{R_0}. \tag{3.41}$$

Since $F(\mathbf{X})$ was entire, ρ may be taken arbitrarily large. Instead, we choose first R_0 and then take α such that $0 < r < 1$. With this choice of α we may majorize $P(\mathbf{X}, \mathbf{Y}; \zeta)$ in terms of the function

$$(1 - |Z|)^{-3/2} = \sum_{n \geq 1} \frac{1}{B(n, \frac{1}{2})} |Z|^{n-1}.$$

The estimate (3.41) permits the following majorization:

$$P(\mathbf{X}, \mathbf{Y}; \zeta) \ll \left\{ \frac{1}{(1 - r|\mathbf{N} \cdot (\mathbf{Y} - \mathbf{X})|)^{3/2}} + \frac{1}{(1 - r|\mathbf{N} \cdot \mathbf{Y}|)^{3/2}} \right\} \cdot \log |\mathbf{N} \cdot (\mathbf{X} - \mathbf{Y})| \tag{3.42}$$
$$+ \frac{1}{(1 - r[1 + |\mathbf{N} \cdot \mathbf{Y}|])^{3/2}}.$$

Since $r > 0$ can be made as close to zero as desired, the only term to be inspected is $\log |\mathbf{N} \cdot (\mathbf{X} - \mathbf{Y})|$, whose normal derivative is majorized by $|\mathbf{N} \cdot (\mathbf{X} - \mathbf{Y})|^{-1}$. For each fixed value of $\zeta(|\zeta| = 1)$, this latter term has just simple, polar, point singularities as $\mathbf{X} \to \partial D_0$. These singularities are integrable and do not contribute to the residue.

We next check to see whether (3.40) is invertible as an operator equation for $\rho(\mathbf{Y})$. It is well known that the kernel in the first integral of (3.40) satisfies the sufficiency criteria for compactness. From our majorization of $P(\mathbf{X}, \mathbf{Y}; \zeta)$ as an analytic function of $\mathbf{X}$, $\mathbf{Y}$ and ζ it is clear that the second integral's kernel also satisfies our compactness criteria. Indeed the kernel of the second integral has only a weak singularity. To show that the Fredholm equation (3.40) has a unique solution we consider the homogeneous transposed equation

$$\mu(\mathbf{Y}) = \frac{1}{2\pi} \int_{\partial D_0} \mu(\mathbf{X}) \frac{\partial}{\partial \nu_x} \left(\frac{1}{|\mathbf{X} - \mathbf{Y}|} \right) d\omega_x + \int_{\partial D_0} \mu(\mathbf{X}) \frac{\partial}{\partial \nu_x} \mathscr{K}(\mathbf{X}, \mathbf{Y})\, d\omega_x.$$

This implies that the "double layer" solution

$$\psi(\mathbf{Y}) = \frac{1}{2\pi} \int_{\partial D_0} \mu(\mathbf{X}) \frac{\partial}{\partial \nu_x} \left(\frac{1}{|\mathbf{X} - \mathbf{Y}|} \right) d\omega_x + \int_{\partial D_0} \mu(\mathbf{X}) \frac{\partial}{\partial \nu_x} \mathscr{K}(\mathbf{X}, \mathbf{Y})\, d\omega_x$$

assumes the boundary value zero; and consequently if we extend $F(\mathbf{X})$ to D' such that $F(\mathbf{X}) < 0$ and is of class $\mathscr{C}^1$, then the exterior problem has the unique solution $\psi(\mathbf{X}) \equiv 0$. Since $\partial \mathscr{K}(\mathbf{X}, \mathbf{Y})/\partial \nu_x$ has a weak singularity, it may be easily shown following the approach of [2, pp. 364–366] that the normal derivative $\partial \psi / \partial \nu$ is continuous across ∂D_0. Hence for the interior problem, $\psi(\mathbf{X})$ obeys the boundary condition $\partial \psi / \partial \nu(\mathbf{X}) = 0$, $\mathbf{X} \in \partial D_0$. Since $F(\mathbf{X}) < 0$ in D, it follows immediately from the identity

$$\int_{D_0} [\nabla \psi \cdot \nabla \psi - F \psi^2]\, d\mathbf{X} = -\int_{\delta D_0} \psi \frac{\partial \psi}{\partial \nu}\, d\omega,$$

that $\psi(\mathbf{X}) \equiv 0$ in D, and hence the only solution to the transposed, homogeneous equation is $\mu(\mathbf{X}) \equiv 0$. We conclude that the Neumann data $f(\mathbf{X})$ of (3.40) is orthogonal to each solution of the homogeneous transposed equation, and hence (3.40) has a unique solution.

We terminate the above discussion with the following theorem.

THEOREM 2. *Let $\psi(\mathbf{X})$ be a solution of (3.1) which is in $\mathscr{C}^0(D + \partial D)$ and let D be appropriate. Then there exists a unique harmonic function in D such that $\psi(\mathbf{X})$ may be represented in D as $\psi(\mathbf{X}) = (\mathbf{T}f)(\mathbf{X})$ with*

$$f(u, \zeta) = -\frac{1}{2\pi}\int_{\mathscr{L}} g(u[1 - t^2], \zeta)\frac{dt}{t^2},$$

where $\mathscr{L}$ is a smooth curve from -1 to $+1$ not passing through the origin and $g(u, \zeta) = (\mathbf{B}_3^{-1}H)(u, \zeta)$.

Proof. The Dirichlet problem for (3.1) with $F(\mathbf{X}) < 0$, $\mathbf{X} \in D$, and $F(\mathbf{X}) \in \mathscr{C}^1(D + \partial D)$ is well-posed. This serves to define a Neumann problem for $D_0 \Subset D$ with Lyapunov boundary ∂D_0. From the above discussion there exists a unique solution $\rho(\mathbf{Y})$ to the integral equation (3.40) arising from such a Neumann problem. Hence there exists a unique harmonic function given by (3.31). This harmonic function is related to $g(u, \zeta)$ by (3.10), and $g(u, \zeta)$ is related to $f(u, \zeta)$ by (3.9). The integral for $f(u, \zeta)$ given in the hypothesis is simply the inverse integral transform for (3.9).

COROLLARY. *Let $F(\mathbf{X}) < 0$ in D, let ∂D be appropriate, and suppose the Dirichlet data $\psi(\mathbf{X}) = g(\mathbf{X})$ is sufficiently smooth. Then there exists a unique harmonic function $H(\mathbf{X})$ in D such that*

$$\psi(\mathbf{X}) = \frac{-1}{2\pi}\mathbf{T}\left(\int_{\mathscr{L}} (B_3^{-1}H)(u[1 - t^2], \zeta)\frac{dt}{t^2}\right)$$

in $D + \partial D$.

Remark. This is obviously the case when $g(\mathbf{X})$ is the restriction to ∂D of a $\mathscr{C}^1(D + \partial D)$ solution of (3.1).

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SIAM J. MATH. ANAL.
Vol. 2, No. 1, February 1971

RECURSIVE ALGORITHMS FOR THE SUMMATION OF CERTAIN SERIES*

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1. Introduction. In this note we study the computation of the series $S_N = \sum_{n=0}^{N} d_n p_n$ by regarding it as a solution of an inhomogeneous difference equation. Here p_n itself satisfies the homogeneous difference equation

$$a_n p_{n+1} + b_n p_n + c_n p_{n-1} = 0, \tag{1a}$$

or, equivalently,

$$p_{n+1} + \bar{b}_n p_n + \bar{c}_n p_{n-1} = 0, \tag{1b}$$

where $\bar{b}_n = b_n/a_n$ and $\bar{c}_n = c_n/a_n$.

An algorithm to sum the series was first given by Clenshaw [1] and has been more recently discussed by Smith [2], Hart et al. [3, p. 70], Elliott [4], Ng [5] and Luke [6, pp. 325–329]. The algorithm consists of constructing a sequence $\{B_n\}$ or $\{\bar{B}_n\}$, $n = N - 1, \cdots, -1$ by means of the recurrence

$$c_{n+1} B_{n+1} + b_n B_n + a_{n-1} B_{n-1} = d_n, \tag{2a}$$

or

$$\bar{c}_{n+1} \bar{B}_{n+1} + \bar{b}_n \bar{B}_n + \bar{B}_{n-1} = d_n \tag{2b}$$

with initial conditions $B_N = B_{N+1} = 0$ or $\bar{B}_N = \bar{B}_{N+1} = 0$. To see how this recurrence allows the evaluation of the series, multiply (2a) by p_n and write out a "system of equations," one for each n:

$$\begin{array}{llll}
B_{N-1}a_{N-1}p_N & & & = d_N p_N, \\
B_{N-2}a_{N-2}p_{N-1} & + B_{N-1}b_{N-1}p_{N-1} & & = d_{N-1}p_{N-1}, \\
B_{N-3}a_{N-3}p_{N-2} & + B_{N-2}b_{N-2}p_{N-2} & + B_{N-1}C_{N-1}p_{N-2} & = d_{N-2}p_{N-2}, \\
\vdots & & & \\
B_{-1}a_{-1}p_0 & + B_0 b_0 p_0 & + B_1 c_1 p_0 & = d_0 p_0.
\end{array}$$

The result

$$\sum_{n=0}^{N} d_n p_n = a_{-1} B_{-1} p_0 + B_0(a_0 p_1 + b_0 p_0) \tag{3a}$$

is obtained by adding these equations and using (1a). Similarly, we have, from (1b) and (2b), the result

$$\sum_{n=0}^{N} d_n p_n = \bar{B}_{-1} p_0 + \bar{B}_0(p_1 + \bar{b}_0 p_0). \tag{3b}$$

* Received by the editors January 26, 1970, and in revised form May 25, 1970.

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This paper presents the results of one phase of research carried out at the Jet Propulsion Laboratory and sponsored by the National Aeronautics and Space Administration under Contract NAS7-100.

As we point out below, (2a) and (2b) are only two of many possible forms of a single difference equation. It is useful to have many forms since one form may be computationally more efficient than another depending on the particular function p_n. For example, when p_n is the Laguerre polynomial L_n, use of (2a) requires N divisions less than does (2b).

The fact that (1) possesses another solution (say, q_n) is important. For values of n for which $|q_n| \gg |p_n|$ large cancellation error occurs in (3). Elliott [4] obtained the last result by expressing $\bar{B}_n$ in terms of p_n, q_n and d_n. In the following section we show that these results follow from the general solution of the inhomogeneous form of (1a):

$$a_n L_{n+1} + b_n L_n + c_n L_{n-1} = d_n. \tag{4}$$

This approach avoids introducing a new difference equation (such as (2a) and (2b)) and subsumes the well-known nested multiplication as a special case, as shown in § 3.

2. General solution of the inhomogeneous difference equation. We now discuss the solution of (4) and show (2a) and (2b) to be two of its transforms. The general solution of (4) can be written as

$$L_n = \alpha p_n + \beta q_n + M_n, \tag{5}$$

where M_n is a particular solution of (4), and α and β are constants. Assuming $d_n = 0, n < 0, n > N$, we can express the particular solution M_n in terms of Green's functions, viz.,

$$M_n = \sum_{k=0}^{N} G_{k,n} d_k, \tag{6}$$

where a Green's function $G_{k,n}$ is a solution of

$$a_n G_{k,n+1} + b_n G_{k,n} + c_n G_{k,n-1} = \delta_{kn},$$
$$\delta_{kn} = \begin{cases} 0, & k \neq n, \\ 1, & k = n. \end{cases} \tag{7}$$

Since any $G_{k,n}$ is a solution of the homogeneous equations for $k \neq n$, it is only necessary to find the simplest linear combinations of p_n and q_n which satisfy (7). We find, assuming $a_n \neq 0$, $c_n \neq 0$ for all n,

$$G_{kn} = \begin{cases} \dfrac{1}{s_k} q_k p_n & \text{for } k \leqq n, \\[2ex] \dfrac{1}{s_k} q_n p_k & \text{for } k \geqq n, \end{cases} \tag{8}$$

where s_k is a normalization constant which satisfies

$$s_k = a_k(q_k p_{k+1} - p_k q_{k+1}) \tag{9}$$

and therefore is just a Wronskian, and obeys the recurrence relation

$$s_k = \frac{c_k}{a_{k-1}} s_{k-1}. \tag{10}$$

Thus with s_k determined by (9) and (10) we obtain

$$L_n = \alpha p_n + \beta q_n + p_n \sum_{k=0}^{n} \frac{1}{s_k} d_k q_k + q_n \sum_{k=n+1}^{N} \frac{1}{s_k} d_k p_k. \tag{11}$$

Using (11), we find

$$\frac{a_N}{s_N}(p_{N+1}L_N - p_N L_{N+1}) = \beta \tag{12}$$

and

$$-\frac{a_N}{s_N}(q_{N+1}L_N - q_N L_{N+1}) = \alpha + \sum_{k=0}^{N} \frac{1}{s_k} d_k q_k, \tag{13}$$

so that (11) can also be given the form

$$\begin{aligned} L_n = {} & \frac{a_N}{s_N}(p_{N+1}L_N - p_N L_{N+1})q_n - \frac{a_N}{s_N}(q_{N+1}L_N - q_N L_{N+1})p_n \\ & + q_n \sum_{k=n+1}^{N} \frac{1}{s_k} d_k p_k - p_n \sum_{k=n+1}^{N} \frac{1}{s_k} d_k q_k. \end{aligned} \tag{14}$$

Similarly,

$$\frac{a_{-1}}{s_{-1}}(p_0 L_{-1} - p_{-1}L_0) = \beta + \sum_{k=0}^{N} \frac{1}{s_k} d_k p_k, \tag{15}$$

$$-\frac{a_{-1}}{s_{-1}}(q_0 L_{-1} - q_{-1}L_0) = \alpha, \tag{16}$$

which, combined with (12) and (13), give

$$\sum_{k=0}^{N} \frac{1}{s_k} d_k p_k = \frac{a_{-1}}{s_{-1}}(L_{-1}p_0 - L_0 p_{-1}) - \frac{a_N}{s_N}(L_N p_{N+1} - L_{N+1}p_N), \tag{17}$$

$$\sum_{k=0}^{N} \frac{1}{s_k} d_k q_k = \frac{a_{-1}}{s_{-1}}(L_{-1}q_0 - L_0 q_{-1}) - \frac{a_N}{s_N}(L_N q_{N+1} - L_{N+1}q_N). \tag{18}$$

If $c_k = a_{k-1}$ for all k, then s_k is independent of k from (10) and so (17) and (18) take on a particularly simple form. We note that there is no loss in generality if we assume the condition $c_k = a_{k-1}$. For were it not true, we could still write (1a) as

$$\lambda_k(a_k p_{k+1} + b_k p_k + c_k p_{k-1}) = 0$$

and choose λ_k to satisfy

$$c'_k \equiv \lambda_k c_k = \lambda_{k-1} a_{k-1} \equiv a'_{k-1}. \tag{19}$$

With (1a) in this form, (17) becomes

$$\sum_{k=0}^{N} d_k p_k = (c_0 L_{-1})p_0 - L_0(c_0 p_{-1}) - a_N(L_N p_{N+1} - L_{N+1}p_N), \tag{17a}$$

where $c_0 L_{-1} \equiv -[a_0 L_1 + b_0 L_0] + d_0$ even when $c_0 = 0$. Table 1 shows that condition (19) is satisfied for nearly all the familiar special functions.

TABLE 1

Functions	λ_n
Trigonometric*	1
Chebyshev	1
Legendre	1
Laguerre	1
Hermite	$(2^n n!)^{-1}$
Bessel	1

* The summation of trigonometric series by recurrence is also known as Goertzel's (or Watt's) algorithm [7].

Assuming now that $c_n = a_{n-1}$ for all n we see that the algorithm described in the previous section is equivalent to using (4) in the backward direction with initial values $L_N = L_{N+1} = 0$, and evidently (2a) and (2b) are transformations of (4). In fact, $B_n = L_n$ and $\bar{B}_n = a_n L_n$. These, however, are only special cases of the general transformation $\hat{B}_n = \mu_n L_n$. For the general transformation we have

$$a_n \frac{\mu_n}{\mu_{n+1}} \hat{B}_{n+1} + b_n \hat{B}_n + c_n \frac{\mu_n}{\mu_{n-1}} \hat{B}_{n-1} = \mu_n d_n \equiv \hat{d}_n \tag{20}$$

and

$$\sum_{k=0}^{N} d_k p_k = a_{-1} \frac{\hat{B}_{-1}}{\mu_{-1}} p_0 - \frac{\hat{B}_0}{\mu_0} c_0 p_{-1}. \tag{21}$$

It might also be noted that (17a), with $L_n = B_n$ or $L_n = \bar{B}_n/a_n$, reduces to (3a) and (3b), respectively.

Finally, we give two examples showing possible choices of μ_n. First consider the computation of

$$\sum_{n=0}^{N} \frac{d_n}{n!} P_n(x),$$

where the P_n's are the Legendre polynomials with $a_n = n + 1$, $b_n = -(2n + 1)x$, $c_n = n$. The sum can be computed through any one of the following three recurrences:

$$B_{n-1} = \frac{1}{n}\left[(2n + 1)xB_n - (n + 1)B_{n+1} + \frac{d_n}{n!}\right], \quad B_n = L_n, \quad n = N, \cdots, 1, \tag{22}$$

$$(c_0 B_{-1}) = xB_0 - B_1 + d_0;$$

$$\bar{B}_{n-1} = \left[\frac{(2n + 1)x}{n + 1}\bar{B}_n - \frac{n + 1}{n + 2}\bar{B}_{n+1} + \frac{d_n}{n!}\right], \tag{23}$$

$$\bar{B}_n = (n + 1)L_n, \quad n = N, \cdots, 0;$$

$$\hat{B}_{n-1} = \frac{1}{n^2}[(2n+1)x\hat{B}_n - \hat{B}_{n+1} + d_n], \tag{24}$$
$$\hat{B}_n = n!L_n, \quad n = N, \cdots, 1,$$
$$\left(c_0\frac{\hat{B}_{-1}}{\mu_{-1}}\right) = x\hat{B}_0 - \hat{B}_1 + d_0.$$

Similarly, the sum $\sum_{n=0}^{N} d_n I_n P_n(x)$, where P_n's are the Legendre polynomials and the I_n's are the modified Bessel functions, may be computed through any one of the following three recurrences:

$$B_{n-1} = \frac{1}{n}[(2n+1)xB_n - (n+1)B_{n+1} + d_n I_n],$$
$$B_n = L_n, \quad n = N, \cdots, 1, \tag{25}$$
$$(c_0 B_{-1}) = xB_0 - B_1 + d_0 I_0;$$

$$\bar{B}_{n-1} = \left[\frac{(2n+1)x}{n+1}\bar{B}_n - \frac{n+1}{n+2}\bar{B}_{n+1} + d_n I_n\right], \tag{26}$$
$$\bar{B}_n = (n+1)L_n, \quad n = N, \cdots, 0;$$

$$\hat{B}_{n-1} = \frac{r_{n-1}}{n}[(2n+1)x\hat{B}_n - (n+1)r_n\hat{B}_{n+1} + d_n],$$
$$\hat{B}_n = L_n/I_n, \quad n = N, \cdots, 1, \tag{27}$$
$$c_0\hat{B}_{-1} = r_{-1}[x\hat{B}_0 - r_0\hat{B}_1 + d_0],$$

where $r_n = I_{n+1}/I_n$.

For (25) and (26), any version of Miller's algorithm may be used to generate the sequence of I_n's; for (27), Gautschi's [8] formulation of Miller's algorithm is most convenient for it generates the r_n's directly.

In this note we have mainly shown that a well-known algorithm can be understood in a more general way. We have not, however, shown how this algorithm may be significantly improved. It would be most useful to further investigate how μ_n should be chosen to minimize errors and save arithmetic operations.

3. Computation of power series utilizing first order difference equations. Power series are often computed using first order difference equations, and an analysis similar to the one just presented can be made. We are now interested in the equation

$$L_{n+1} - b_n L_n = a_n. \tag{28}$$

The solution of the homogeneous equation is given as αp_n with $p_n = \prod_{k=0}^{n-1} b_k$ and $p_0 = 1$. A particular solution of (28) is

$$M_n = \sum_{k=0}^{n-1} G_{kn} a_k, \quad n > 1, \quad M_0 = 0, \quad M_1 = a_0, \tag{29}$$

where $G_{kn} = \prod_{m=k+1}^{n-1} b_m$ is an analogue of the Green's function described in § 2.

The general solution[1] can then be written as

$$L_n = \alpha p_n + \sum_{k=0}^{n-1} a_k \prod_{m=k+1}^{n-1} b_m. \tag{30}$$

One immediate application of these results is the summation of power series $\sum_{n=0}^{N} c_n x^n$. For by setting $b_n = x$, $a_n = c_{N-n}$, $L_0 = 0$, forward recurrence yields the result

$$L_{N+1} = \sum_{n=0}^{N} c_n x^n. \tag{31}$$

We note here, as was similarly noted in § 2, that the sum (31) can be evaluated using different forms of the difference equation (28). Multiplying by μ_n we obtain

$$D_{n+1} - \frac{\mu_n}{\mu_{n-1}} b_n D_n = \mu_n a_n, \tag{32}$$

where

$$D_n = \mu_{n-1} L_n.$$

So, if for example we want to sum $\sum_{n=0}^{N} c_n x^n/n!$, then with $b_n = x/(N - n + 1)$, $a_n = c_{N-n}$, we obtain

$$L_{N+1} = \sum_{n=0}^{N} c_n \frac{x^n}{n!}, \tag{33}$$

which can also be obtained with $a_n = c_{N-n}/(N - n)!$ and $b_n = x$. The former is more convenient since it saves N multiplications.

Acknowledgment. We are indebted to the editor and the referee for helpful comments.

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SIAM J. MATH. ANAL.
Vol. 2, No. 1, February 1971

TAYLOR'S SERIES GENERALIZED FOR FRACTIONAL DERIVATIVES AND APPLICATIONS*

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Abstract. The familiar Taylor's series expansion of the function $f(z)$ has for its general term $D^n f(z_0)(z - z_0)^n/n!$. A new generalization of Taylor's series in which the general term is $D^{an+\gamma} f(z_0)(z - z_0)^{an+\gamma}/\Gamma(an + \gamma + 1)$, where $a > 0$ and γ is an arbitrary complex number, is examined. This new series is extended further to a form which includes the familiar Lagrange's expansion as a special case. The derivatives appearing in this series are of order $an + \gamma$ and are called "fractional derivatives." Examples of the use of this new series for discovering generating functions are given.

1. Introduction. A fractional derivative $D^{\alpha}_{g(z)} f(z)$ is an extension of the familiar nth derivative $D^n_{g(z)} f(z) = d^n f(z)/(dg(z))^n$ of the function $f(z)$ with respect to $g(z)$ to nonintegral values of n. The literature contains many examples of the use of fractional derivatives in the solution of problems in ordinary differential equations [8], partial differential equations [4], [13] and integral equations [3].

The study of the special functions of mathematical physics is also facilitated by the introduction of fractional differential operators. Consider, for example, the various representations of the Bessel function $J_v(z)$ of order v. Power series and definite integral representations are the most common; however, the less familiar derivative representation

$$J_v(z) = \pi^{-1/2}(2z)^{-v} D_{z^2}^{-v-1/2} \frac{\cos z}{z} \tag{1.1}$$

warrants further attention. When $-v - 1/2$ is a natural number, (1.1) reveals that $D_{z^2}^{-v-1/2}$ is the usual elementary differential operator, and thus $J_v(z)$ is an elementary function. When $-v - 1/2$ is not a natural number, the operator $D_{z^2}^{-v-1/2}$ still behaves very much like the familiar differential operator from the elementary calculus. The operation $D^a D^b = D^{a+b}$, the Leibniz rule [9], [10], the chain rule [9], [11] and other generalizations of the manipulations so familiar from the elementary calculus are valid for nonintegral values of a and b. These manipulations permit us to find easily many relations for the special functions from representations similar to (1.1) which would not otherwise seem obvious [7], [8], [9], [10], [11]. Table 1 gives a short list of fractional derivative representations for the special functions.

In this paper the Taylor's series is generalized to include fractional derivatives and thus provides an additional tool which is particularly convenient for the study of the special functions through their fractional derivative representations. There are two equivalent forms of our general result:

$$\begin{aligned} &\sum_{k \in K} a^{-1} \omega^{-\gamma k} f(\theta^{-1}(\theta(z)\omega^k)) \\ &\quad = \sum_{n=-\infty}^{\infty} \frac{D_{z-b}^{an+\gamma}[f(z)\theta'(z)[(z - z_0)/\theta(z)]^{an+\gamma+1}]|_{z=z_0} \, \theta(z)^{an+\gamma}}{\Gamma(an + \gamma + 1)} \end{aligned} \tag{1.2}$$

* Received by the editors March 26, 1970, and in revised form June 8, 1970.

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and

$$\sum_{k \in K} a^{-1}\omega^{-\gamma k} f(\theta^{-1}(\theta(z)\omega^k)) = \sum_{n=-\infty}^{\infty} \frac{\theta(z)^{an+\gamma}}{2\pi i} \int_b^{(z_0^+)} \frac{f(\xi)\theta'(\xi)\,d\xi}{\theta(\xi)^{an+\gamma+1}}. \tag{1.2a}$$

There are several restrictions which must be imposed on the functions and parameters in (1.2) and (1.2a), all of which are listed in the hypothesis of Theorem 4.1. For the moment, it suffices to notice the following:

(i) The order of the derivatives in (1.2) is $an + \gamma$, where n is the integral index of summation, $a > 0$, and γ is an arbitrary complex number.

(ii) b is a fixed point in the z-plane and $\{z \mid |\theta(z)| = |\theta(b)|\}$ defines a simple closed curve C on which the series (1.2) and (1.2a) converge. $\theta(z)$ is an analytic function inside and on C. $\theta(z)$ has only one zero inside C, located at $z = z_0$, and that zero is simple.

(iii) $\omega = \exp(2\pi i/a)$, and the finite set of integers K is defined by $K = \{k \mid k$ is integral, and $\arg\theta(b) < \arg\theta(z) + 2\pi k/a < \arg\theta(b) + 2\pi\}$.

While the general formulas (1.2) and (1.2a) are new, several special cases are familiar from the literature.

Case 1. If $a = 1, \gamma = 0$ and $\theta(z) = z - z_0$ in (1.2), we have the familiar Taylor's series

$$f(z) = \sum_{n=0}^{\infty} D^n f(z_0)(z - z_0)^n/n!.$$

Case 2. We obtain Lagrange's expansion [16, p. 132] from (1.2) (after an integration by parts) by taking $a = 1, \gamma = 0$ and $\theta(z) = \theta_1(z)(z - z_0)$:

$$f(z) = \sum_{n=0}^{\infty} D^{n-1}\{f'(z_0)/\theta_1(z_0)^n\}\,\theta(z)^n/n!$$

Case 3. If we take $a = 1$ and $\gamma = 0$ in (1.2a), we obtain Teixeira's extended form of Burmann's theorem [16, p. 131]:

$$f(z) = \sum_{n=-\infty}^{\infty} \frac{\theta(z)^n}{2\pi i} \oint \frac{f(\xi)\theta'(\xi)\,d\xi}{\theta(\xi)^{n+1}}.$$

Case 4. We obtain the least familiar special case of (1.2) which can be found in the literature by taking $a = 1, \theta(z) = z - z_0$, and γ arbitrary. It is called the Taylor–Riemann series:

$$f(z) = \sum_{n=-\infty}^{\infty} \frac{D_{z-b}^{n+\gamma} f(z)|_{z=z_0}(z - z_0)^{n+\gamma}}{\Gamma(n + \gamma + 1)}. \tag{1.3}$$

This series was first considered formally by Riemann [12] in 1847, in a manuscript probably never intended for publication. Riemann did not prove (1.3), but used its structure to suggest a definition of fractional differentiation. The special cases of (1.3) in which $f(z)$ is e^z and z^p were studied by Heaviside [6, Chap. 7, 8] and Watanabe [14]. The first critical discussion of (1.3) for arbitrary functions $f(z)$ was not given until 1945 when G. H. Hardy [5] considered (1.3) as an asymptotic expansion of $f(z)$ and as a series summable Borel to $f(z)$. The first analysis of the pointwise convergence of the series (1.3) to the function $f(z)$ in the z-plane seems to be [9, Chap. 3]. The nature of the pointwise convergence of (1.3) in the z-plane is given as a special case of the more general formula (1.2) in Theorem 4.1 of this paper.

If we restrict a to the interval $0 < a \leqq 1$, the left-hand side of (1.2) contains only the term in which $k = 0$, and we obtain the particularly simple series

$$f(z)a^{-1} = \sum_{n=-\infty}^{\infty} \frac{D_{z-b}^{an+\gamma}[f(z)\theta'(z)[(z-z_0)/\theta(z)]^{an+\gamma+1}]|_{z=z_0}\theta(z)^{an+\gamma}}{\Gamma(an+\gamma+1)}.$$

To the best of the author's knowledge, neither this series nor the more general series (1.2) have appeared before in the literature.

Finally, a few examples of the generalized Taylor's series are studied for specific functions $f(z)$. We find that (1.2) is particularly useful for obtaining generating functions for the special functions of mathematical physics when these special functions are represented by fractional derivatives.

2. Fractional derivatives and special functions. In this section we review the definition of fractional differentiation and give examples of common special functions of mathematical physics represented by fractional derivatives of elementary functions.

The most common definition for the fractional derivative of $f(z)$ of order α found in the literature is the "Riemann–Liouville integral" [2], [3], [4], [5], [7], [8], [9], [10], [11], [13], [14]:

$$D_z^\alpha f(z) = \Gamma(-\alpha)^{-1} \int_0^z f(t)(z-t)^{-\alpha-1}\,dt$$

where $\mathrm{Re}(\alpha) < 0$. The concept of a fractional derivative $D_{g(z)}^\alpha f(z)$ with respect to an arbitrary function $g(z)$ was apparently introduced for the first time in the author's papers [9], [10], while the idea appeared earlier for certain specific functions $g(z)$ in [4]. The most convenient form of the definition for our purposes is given through a generalization of Cauchy's integral formula. A thorough motivation for the following precise definition is found in [9], [10].

DEFINITION 2.1. Let $f(z)$ be analytic in the simply connected region R. Let $g(z)$ be regular and univalent on R, and let $g^{-1}(0)$ be an interior or boundary point of R. Assume also that $\oint_C f(z)\,dz = 0$ for any simple closed contour C in $RU\{g^{-1}(0)\}$ through $g^{-1}(0)$. Then if α is not a negative integer and z is in R, we define the *fractional derivative of order* α of $f(z)$ with respect to $g(z)$ to be

$$D_{g(z)}^\alpha f(z) = \frac{\Gamma(\alpha+1)}{2\pi i} \int_{g^{-1}(0)}^{(z+)} \frac{f(\zeta)g'(\zeta)\,d\zeta}{(g(\zeta)-g(z))^{\alpha+1}}. \tag{2.1}$$

For nonintegral α, the integrand has a branch line which begins at $\zeta = z$ and passes through $\zeta = g^{-1}(0)$. The notation on this integral implies that the contour of integration starts at $g^{-1}(0)$, encloses z once in the positive sense, and returns to $g^{-1}(0)$ without cutting the branch line or leaving $RU\{g^{-1}(0)\}$. (See Fig. 1.)

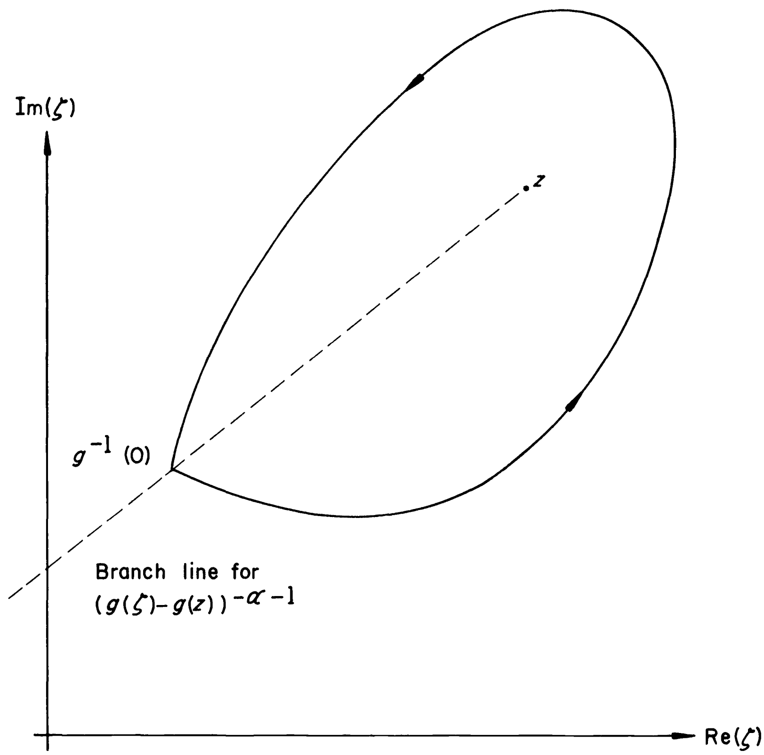

FIG. 1. *Branch line and contour of integration for the Definition* 2.1 *of fractional differentiation*

It is particularly interesting to set $g(z) = z - a$, for we find that

$$D^{\alpha}_{z-a}f(z) = \frac{\Gamma(\alpha+1)}{2\pi i}\int_a^{(z+)} f(\zeta)(\zeta - z)^{-\alpha-1}\,d\zeta. \tag{2.2}$$

While ordinary derivatives with respect to z and $z - a$ are equal, (2.2) shows that this is not the case for fractional derivatives, since the value of the contour integral depends on the point $\zeta = a$ at which the contour crosses the branch line.

The equivalence of the two forms of the generalized Taylor's series (1.2) and (1.2a) is seen at once from (2.2).

Contour integrals of the type (2.1) occur often in the representations of special functions. Table 1 gives a brief list of fractional derivative representations for a few selected functions. These are particularly convenient for use with the generalized Taylor's series (1.2). Fractional derivative representations of special functions are also found in [8] and can be easily constructed from the tables in [2].

3. Motivation for the generalized Taylor's theorem. The generalized Taylor's theorem features a "finite sum over k" on the left-hand side of (1.2). Why? An intuitive answer to this question is provided in this section through a formal examination of (1.2) in the special case in which a and γ are integers and $\theta(z) = z$. The relationship between the generalized Taylor's series and the Fourier series is then suggested by the consideration of a second formal example in which

TABLE 1
Special functions expressed as fractional derivatives

Name	Derivative Representation
Hypergeometric function	$F(a, b; c; z) = \dfrac{\Gamma(c)z^{1-c}}{\Gamma(b)} D_z^{b-c} z^{b-1}(1 - z)^{-a}$
Confluent hypergeometric function	${}_1F_1(a; c; z) = \dfrac{\Gamma(c)z^{1-c}}{\Gamma(a)} D_z^{a-c} e^z z^{a-1}$
Bessel function	$J_v(z) = \pi^{-1/2}(2z)^{-v} D_{z^2}^{-v-1/2} \dfrac{\cos z}{z}$
Modified Bessel function	$I_v(z) = \pi^{-1/2}(2z)^{-v} D_{z^2}^{-v-1/2} \dfrac{\cosh z}{z}$
Struve function	$\mathbf{H}_v(z) = \pi^{-1/2}(2z)^{-v} D_{z^2}^{-v-1/2} \dfrac{\sin z}{z}$
Modified Struve function	$\mathbf{L}_v(z) = \pi^{-1/2}(2z)^{-v} D_{z^2}^{-v-1/2} \dfrac{\sinh z}{z}$
Legendre function of the first kind	$P_v(z) = D_{1-z}^{v}(1 - z^2)^v/(\Gamma(v + 1)2^v)$
Associated Legendre function of the first kind	$P_v^u(z) = (1 - z^2)^{u/2} D_{1-z}^{v}(1 - z^2)^v/(\Gamma(v + 1)2^v)$
Laguerre function	$L_v^{(a)}(z) = \dfrac{\Gamma(a + v + 1)z^{-a}}{\Gamma(v + 1)\Gamma(-v)} D_z^{-a-v-1} e^z z^{-v-1}$
Incomplete gamma function	$\gamma(a, z) = \Gamma(a)e^{-z} D_z^{-a} e^z$

$0 < a \leqq 1$. Together, these two examples provide intuitive insight into the structure of the generalized Taylor's series and give preliminary assurance of its validity. The complete proof is postponed until the next section.

Case 1. Let $\theta(z) = z$ and a and γ be integers in (1.2). We then obtain

$$\sum_{k=0}^{a-1} \omega^{-\gamma k} f(z\omega^k) = a \sum_{n=0}^{\infty} f_{an+\gamma} z^{an+\gamma}, \tag{3.1}$$

where we have written $f_{an+\gamma}$ for $D^{an+\gamma} f(0)/(an + \gamma)!$, and $\omega = \exp(2\pi i/a)$. The examination of the special case in which $a = 3$ and $\gamma = 1$ is sufficient to suggest the manner in which the general case proceeds:

$$\begin{aligned}
f(z) &= f_0 + f_1 z + f_2 z^2 + f_3 z^3 + f_4 z^4 + \cdots. \\
\omega^{-1} f(z\omega) &= \omega^{-1} f_0 + f_1 z + \omega f_2 z^2 + \omega^2 f_3 z^3 + f_4 z^4 + \cdots, \\
\omega^{-2} f(z\omega^2) &= \omega^{-2} f_0 + f_1 z + \omega^2 f_2 z^2 + \omega^4 f_3 z^3 + f_4 z^4 + \cdots.
\end{aligned}$$

Summing these columns we see at once that the right-hand side is $3\sum_{n=0}^{\infty} f_{3n+1} z^{3n+1}$

as (3.1) predicts. Equation (3.1) is true for arbitrary integral a and γ by an equivalent calculation. This example shows that the finite sum over k in the generalized Taylor's theorem is natural and to be expected. (If we think of the way in which $\cosh(z)$ is related to e^z, we see at once that this is the special case of (3.1) in which $a = 2$, $\gamma = 0$, and $f(z) = e^z$.)

Case 2. Let $0 < a \leqq 1$ in (1.2a). We then have

$$\theta(z)^{-\gamma} f(z) = \sum_{n=-\infty}^{\infty} \frac{a}{2\pi i} \int_b^{(z_0^+)} \frac{f(\xi)\theta'(\xi)\, d\xi}{\theta(\xi)^{an+\gamma+1}} \theta(z)^{an}. \tag{3.2}$$

Let $\theta(\xi) = |\theta(b)| \exp(i\phi_0)$, $\theta(z) = |\theta(b)| \exp(i\phi)$, and $\theta(z)^{-\gamma} f(z) = F(\phi)$ in (3.2) and we obtain

$$F(\phi) = \sum_{n=-\infty}^{\infty} \frac{a}{2\pi} \int_{\arg \theta(b)}^{\arg \theta(b) + 2\pi} F(\phi_0) \exp(-ian\, \phi_0)\, d\phi_0 \exp(ian\, \phi). \tag{3.3}$$

The right-hand side of (3.3) is the Fourier series expansion of the function

$$F_0(\phi) = \begin{cases} F(\phi) & \text{for } \arg \theta(b) < \phi < \arg \theta(b) + 2\pi, \\ 0 & \text{otherwise} \end{cases}$$

over the interval $|\phi - \arg \theta(b) - \pi| < \pi/a$. Since we are only interested in ϕ such that $\arg \theta(b) < \phi < \arg \theta(b) + 2\pi$, formula (3.3) is valid. This example reveals that the generalized Taylor's theorem for arbitrary nonintegral a is nothing more than the Fourier series of a new function constructed from the function $f(z)$.

A rigorous proof of the generalized Taylor's theorem for arbitrary positive a could be constructed from the Fourier series analysis just given. However, a simpler method employing contour integration (not unlike the usual proof of Laurent's theorem) is given in the next section.

4. Proof of the generalized Taylor's series. Having examined examples which give motivation for the structure of the generalized Taylor's series (1.2) and (1.2a), we proceed to a rigorous derivation.

THEOREM 4.1. *Let a be real and positive, and let $\omega = \exp(2\pi i/a)$. Let $\theta(z)$ be a given function such that* (i) *the curves $C(r) = \{z \mid |\theta(z)| = r\}$ are simple and closed for each r such that $0 < r \leqq p$,* (ii) *$\theta(z)$ is analytic inside and on $C(p)$, and* (iii) *$\theta(z)$ has only one zero inside $C(p)$ and that zero is a simple one located at $z = z_0$. Let $b \neq z_0$ be a fixed point inside $C(p)$. Let $\theta(z)^q = \exp(q \ln \theta(z))$ denote that branch of the function which is continuous and single-valued on the region inside $C(p)$ cut by the branch line $z = z_0 + (b - z_0)r$, $0 \leqq r$, such that $\ln \theta(z)$ is real where $\theta(z) > 0$. Let $f(z)$ satisfy the conditions of Definition* 2.1 *for the existence of $D_{z-b}^{\alpha} f(z)$ for $\{z \mid z$ inside $C(p)$; but $z \neq b + r \exp(i \arg(b - z_0))$, $0 \leqq r\}$. Let $K = \{k \mid k$ integral, and $\arg \theta(b) < \arg \theta(z) + 2\pi k/a < \arg \theta(b) + 2\pi\}$. Then for arbitrary γ and z on $\{z \mid z$ on the curve $C(|\theta(b)|)$, but $\theta(z)^a \neq \theta(b)^a\}$, the generalized Taylor's series* (1.2) *and* (1.2a) *are valid.*

Proof. The maximum modulus theorem insures that the set of simple closed curves $C(r)$, $0 < r \leqq p$, are such that $C(s)$ is contained inside $C(t)$ for $s < t$. Let C_x denote the contour consisting of a straight line segment from $\xi = b$ to $\xi = b + x(b - z_0)$, the curve $C(|\theta(b + x(b - z_0))|)$ traversed once in the positive

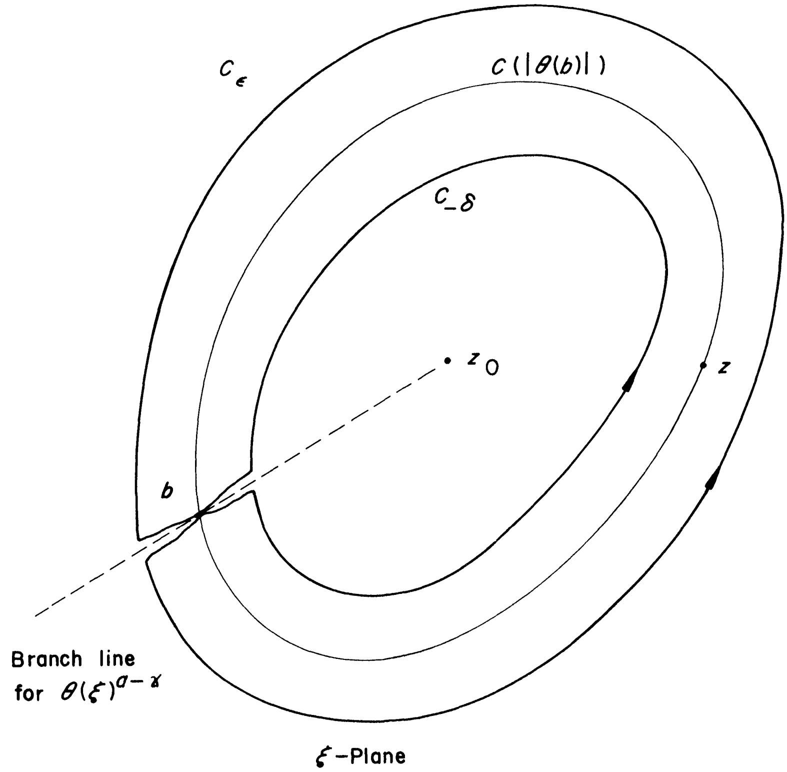

FIG. 2. *Contours of integration used in the proof of Theorem* 4.1

sense, and a straight line segment from $b + x(b - z_0)$ back to b. The contours C_ε and $C_{-\delta}$ are shown in Figure 2.

Consider the integral

$$I = \frac{\theta(z)^\gamma}{2\pi i} \int_{C_\varepsilon - C_{-\delta}} \frac{\theta(\xi)^{a-\gamma-1}\theta'(\xi)f(\xi)\,d\xi}{\theta(\xi)^a - \theta(z)^a}. \tag{4.1}$$

The integrand in (4.1) contains poles at the points where $\theta(\xi)^a = \theta(z)^a$. This means that $\theta(\xi) = \theta(z)\exp(2\pi ki/a)$ for $k \in K$. (The set of integers K is defined in the hypothesis.) Thus there are poles at $\xi = \theta^{-1}(\theta(z)\omega^k)$, $k \in K$, while the integrand of (4.1) is analytic for all other values of ξ inside the closed contour $C_\varepsilon - C_{-\delta}$. Each of these poles is simple because the number of roots of the equation $\theta(\xi) = c$, $|c| < p$, is given by the argument principle as

$$(2\pi i)^{-1} \int_{C(p)} \frac{\theta'(\xi)\,d\xi}{\theta(\xi) - c} = \sum_{n=0}^{\infty} (2\pi i)^{-1} c^n \int_{C(p)} \theta'(\xi)\theta(\xi)^{-n-1}\,d\xi.$$

All terms in this last sum vanish but the first, which equals 1 since $\xi = z_0$ is a simple root of θ. The residue at $\xi = \theta^{-1}(\theta(z)\omega^k)$ is given by

$$\lim_{\xi \to \theta^{-1}(\theta(z)\omega^k)} \left\{ \frac{(\xi - \theta^{-1}(\theta(z)\omega^k))\theta(\xi)^{a-\gamma-1}\theta'(\xi)f(\xi)}{\theta(\xi)^a - \theta(z)^a} \right\};$$

and using l'Hospital's rule this becomes

$$\left.\frac{\theta(\xi)^{a-\gamma-1}f(\xi)}{a\theta(\xi)^{a-1}}\right|_{\xi=\theta^{-1}(\theta(z)\omega^k)} = \frac{f(\theta^{-1}(\theta(z)\omega^k))}{a\omega^{\gamma k}\theta(z)^{\gamma}}.$$

Thus we see from the residue theorem that

$$I = \sum_{k\in K} a^{-1}\omega^{-\gamma k} f(\theta^{-1}(\theta(z)\omega^k)). \tag{4.2}$$

Returning to (4.1) we see that

$$\begin{aligned} I &= \frac{\theta(z)^{\gamma}}{2\pi i}\left\{\int_{C_\varepsilon} - \int_{C_{-\delta}}\right\} \\ &= \frac{\theta(z)^{\gamma}}{2\pi i}\left\{\int_{C_\varepsilon} \frac{\theta(\xi)^{-\gamma-1}\theta'(\xi)f(\xi)\,d\xi}{1-[\theta(z)/\theta(\xi)]^a} + \int_{C_{-\delta}} \frac{\theta(\xi)^{a-\gamma-1}\theta'(\xi)f(\xi)\,d\xi}{\theta(z)^a[1-[\theta(\xi)/\theta(z)]^a]}\right\}. \end{aligned}$$

Expanding the denominators of both integrals in powers of $[\theta(z)/\theta(\xi)]^a$ we obtain

$$\begin{aligned} I = \frac{\theta(z)^{\gamma}}{2\pi i}\Bigg\{ &\sum_{n=0}^{N}\int_{C_\varepsilon} \theta(\xi)^{-an-\gamma-1}\theta'(\xi)f(\xi)\,d\xi\,\theta(z)^{an} \\ &+ \sum_{n=-1}^{-N}\int_{C_{-\delta}} \theta(\xi)^{-an-\gamma-1}\theta'(\xi)f(\xi)\,d\xi\,\theta(z)^{an} \\ &+ R_\varepsilon(N) + R_{-\delta}(N)\Bigg\}, \end{aligned} \tag{4.3}$$

where

$$R_\varepsilon(N) = \int_{C_\varepsilon} \frac{\theta(\xi)^{-\gamma-1}[\theta(z)/\theta(\xi)]^{aN+a}\theta'(\xi)f(\xi)\,d\xi}{1-[\theta(z)/\theta(\xi)]^a}$$

and

$$R_{-\delta}(N) = \int_{C_{-\delta}} \frac{\theta(\xi)^{a-\gamma-1}[\theta(\xi)/\theta(z)]^{aN}\theta'(\xi)f(\xi)\,d\xi}{1-[\theta(\xi)/\theta(z)]^a}.$$

We note that the regularity of θ and f permits us to deform the contours of integration C_ε and $C_{-\delta}$ in (4.3) provided the contours start and end at $\xi = b$ and do not cross the branch line for $\theta(\xi)^{a-\gamma}$ (see Fig. 2). Comparison of (4.3) with the definition of fractional derivative

$$\left. D_{z-b}^{an+\gamma}F(z)\right|_{z=z_0} = \frac{\Gamma(an+\gamma+1)}{2\pi i}\int_b^{(z_0^+)} \frac{F(\xi)\,d\xi}{(\xi-z_0)^{an+\gamma+1}}$$

yields at once

$$\begin{aligned} I = &\sum_{n=-N}^{N} \frac{D_{z-b}^{an+\gamma}[f(z)\theta'(z)((z-z_0)/\theta(z))^{an+\gamma+1}]|_{z=z_0}\,\theta(z)^{an+\gamma}}{\Gamma(an+\gamma+1)} \\ &+ \theta(z)^{\gamma}\frac{R_\varepsilon(N)+R_{-\delta}(N)}{2\pi i}. \end{aligned} \tag{4.4}$$

$R_\varepsilon(N)$ is the sum of three integrals, two over short line segments of length ε and one over the contour $C(|\theta(b + \varepsilon(b - z_0))|)$. Since the integrand contains the term $[\theta(z)/\theta(\xi)]^a$ (which has modulus less than 1 if z is on $C(|\theta(b)|)$) to the power $N + 1$, it is easy to see that for sufficiently small ε and large N, $R_\varepsilon(N)$ can be made arbitrarily small. A similar argument holds for $R_{-\delta}(N)$. Comparing (4.2), (4.3) and (4.4) we see that the theorem is proved.

If a is a natural number, the generalized Taylor's series sometimes converges in a region larger than that described in Theorem 4.1. This special case is examined in the following corollary.

COROLLARY 4.1. *Assume a is a natural number in the hypothesis of the previous theorem and that $f(z) = (z - b)^{\gamma+N}g(z)$, where $g(z)$ is analytic for z inside $C(p)$ and N is an integer. Then the generalized Taylor's series* (1.2) *converges not only for z on $C(|\theta(b)|)$, $\theta(z)^a \neq \theta(b)^a$, but also for all z in the ring-shaped region between $C(p)$ and $C(|\theta(b)|)$.*

Proof. The integrand of I in (4.1) is

$$F(\xi) = \frac{\theta(\xi)^{a-\gamma-1}(\xi - b)^{\gamma+N}g(\xi)}{\theta(\xi)^a - \theta(z)^a}.$$

Since $F(\xi)$ is analytic for ξ in the ring-shaped region between $C(p)$ and $C(|\theta(b)|)$, the two straight line segments of the contour C_ε cancel each other. Thus C_ε can be replaced by any contour $C(r)$ between $C(|\theta(b)|)$ and $C(p)$. Since z need no longer be on $C(|\theta(b)|)$ for $R_\varepsilon(N)$ to tend to zero in the proof of the previous theorem, the corollary is proved.

The generalized Taylor's series (1.2) involves a sum from $n = -\infty$ to ∞. Certain special cases of the sum over $n = 0$ to ∞ have appeared before [1, vol. 3, pp. 206–224]. In the following corollary we give a contour integral representation of this general sum.

COROLLARY 4.2. *With the hypothesis of Theorem* 4.1, *the formula*

$$\frac{\theta(z)^\gamma}{2\pi i}\int_C \frac{\theta(\xi)^{a-\gamma-1}\theta'(\xi)f(\xi)\,d\xi}{\theta(\xi)^a - \theta(z)^a}$$

$$= \sum_{n=0}^{\infty} \frac{D_{z-b}^{an+\gamma}[f(z)\theta'(z)((z - z_0)/\theta(z))^{an+\gamma+1}]|_{z=z_0}\,\theta(z)^{an+\gamma}}{\Gamma(an + \gamma + 1)} \tag{4.5}$$

is valid for all z inside the closed curve $C(|\theta(b)|)$. The contour of integration C starts at $\xi = b$, encloses the curve $C(|\theta(z)|)$ in the positive sense, and returns to $\xi = b$.

Proof. The corollary follows at once from the observation that the series (4.5) is generated by the integral (4.1) over the contour C_ε in the proof of Theorem 4.1.

We have seen that the contour integral definition of fractional differentiation (Definition 2.1) provides a convenient tool from which a proof of the generalized Taylor's series is constructed. In the next section this series is applied to the study of generating functions and other series expansions.

5. The discovery of generating functions and other examples. In this section we examine several examples of series which can be obtained from the generalized Taylor's series (1.2) by choosing specific functions for $f(z)$ and $\theta(z)$. A novel form of the binomial theorem is obtained as well as several generating functions for the special functions of mathematical physics. In fact, these examples reveal that the generalized Taylor's series is a very powerful tool for the discovery of generating functions when combined with the fractional derivative representations of the special functions such as those listed in Table 1.

In the examples which follow, ω and K are defined in the statement of Theorem 4.1, and the fractional derivatives encountered are computed with the aid of Table 1 and the extensive table in [2, vol. 2, pp. 185–200].

Example 1. Let $f(z) = z^p$, $\theta(z) = z - z_0$, and $b = 0$ in the generalized Taylor's series (1.2). We then obtain

$$\sum_{n=-\infty}^{\infty} \binom{p}{an+\gamma} t^{an+\gamma} = \begin{cases} a^{-1}(1+t)^p & \text{for } 0 < a \leqq 1, \\ a^{-1} \sum_{k\in K} \omega^{-\gamma k}(1+t\omega^k)^p & \text{for } 1 \leqq a \end{cases} \tag{5.1}$$

for $|t| = 1$, after making the substitution $t = (z - z_0)/z_0$. The special case in which $a = 1$ and $\gamma = 0$ is the familiar binomial expansion of $(1 + t)^p$. The case in which $a = 1$ and γ is arbitrary is an unusual form of the binomial theorem first stated by Riemann [12] and mentioned later by Heaviside [6, Chap. 7, 8], Watanabe [14] and Hardy [5]. The general case in which $0 < a \leqq 1$ and γ is arbitrary appears to be new.

Example 2. Let $f(z) = \sin\sqrt{z}$, $\theta(z) = z - z_0$, and $b = 0$ in (1.2). We then obtain the generating function

$$\sum_{n=-\infty}^{\infty} \frac{J_{1/2-an-\gamma}(x)t^{an+\gamma}}{\Gamma(an+\gamma+1)} = \begin{cases} a^{-1}\sqrt{2/(\pi x)}\sin\sqrt{x^2+2xt} & \text{for } 0 < a \leqq 1, \\ a^{-1}\sqrt{2/(\pi x)}\sum_{k\in K}\omega^{-\gamma k}\sin\sqrt{x^2+2xt\omega^k} & \text{for } 1 < a. \end{cases} \tag{5.2}$$

We have set $\sqrt{z_0} = x$ and $(z - z_0)/(2\sqrt{z_0}) = t$. The series (5.2) converges for $2|t| = |x|$. The special case of (5.2) in which $a = 1$ is well known [1, vol. 2, p. 100], while the general form for which $0 < a$ appears to be new.

Example 3. Let $f(z) = z^{(2\gamma-2\delta+1)/4}J_{1/2-\delta+\gamma}(\sqrt{z})$, $\theta(z) = z - z_0$, and $b = 0$ in (1.2). We then have

$$\sum_{n=-\infty}^{\infty} \frac{J_{1/2-\delta-an}(x)t^{an+\gamma}}{\Gamma(an+\gamma+1)} = \begin{cases} a^{-1}(1+2t/x)^{(2\gamma-2\delta+1)/4}J_{\gamma-\delta+1/2}(\sqrt{x^2+2tx}) & \text{for } 0 < a \leqq 1, \\ a^{-1}\sum_{k\in K}\omega^{-\gamma k}(1+2t\omega^k/x)^{(2\gamma-2\delta+1)/4}J_{\gamma-\delta+1/2}(\sqrt{x^2+2tx\omega^k}) & \text{for } 1 < a. \end{cases} \tag{5.3}$$

This series converges for $2|t| = |x|$. We have set $t = (z - z_0)/(2\sqrt{z_0})$, and $\sqrt{z_0} = x$. Equation (5.2) of the previous example is the special case of (5.3) in which $\delta = \gamma$. The special case of (5.3) in which $a = 1$ and $\gamma = 0$ is known as Lommel's formula [15, p. 140]. The general formula in which $0 < a$ appears to be new.

Example 4. In this example we take $\theta(z) = (z - 1)e^z$, so that we are using the Lagrange's expansion form of the generalized Taylor's series (1.2). Let $f(z) = e^{cz}z^{p-1}$ and $b = 0$. We then obtain

$$a^{-1}e^{cz}z^{p-1} = \sum_{n=-\infty}^{\infty} \binom{p}{an + \gamma} {}_1F_1(p + 1; p - an - \gamma + 1; \quad c - \gamma - an) \cdot ((z - 1)e^z)^{an+\gamma}$$

for $0 < a \leqq 1$ and $|(z - 1)e^z| = 1$.

Example 5 (The discovery of generating functions). Examples 2, 3 and 4 above show that generating functions for the special functions of mathematical physics can readily be obtained from the generalized Taylor's series by a simple substitution of the fractional derivative representations for these special functions. Using this method, the author, who is not familiar with the clever manipulations of series so often encountered in this subject, was able to derive every generating function for the Bessel functions listed in the standard reference [1, vol. 3, Chap. 19]. These were obtained in a few hours from the fractional derivative representations of the Bessel functions.

Further examples of the use of the generalized Taylor's series in finding generating functions are given in [9].

6. Acknowledgment. The author wishes to express his appreciation to Professor S. N. Karp of the Courant Institute of Mathematical Sciences at New York University for many hours of helpful discussion.

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SIAM J. MATH. ANAL.
Vol. 2, No. 1, February 1971

A NOTE ON THE SPECTRAL MAPPING THEOREM*

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Summary. Let A be a linear operator on a complex normed linear space (into itself). We denote by $N(A)$ and $R(A)$ the null-space and the range of A, respectively. The purpose of this paper is to show that

$$N\left(\prod_{i=1}^{s}(A-\alpha_i I)^{n_i}\right)=\sum_{i=1}^{s}\oplus N((A-\alpha_i I)^{n_i})\quad\text{(direct sum)}$$

and

$$R\left(\prod_{i=1}^{s}(A-\alpha_i I)^{n_i}\right)=\bigcap_{i=1}^{s}R((A-\alpha_i I)^{n_i}),$$

where the α_i are different complex numbers and I is the identity operator, and to note that some theorems in spectral theory follow directly from these formulas.

1. Results. Let A be a linear operator mapping a complex normed linear space into itself. We denote by A_λ the operator $A-\lambda I$, where λ is a complex number and I is the identity operator. $N(A)$ and $R(A)$ stand for the null-space and the range of A, respectively. We shall prove the following theorem.

THEOREM. *For s different complex numbers α_i and positive integers n_i, $1\leqq i\leqq s$, we have*

$$N(A_{\alpha_1}^{n_1}\cdots A_{\alpha_s}^{n_s})=N(A_{\alpha_1}^{n_1})\oplus\cdots\oplus N(A_{\alpha_s}^{n_s})\quad\textit{(direct sum)}$$

and

$$R(A_{\alpha_1}^{n_1}\cdots A_{\alpha_s}^{n_s})=R(A_{\alpha_1}^{n_1})\cap\cdots\cap R(A_{\alpha_s}^{n_s}).$$

Proof. It is sufficient to prove that $N(A_{\alpha_1}^{n_1}\cdots A_{\alpha_s}^{n_s})=N(A_{\alpha_1}^{n_1})\oplus N(A_{\alpha_2}^{n_2}\cdots A_{\alpha_s}^{n_s})$ and $R(A_{\alpha_1}^{n_1}\cdots A_{\alpha_s}^{n_s})=R(A_{\alpha_1}^{n_1})\cap R(A_{\alpha_2}^{n_2}\cdots A_{\alpha_s}^{n_s})$. For convenience we put $A_1=A_{\alpha_1}^{n_1}$, $A_2=A_{\alpha_2}^{n_2}\cdots A_{\alpha_s}^{n_s}$ and $N_i=N(A_i)$, $i=1,2$. We first show that $N(A_1A_2)=N_1+N_2$. Since the $\supseteqq$ part is clear, we have only to prove the $\subseteqq$ part. Let $x\in N(A_1A_2)$ and consider two polynomials $f_1(z)=(z-\alpha_1)^{n_1}$ and $f_2(z)=\prod_{i=2}^{s}(z-\alpha_i)^{n_i}$. Then $A_1x=f_1(A)x\in N_2$ and $A_2x=f_2(A)x\in N_1$. On the other hand, $f_1(z)$ and $f_2(z)$ are relatively prime and hence there exist two polynomials $h_i(z)$, $i=1,2$, such that $h_1(z)f_1(z)+h_2(z)f_2(z)=1$. Thus we have

$$x=h_1(A)f_1(A)x+h_2(A)f_2(A)x\in N_2+N_1.$$

At the same time the above argument shows that $N_1\cap N_2=\{0\}$ since, if $x\in N_1\cap N_2$, we have $f_1(A)x=f_2(A)x=0$. This implies that $N(A_1A_2)=N_1\oplus N_2$.

To prove the second assertion, let $x\in R(A_1)\cap R(A_2)$ and $x=A_1x_1=A_2x_2$ for some x_i, $i=1,2$. Then we have

$$\begin{aligned}x&=h_1(A)A_1x+h_2(A)A_2x\\&=h_1(A)A_1(A_2x_2)+h_2(A)A_2(A_1x_1)\\&=A_1A_2(h_1(A)x_2+h_2(A)x_1)\in R(A_1A_2)\end{aligned}$$

* Received by the editors February 12, 1970.

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since $h_1(A)$, $h_2(A)$, A_1 and A_2 commute. Hence we obtain $R(A_1A_2) \supseteqq R(A_1) \cap R(A_2)$. Since the $\subseteqq$ part is obvious, the proof is complete.

Remark. The first assertion of the theorem is stated in Zaanen's book [3, Chap. 11, § 6, Theorem 2] under the assumption that A^p is compact for some p. However, our proof is direct and contains his theorem as a special case.

2. Corollaries. We shall show that some theorems in spectral theory follow directly from the above results. Before doing this, some preliminaries are necessary. In the following we assume that A is bounded. The algebraic multiplicity of $\lambda \in P\sigma(A)$ (the point spectrum of A) is defined as the dimension of the space $\bigcup_{n=1}^{\infty} N(A_\lambda^n)$, while the geometric multiplicity of λ is defined as the dimension of $N(A_\lambda)$. If there exists a positive integer m such that $N(A_\lambda^{m-1}) \subsetneqq N(A_\lambda^m) = N(A_\lambda^{m+1})$, m is called the index of λ for A. We define the index of $\alpha \notin P\sigma(A)$ for A to be zero. It is well known that, if A is compact, each nonzero eigenvalue of A has a finite index and finite algebraic multiplicity. Now we have the following corollary.

COROLLARY 1 (The (point) spectral mapping theorem). *Let $f(z)$ be any polynomial with complex coefficients. Then:*

(i) $P\sigma(f(A)) = f(P\sigma(A))$.

(ii) *The algebraic multiplicity of $\mu \in P\sigma(f(A))$ is equal to the sum of the algebraic multiplicities of the λ_i such that $\lambda_i \in P\sigma(A)$ and $f(\lambda_i) = \mu$. Here we admit an equality $\infty = \infty$.*

(iii) *If A^p is compact for some positive integer p, the algebraic multiplicity of any $\lambda(\neq 0) \in P\sigma(A)$ is finite. Hence, in this case, the algebraic multiplicity of $\mu(\neq f(0)) \in P\sigma(f(A))$ is finite.*

(iv) *If A^p is compact for some p, the geometric multiplicity of $\mu(\neq f(0)) \in P\sigma(f(A))$ is equal to the sum of the geometric multiplicities of the λ_i such that $\lambda_i \in P\sigma(A)$ and $f(\lambda_i) = \mu$ if and only if, for each i, λ_i has index one for A or λ_i is a simple root of the equation $f(z) = \mu$.*

Proof. Let $\mu \in P\sigma(f(A))$ and $f(z) - \mu = \prod_{i=1}^{s} (z - \alpha_i)^{n_i}$. Then the assertions are immediate consequences of the theorem in § 1. For example,

$$\begin{aligned} \mu \in P\sigma(f(A)) &\Leftrightarrow 1 \leqq \dim N(f(A) - \mu I) = \sum_{i=1}^{s} \dim N((A - \alpha_i I)^{n_i}) \\ &\Leftrightarrow \dim N((A - \alpha_i I)^{n_i}) \geqq 1 \quad \text{for some } i \\ &\Leftrightarrow \dim N(A - \alpha_i I) \geqq 1 \quad \text{for some } i \\ &\Leftrightarrow \alpha_i \in P\sigma(A) \quad \text{for some } i. \end{aligned}$$

This proves (i). The remaining parts can be proved along similar lines.

Remark. An elegant proof for (i) may be found in [1]. It is not clear, however, whether this proof works in the case of eigenvalues of (algebraic or geometric) multiplicity $\geqq 2$. Our results clarify this point. Theorem 1 in [3, Chap. 11, § 3] and Theorem 3 in [3, Chap. 11, § 6] are now special cases of Corollary 1.

COROLLARY 2 [3, Chap. 11, § 6, Theorem 1]. *Let A^p be compact for some p. If $\alpha_1, \cdots, \alpha_k$ are different complex numbers, all $\neq 0$, with indices $m_1, \cdots, m_k$ for A, then, putting $A_i = A_{\alpha_i}^{m_i}$, we have that the whole space X is the direct sum of $N(A_1), \cdots, N(A_k)$ and $\bigcap_{i=1}^{k} R(A_i)$.*

Proof. If $m_1 = \cdots = m_k = 0$, the assertion is clear. Thus we may assume that $m = \max_{1 \leqq i \leqq k} m_i > 0$. Then there exists a (unique) polynomial $f(z)$ such that $f(0) = 0$ and $f(A) - \mu I = A_1 A_2 \cdots A_k$ for some complex number $\mu \neq 0$. Then, $f(A)^p$ is compact and μ has index one for $f(A)$. Hence, by the theorem, we have

$$\begin{aligned} X &= N(f(A) - \mu I) \oplus R(f(A) - \mu I) \quad [3, \text{Chap. } 11, \S\, 3, \text{Theorem } 8] \\ &= N(A_1 \cdots A_k) \oplus R(A_1 \cdots A_k) \\ &= N(A_1) \oplus \cdots \oplus N(A_k) \oplus \bigcap_{i=1}^{k} R(A_i). \end{aligned}$$

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SIAM J. MATH. ANAL.
Vol. 2, No. 1, February 1971

A GENERALIZED WEIERSTRASS TRANSFORMATION FOR THE CASE OF SEVERAL INDEPENDENT VARIABLES*

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Abstract. The ordinary Weierstrass transformation is extended to a class of generalized functions of n independent variables as follows. A testing function space η_μ is constructed, which is a countably normed space and contains as a member the Weierstrass kernel, $K(x - \tau, 1)$, considered as a function of τ. Then, the generalized Weierstrass transform $F(s)$ of any member of the dual space η'_μ is obtained by applying f to the kernel function: $F(s) = \langle f(\tau), K(x - \tau, 1)\rangle$. Next, a theorem is given concerning the convergence behavior of the one-dimensional inversion formulas of P. G. Rooney for the ordinary transformation. On the basis of this result we are able to extend these inversion formulas to the one-dimensional generalized transformation and then construct an inversion formula for the n-dimensional case. Finally, an application to the heat equation for an n-dimensional medium is given.

1. Introduction. The Weierstrass transformation $F(x)$ of a suitably restricted function $f(\tau)$ is given by

$$F(x) = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{+\infty} f(\tau) \exp[-(x - \tau)^2/4]\, dt \tag{1}$$

[4, Chap. VIII]. Recently, the transformation (1) was extended by Zemanian to a certain class of generalized functions [3] and an inversion theory presented based on the Hirschman–Widder formula [4]. In this paper we extend the transformation to a class of generalized functions which is smaller than that studied in [3], but for which we shall be able to prove additional results. In particular, we extend the transformation to generalized functions of n independent variables and develop an inversion theory based on the series representation due to Rooney [1], [2]:

$$f(x) = \sum_{\rho=0}^{\infty} \frac{(-1)^\rho H_\rho(z/2)}{\rho!} F^{(\rho)}(x + z), \tag{2}$$

where z is any fixed real number, $F^{(\rho)}(x) = d^\rho F/dx^\rho$, and H_ρ is the Hermite polynomial of degree ρ.

The theory of the Weierstrass transformation is closely related to the solution of the heat equation. Generally, these problems involve several independent variables and require a multidimensional analysis. The Weierstrass transform, however, has been studied extensively for the case of one independent variable, but little has been done regarding higher dimensions. In this paper we develop a transformation theory for generalized functions of several variables and construct a corresponding inversion formula based on the Abel sums of (2), which is valid when convergence is taken in the sense of weak convergence in Schwartz's space $\mathscr{D}'$. Furthermore, it is shown that in the case of one independent variable, the ordinary sum of (2) also inverts our generalized transform, convergence again being taken in $\mathscr{D}'$. Finally, an application to the heat equation for an n-dimensional medium is given.

As in [3], our procedure for extending (1) to generalized functions is to construct a testing function space η_μ, which is a countably normed space and contains

* Received by the editors December 30, 1969 and in revised form May 4, 1970.

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as a member the kernel

$$\frac{1}{(\sqrt{4\pi})^n} \exp\left[-|x-\tau|^2/4\right]$$

considered as a function of τ. Then the Weierstrass transform $F(x)$ of any member f of the dual space η'_μ is obtained by applying f to the kernel function:

$$F(x) = \frac{1}{(\sqrt{4\pi})^n} \langle f(\tau), \exp\left[-|x-\tau|^2/4\right] \rangle.$$

We shall make use of the following notation. R^n and C^n are respectively the real and complex n-dimensional Euclidean spaces. m will always denote a nonnegative integer in R^n while $|m|$ is used to signify the sum $m_1 + m_2 + \cdots + m_n$ and should not be confused with the standard "magnitude symbol" which will be applied to elements of R^n and C^n. Also, $D_x^m g(x) = g^{(m)}(x) = (\partial^{|m|}/\partial x_1^{m_1} \cdots \partial x_n^{m_n}) g(x)$ while $D_{x_i}^{m_i}$ is just $\partial^{m_i}/\partial x_i$. In the following, all symbols should be interpreted in their n-dimensional sense, except where indicated otherwise.

2. The testing function space η_μ. Let μ be a fixed positive number in R^1. We define η_μ as the linear space of all smooth functions ϕ (i.e., having continuous derivatives of all orders everywhere) from R^n into C^1 which are such that for each fixed m,

$$\gamma_m(\phi) = \gamma_{n,m,\mu}(\phi) = \sup_{x \in R^n} \left| \exp\left[\frac{|x|^2}{8+\mu}\right] D_x^m \phi(x) \right| < \infty.$$

We assign to η_μ the topology generated by the set of seminorms $\{\gamma_m\}_{0 \leqq |m| < \infty}$. It is easily verified that η_μ is sequentially complete and that differentiation is a continuous linear mapping of η_μ into itself. η'_μ denotes the dual of η_μ. It is a linear space under the customary definitions of addition and multiplication by a complex number. By a standard result from the theory of topological spaces [6, pp. 12–13], η'_μ is also sequentially complete. For $f \in \eta'_\mu$ and $\phi \in \eta_\mu$, we denote the number that f assigns to ϕ by $\langle f, \phi \rangle$. The generalized derivative $D^m f$ of f is defined by $\langle D^m f, \phi \rangle = \langle f, (-1)^m D^m \phi \rangle$. It now follows that differentiation is a continuous linear mapping of η'_μ into itself.

We now list some other readily established properties of η_μ and η'_μ.

(I) Let $\mathscr{D}$ denote Schwartz's space of all smooth functions having compact support and let $\mathscr{D}'$ be its dual [5]. Then $\mathscr{D} \subset \eta_\mu$ and the topology of $\mathscr{D}$ is stronger than that induced on it by η_μ. Consequently, the restriction of $f \in \eta'_\mu$ to $\mathscr{D}$ is in $\mathscr{D}'$.

(II) Set $k(x_i - \tau_i, t) = (4\pi)^{-1/2} \exp[-(x_i - \tau_i)^2/4t]$ and $K(x - \tau, t) = \prod_{i=1}^n k(x_i - \tau_i, t)$. Then for every fixed $x \in R^n$ and $t \in R^1$, where $0 < t \leqq 1$, $K(x - \tau, t)$ considered as a function of τ is in η_μ.

(III) A locally Lebesgue-integrable function $f(\tau)$ which is such that $f(\tau) \exp[-\tau^2/(8+\mu)]$ is absolutely integrable on R^n, generates a member of η'_μ through the usual definition

$$\langle f, \phi \rangle = \int_{R^n} f(\tau)\phi(\tau)\, d\tau, \quad \phi \in \eta_\mu.$$

(IV) Let $f \in \eta'_\mu$ and $u(x, t) = \langle f(\tau), K(x - \tau, t) \rangle$, $0 < t \leqq 1$. Then $D_x^m u(x, t) = \langle f(\tau), D_x^m K(x - \tau, t) \rangle$.

(V) Let $f \in \eta'_\mu$ and $\psi(x)$ be a smooth function over the finite region $\Omega \subset R^n$. Also, let $\theta(\tau, x)$ be a smooth function for $x \in \Omega$, $\tau \in R^n$, such that for each m,

$$\lim_{|\tau| \to \infty} \exp\left[\frac{|\tau|^2}{8+\mu}\right] D_\tau^m \theta(\tau, x) = 0$$

uniformly for $x \in \Omega$. Then

$$\int_\Omega \psi(x) \langle f(\tau), \theta(\tau, x)\rangle \, dx = \langle f(\tau), \int_\Omega \psi(x)\theta(\tau, x)\, dx\rangle .$$

Notes (IV) and (V) can be justified using arguments similar to those found in [3, Theorem 3.1, Lemma 2.1]. For brevity we omit the details here.

3. An inversion formula for the case of one independent variable. Before we can turn to the development of an analogue expression to (2) for the generalized transformation, it is necessary to establish the following convergence property of the series (2). In this section x, z and τ are one-dimensional variables only.

LEMMA 3.1. *Let $f(\tau)$ be a locally Lebesgue-integrable function which is such that $f(\tau)/\exp[\tau^2/(8+\mu)]$ is absolutely integrable on $-\infty < \tau < \infty$. Also, let $F(x)$ be defined as in* (1). *Then, for each fixed real number z and for $N \to \infty$,*

$$\sum_{\rho=0}^{N} \frac{(-1)^\rho H_\rho(z/2)}{\rho!} F^{(\rho)}(x+z) \tag{3}$$

and

$$\frac{1}{\pi}\int_{x-\delta}^{x+\delta} \frac{\sin[\sqrt{N/2}(x-\tau)]}{x-\tau} f(\tau)\, d\tau \tag{4}$$

are uniformly equiconvergent on every compact subset Ω of the real line $-\infty < x < \infty$.

Rooney has shown that (3) and (4) are equiconvergent for each x and z. In the above lemma we extend this result to uniform equiconvergence on finite intervals in x.

Proof. According to [2, pp. 48],

$$\begin{aligned} &\sum_{\rho=0}^{N} \frac{(-1)^\rho H_\rho(z/2)}{\rho!} F^{(\rho)}(x+z) \\ &\quad = \frac{1}{2}\int_{-\infty}^{+\infty} \exp\left[\frac{-(x+z-\tau)^2}{4}\right] K_N[(x+z-\tau)/2, z/2] f(\tau)\, d\tau, \end{aligned} \tag{5}$$

where

$$K_N(x, y) = \sum_{\rho=0}^{N} \frac{H_\rho(x) H_\rho(y)}{2^\rho \rho! \sqrt{\pi}},$$

the kernel for the Hermite series.

Let us replace $f(x)$ by a polynomial $p(x)$ in the theorem. Then (4) converges uniformly to $p(x)$ as $N \to \infty$ on every compact set Ω [7, pp. 127–130]. Furthermore (3) is identically equal to $p(x)$ if N exceeds the degree of $p(x)$. Indeed, if we set

$f(\tau) = \tau^\alpha$ in (5), where α is a nonnegative integer, make the change in variable $\tau = x + z - 2v$, and employ the binomial expansion for $(x + z - 2v)^\alpha$, (5) becomes

$$\sum_{q=0}^{\alpha} \binom{\alpha}{q} (x + z)^{\alpha - q} \int_{-\infty}^{\infty} (-2v)^q e^{-v^2} K_N(v, z/2)\, dv.$$

By expanding $(-2v)^q$ into a Hermite series and invoking [8, pp. 193, (7)], we find the last expression to be identically equal to x^α for all $N > \alpha$.

Now the integral

$$\int_{-\infty}^{\infty} |f(\tau) - p(\tau)| \exp\left[\frac{-\tau^2}{8 + \mu}\right] d\tau \tag{6}$$

can be made arbitrarily small by a proper choice of $p(\tau)$ [9, Theorem 5.7.2]. It is therefore sufficient to show that the difference between (3) and (4) is bounded for each fixed z by

$$M \int_{-\infty}^{\infty} |f(\tau)| \exp\left[\frac{-\tau^2}{8 + \mu}\right] d\tau,$$

where M is a constant independent of all N and $x \in \Omega$. For, if we let $I_1(f)$ denote the difference between (3) and (4) and choose $p(\tau)$ so that (6) is less than $\varepsilon/2$ for any given $\varepsilon > 0$, we may then choose N so large that $|I_1(p)| < \varepsilon/2$ and therefore

$$|I_1(f)| \leqq |I_1(f - p)| + |I_1(p)| < \varepsilon$$

uniformly for all $x \in \Omega$.

Let us set

$$I_2(f) = \int_{x-\delta}^{x+\delta} \frac{1}{2} \exp\left[\frac{-(x + z - \tau)^2}{4}\right] K_N\left(\frac{x + z - \tau}{2}, \frac{z}{2}\right) - \frac{1}{\pi} \frac{\sin[\sqrt{N/2}(x - \tau)]}{x - \tau} f(\tau)\, d\tau$$

and

$$I_3(f) = \left\{\int_{-\infty}^{x-\delta} + \int_{x+\delta}^{\infty}\right\} \frac{1}{2} \exp\left[\frac{-(x + z - \tau)^2}{4}\right] K_N\left(\frac{x + z - \tau}{2}, \frac{z}{2}\right) f(\tau)\, d\tau.$$

Then $I_1(f) = I_2(f) + I_3(f)$. Now, for each fixed z, $x \in \Omega$, and τ restricted to a finite interval, there exists a constant C independent of x, τ and N such that [9, 9.5.24]

$$\frac{1}{2} \exp\left[\frac{-(x + z - \tau)^2}{4}\right] K_N\left(\frac{x + z - \tau}{2}, \frac{z}{2}\right) - \frac{1}{\pi} \frac{\sin[\sqrt{N/2}(x - \tau)]}{x - \tau} \leqq C.$$

Therefore,

$$|I_2(f)| \leqq C \int_{x-\delta}^{x+\delta} |f(\tau)|\, d\tau \leqq Q \int_{x-\delta}^{x+\delta} |f(\tau)| \exp\left[\frac{-\tau^2}{8 + \mu}\right] d\tau, \tag{7}$$

where

$$Q = C \sup_{x\in\Omega} \left\{ \sup_{x-\delta<\tau<x+\delta} \exp\left[\frac{\tau^2}{8+\mu}\right]\right\}.$$

According to [9, Theorem 8.91.3], for any fixed real number $a > 0$ and $|x| \geqq a$,

$$|x^{-1}H_N(x)| < B(2^N N!)^{1/2} N^{-1/4} e^{x^2/2}, \tag{8}$$

where B is a constant independent of x and N. Applying (8) to the Christoffel–Darboux formula [10, vol. 11, 10.13 (11)], we obtain

$$|K_N(x, y)| < \frac{2B^2|xy|}{\sqrt{\pi}|x - y|} \exp\frac{(x^2 + y^2)}{2} \tag{9}$$

for $|x| \geqq a$, $|y| \geqq a$, $a > 0$ and $N \geqq 1$.

On the other hand, let us consider the case where $y = 0$. Since $H_{2N+1}(0) = 0$ and $H_{2N}(0) = (-1)^N(2N)!/N!$ [10, vol. II, 10.13 (15)], it follows that

$$K_{2N}(x, 0) = K_{2N+1}(x, 0) = \frac{(-1)^N H_{2N+1}(x)}{\sqrt{\pi}2^{2N+1}N!x}.$$

Therefore, by (8) and Legendre's duplication formula [10, vol. I, 1.2 (11)] we have that

$$\begin{aligned}|K_{2N}(x, 0)| &= |K_{2N+1}(x, 0)| \\ &\leqq \frac{B}{\pi^{3/4}}\left[\frac{\Gamma(N + 3/2)}{N!(2N + 1)^{1/2}}\right]^{1/2} e^{x^2/2}, \quad |x| \geqq a.\end{aligned}$$

According to Stirling's formula, $\Gamma(N + 3/2)/N! = O(\sqrt{N})$ as $N \to \infty$, and the right-hand side of the last inequality is therefore bounded by a constant multiple of $e^{x^2/2}$ for all N. By this result and also (9), we conclude that for each fixed y and as $N \to \infty$,

$$K_N(x, y) = O(|x|e^{x^2/2}) \tag{10}$$

uniformly for all x such that $|x - y| \geqq a$, $a > 0$.

If z is fixed and A denotes the maximum value of $|x + z|$, $x \in \Omega$, then as $N \to \infty$,

$$\begin{aligned}\exp\left[\frac{-(x + z - \tau)^2}{4}\right] K_N\left(\frac{x + z - \tau}{2}, \frac{z}{2}\right) &= O\left(\exp\frac{-(x + z - \tau)^2}{8}\right) \\ &= O\left(\exp\left[\frac{A\tau}{4} - \frac{\tau^2}{8}\right]\right) = O\left(\exp\frac{-\tau^2}{8 + \mu}\right)\end{aligned}$$

uniformly for all $x \in \Omega$ and $\tau \in (-\infty, x - \delta] \cup [x + \delta, \infty)$, where δ is chosen greater than the length of some fixed interval containing Ω. Hence, there exists a constant E independent of $x \in \Omega$ such that

$$|I_3(f)| \leqq E\left\{\int_{-\infty}^{x-\delta} + \int_{x+\delta}^{\infty}\right\} |f(\tau)| \exp\left[\frac{-\tau^2}{8 + \mu}\right] d\tau.$$

Setting $M = \max(Q, E)$, we conclude that

$$|I_1(f)| \leqq |I_2(f)| + |I_3(f)| \leqq M \int_{-\infty}^{\infty} |f(\tau)| \exp\left[\frac{-\tau^2}{8+\mu}\right] d\tau$$

for all $x \in \Omega$.

We shall need the following result which makes use of Lemma 3.1.

LEMMA 3.2. *Let $\phi \in \mathscr{D}$ and $\Phi(x) = \langle \phi(x), k(x - \tau, 1)\rangle$. Then for each fixed z,*

$$\sum_{\rho=0}^{\infty} \frac{(-1)^\rho H_\rho(z/2)}{\rho!} \Phi^{(\rho)}(x + z) = \phi(x),$$

where the series converges in η_μ.

Proof. Using (IV) and integrating by parts m times, we have

$$D_x^{m+\rho}\Phi(x) = \langle \phi^{(m)}(\tau), D_x^\rho k(x - \tau, 1)\rangle$$

Hence the lemma will be proved if we show that for each fixed m and z, and for any $\varepsilon > 0$, there exist numbers T_ε and N_ε such that

$$\exp\left[\frac{x^2}{8+\mu}\right]\left|\phi^{(m)}(x) - \sum^{N} \frac{(-1)^\rho H_\rho(z/2)}{p!} \int_{-\infty}^{+\infty} \phi^{(m)}(\tau) D_x^\rho k(x + z - \tau, 1)\, d\tau\right| < \varepsilon \tag{11}$$

uniformly in x on each of the intervals $T_\varepsilon < |x| < \infty$ and $|x| \leqq T_\varepsilon$, for all $N > N_\varepsilon$.

Consider $T_\varepsilon < |x| < \infty$. We are free to choose T_ε so large that $|x - \tau| \geqq a > 0$ for all $\tau \in \operatorname{supp} \phi$. Then, $\phi^{(m)}(x) \equiv 0$ on $T_\varepsilon < |x| < \infty$. Using (5), we may rewrite (11) as

$$\exp\left[\frac{x^2}{8+\mu}\right]\left|\frac{1}{2}\int_{-\infty}^{+\infty} \phi^{(m)}(\tau) \exp\left[\frac{-(x + z - \tau)^2}{4}\right] \cdot K_N((x + z - \tau)/2, z/2)\, d\tau\right|. \tag{12}$$

According to (10), for each fixed z,

$$K_N((x + z - \tau)/2, z/2) = O\left(|x| \exp\frac{(x + 2 - \tau)^2}{8}\right)$$

as $N \to \infty$ uniformly for all $\tau \in \operatorname{supp} \phi$ and $|x| \in [T_\varepsilon, \infty]$. Hence for each m and z, (12) is uniformly bounded for all $|x| \in [T_\varepsilon, \infty]$ by

$$O\left(|x| \exp\left[\frac{-\mu x^2}{8(8+\mu)} + \frac{A|x|}{4}\right]\right), \tag{13}$$

where A denotes the maximum value of $|\tau - z|$, $\tau \in \operatorname{supp} \phi$. The expression (13) tends to zero as $|x| \to \infty$; therefore we may choose T_ε so large that (13), and consequently (12), will be uniformly bounded by ε for all $|x| \in [T_\varepsilon, \infty]$ and all N.

Next consider $|x| \leqq T_\varepsilon$. If we replace f by $\phi^{(m)} \in \mathscr{D}$ in the hypothesis of Lemma 3.1, then by a well-known result [7, pp. 127–130] the Fourier integral (4) converges uniformly to $\phi^{(m)}$ for each m and all $x \in \Omega$. Therefore we may choose N_ε so large that (12) can be made arbitrarily small for all $N > N_\varepsilon$. This completes the proof of Lemma 3.2.

We now are prepared to establish an inversion formula for the one-dimensional generalized transform.

THEOREM 3.1. *Let* $f \in \eta'_\mu$ *and* $F(x) = \langle f(\tau), k(x - \tau, 1)\rangle$; *then for any real number* z,

$$\sum_{\rho=0}^{\infty} \frac{(-1)^\rho H_\rho(z/2)}{\rho!} F^{(\rho)}(x + z) = f(x),$$

where the series converges in the sense of weak convergence in $\mathscr{D}'$.

Proof. Let $\phi \in \mathscr{D}$; then

$$\begin{aligned}
&\left\langle \frac{(-1)^\rho H_\rho(z/2)}{\rho!} F^{(\rho)}(x + z), \phi(x)\right\rangle \\
(14) \qquad &= \left\langle F(x), \frac{H_\rho(z/2)}{\rho!} \phi^{(\rho)}(x - z)\right\rangle \\
&= \left\langle \langle f(\tau), k(x - \tau, 1)\rangle_\tau, \frac{H_\rho(z/2)}{\rho!} \phi^{(\rho)}(x - z)\right\rangle_x .
\end{aligned}$$

(The notation $\langle \cdot, \cdot \rangle_x$ indicates that x is the independent variable for the quantities in the inner product.) Choose the real numbers A and B such that the support of $\phi(x - z)$ is contained in $A < x < B$. Then, the inner product on x in (14) can be rewritten as an integral on $A \leqq x \leqq B$. Furthermore, $\phi^{(\rho)}(x - z)$ is smooth on $A \leqq x \leqq B$, and for every m, $\exp[x^2/(8 + \mu)]D_x^m k(x - \tau, 1) \to 0$ as $|x| \to \infty$, uniformly for $A \leqq x \leqq B$. In view of note (V) we may interchange the order of inner products on x and τ in (14) to obtain

$$\left\langle f(\tau), \frac{H_\rho(z/2)}{\rho!} \langle \phi^{(\rho)}(x - z), k(x - \tau, 1)\rangle_x \right\rangle_\tau .$$

Consequently, we may write that

$$\begin{aligned}
&\sum_{\rho=0}^{N} \left\langle \frac{(-1)^\rho H_\rho(z/2)}{\rho!} F^{(\rho)}(x + z), \phi(x)\right\rangle \\
(15) \qquad &= \left\langle f(\tau), \sum_{\rho=0}^{N} \frac{(-1)^\rho H_\rho(-z/2)}{\rho!} \Phi^{(\rho)}(\tau - z)\right\rangle ,
\end{aligned}$$

which by Lemma 3.2 converges to $\langle f(\tau), \phi(\tau)\rangle$ as $N \to \infty$.

4. An inversion formula for the case of n independent variables. In this section we construct an inversion formula for multidimensional transformation by taking the Abel sum of (2). First we prove the following lemma.

LEMMA 4.1. *For each fixed* x_i, z_i, *and* t, $0 < t < 1$, *the series*

$$(16) \qquad \sum_{\rho=0}^{\infty} \exp\left[\frac{-(x_i + z_i\sqrt{t} - \tau_i)^2}{4}\right] \frac{H_\rho((x_i + z_i\sqrt{t} - \tau_i)/2) H_\rho(z_i/2) t^{\rho/2}}{2^\rho \rho! \sqrt{4\pi}}$$

considered as a function of τ_i, *converges in* $\eta_\mu(R^1)$ *to*

$$(17) \qquad k(x_i - \tau_i, 1 - t) = [4\pi(1 - t)]^{-1/2} \exp\left[\frac{-(x_i - \tau_i)^2}{4(1 - t)}\right].$$

Proof. It is a well-known result [10, vol. II, 10.13 (22)] that the series (16) converges pointwise, in the usual sense, to (17). Hence we need merely show that the partial sums of (16) form a convergent sequence in $\eta_\mu(R^1)$. If we differentiate term by term the Nth partial sum of the series (16), and multiply the result by $\exp[\tau_i^2/(8+\mu)]$, we obtain

$$\exp\left[\frac{\tau_i^2}{8+\mu}\right]\sum_{\rho=0}^{N}\exp\left[\frac{-(y_i-\tau_i)^2}{4}\right]\frac{H_{\rho+m}((y_i-\tau_i)/2)H_\rho(z_i/2)t^{\rho/2}}{2^{\rho+m}\rho!\sqrt{4\pi}}, \tag{18}$$

where we have set $y_i = x_i + z_i\sqrt{t}$ for simplicity. Since $|H_\rho(x)| < B2^{\rho/2}\sqrt{\rho!}e^{x^2/2}$, where B is a constant [7, p. 324], expression (18) is no greater than

$$\exp\left[\frac{\tau_i^2}{8+\mu}-\frac{(y_i-\tau_i)^2-z_i^2}{8}\right]\frac{B}{\sqrt{4\pi}2^{m/2}}\sum_{\rho=0}^{N}[(\rho+m)(\rho+m-1)\cdots(\rho+1)]^{1/2}t^{\rho/2}, \tag{19}$$

the exponential can be bounded by a constant C which is independent of τ_i, $-\infty < \tau_i < \infty$. Expression (19) is therefore uniformly bounded as $N \to \infty$ by

$$\frac{B^2C}{\sqrt{4\pi}2^{m/2}}\sum_{\rho=0}^{\infty}\rho^{m/2}t^{\rho/2},$$

which converges for each fixed t, $0 < t < 1$, and the result follows directly.

LEMMA 4.2. *Let*

$$\left\{\{\psi_{\rho_i}\}_{\rho_i=1}^{\infty}\right\}_{i=1}^{n}$$

be a set of n convergent sequences in $\eta_\mu(R^1)$, *and* $\{\psi_i\}_{i=1}^n$ *denote their respective limits. If we set*

$$\Psi_N(\tau) = \prod_{i=1}^{n}\psi_{N_i}(\tau_i)$$

and

$$\Psi(\tau) = \prod_{i=1}^{n}\psi(\tau_i),$$

where N_i *denotes the ith integer component of the integer* $N \in R^n$, *then* $\Psi_N(\tau) \to \Psi(\tau)$ *in* $\eta_\mu(R^n)$ *as* $N \to \infty$. *Here* $N \to \infty$ *means* $N_1, \cdots, N_n$ *all tend to infinitely independently.*

Proof. Briefly, this result can be established by expressing the difference $\Psi(\tau) - \Psi_N(\tau)$ as

$$\Psi(\tau) - \Psi_N(\tau) = \sum_{j=1}^{n}[\psi(\tau_j) - \psi_{N_j}(\tau_j)]\left\{\prod_{i=1}^{j-1}\psi_{N_i}(\tau_i)\right\}\left\{\prod_{k=j+1}^{n}\psi(\tau_k)\right\}$$

and verifying that

$$\gamma_m\{\Psi(\tau) - \Psi_N(\tau)\} \leqq \sum_{j=1}^{n}\gamma_{m_j}\{\Psi(\tau_j) - \Psi_{N_j}(\tau_j)\}\left\{\prod_{i=1}^{j-1}\gamma_{m_i}\{\Psi_{N_i}(\tau_i)\}\right\}\left\{\prod_{k=j+1}^{n}\gamma_{m_k}\{\Psi(\tau_k)\}\right\},$$

where γ_{m_j} denotes the seminorms for $\eta_\mu(R^1)$. The result then follows immediately.

In the following, f and ϕ are functions of n independent variables; i.e., x, z and $\tau \in R^n$. Moreover m and N will represent nonnegative integers in R^n while ρ is an integer in R^1. We define the following differential operator of infinite order:

$$M_{x,z,t} \triangleq \lim_{N\to\infty} \prod_{i=1}^{n} \left(\sum_{\rho=0}^{N_i} \frac{t^{\rho/2}(-1)^\rho H_\rho(z_i/2)}{\rho!} D_{x_i}^\rho \right). \tag{20}$$

THEOREM 4.1. *Let $f \in \eta'_\mu$ and $F(x) = \langle f(\tau), K(x-\tau, 1)\rangle$; then for each fixed z,*

$$f(x) = \lim_{t\to 1-} M_{x,z,t} F(x + z\sqrt{t})$$

in the sense of weak convergence in $\mathscr{D}'$.

Proof. Let $M_{x,z,t}^N$ denote the Nth partial sums of the operator (20); that is, $M_{x,z,t}F(x) = \lim_{N\to\infty} M_{x,z,t}^N F(x)$. Now

$$\begin{aligned} M_{x,z,t}^N F(x + z\sqrt{t}) &= \langle f(\tau), M_{x,z,t}^N K(x + z\sqrt{t} - \tau, 1)\rangle \\ &= \left\langle f(\tau), \prod_{i=1}^{n} \left(\sum_{\rho=0}^{N_i} \frac{t^{\rho/2}(-1)^\rho H_\rho(z_i/2)}{\rho!} D_{x_i}^\rho k(x_i + z_i\sqrt{t} - \tau_i, 1)\right)\right\rangle . \end{aligned} \tag{21}$$

The summation in (21) is equivalent to the N_i-th partial sum of (16) and therefore by Lemma 4.1 converges in $\eta_\mu(R^1)$ to $k(x_i - \tau_i, 1-t)$ for each $i = 1, 2, \cdots, n$. According to Lemma 4.2, the product in (21) must converge in $\eta_\mu(R^n)$ to

$$\prod_{i=1}^{n} k(x_i - \tau_i, 1 - t).$$

Therefore,

$$\lim_{N\to\infty} M_{x,z,t}^N F(x + z\sqrt{t}) = \langle f(\tau), K(x - \tau, 1 - t)\rangle. \tag{22}$$

To complete the proof of Theorem 4.1, we must show that the right-hand side of (22) converges weakly in $\mathscr{D}'$ to $f(x)$ as $t \to 1-$, that is,

$$\lim_{t\to 1-} \langle\langle f(\tau), K(x-\tau, 1-t)\rangle_\tau, \phi(x)\rangle_x = \langle f(x), \phi(x)\rangle \tag{23}$$

for every $\phi \in \mathscr{D}$. Noting that $\exp[|\tau|^2/(8+\mu)]D_\tau^m K(x-\tau, 1-t)$ tends to zero uniformly on $x \in \Omega \supset \operatorname{supp}\phi$ as $|\tau| \to \infty$, we may again use note (V) in § 2 to interchange the order of inner products in (23) to obtain

$$\langle f(\tau), \langle \phi(x), K(x-\tau, 1-t)\rangle_x\rangle_\tau.$$

As the last step, we must prove that, as $t \to 1-$, $\langle \phi(x), K(x-\tau, 1-t)\rangle \to \phi(\tau)$ in η_μ. The following argument is similar to one given in [3].

Set $x = \tau + 2y(1-t)^{1/2}$, where τ and t are fixed. Then

$$[(4\pi(1-t)]^{-n/2} \int_{x\in R^n} \exp\left[\frac{|x-\tau|^2}{4(t-1)}\right] dx = \prod_{i=1}^{n} \left(\pi^{-1/2} \int_{-\infty}^{+\infty} e^{-y_i^2}\, dy_i\right) = 1. \tag{24}$$

Therefore, we can set

$$\exp\left[\frac{|\tau|^2}{8+\mu}\right] D_\tau^m \{\langle \phi(x), K(x-\tau, 1-t)\rangle - \phi(\tau)\} = I_1 + I_2, \tag{25}$$

where

$$I_l(\tau) = [4\pi(1-t)]^{-n/2} \exp\left[\frac{|\tau|^2}{8+\mu}\right] \int_{\Delta_l} [\phi^{(m)}(x) - \phi^{(m)}(\tau)] \exp\left[\frac{|x-\tau|^2}{4(t-1)}\right] dx,$$

$$l = 1, 2,$$

and

$$\Delta_1 = \{x : |x - \tau| \leqq \delta, \delta \in R^1 : 0 < \delta < 1\},$$

$$\Delta_2 = R^n - \Delta_1 \quad \text{(that is, the complement of } \Delta_1).$$

Consider $I_1(\tau)$. In view of (24), we have

$$|I_1(\tau)| \leqq \exp\left[\frac{|\tau|^2}{8+\mu}\right] \sup_{x\in\Delta_1} |\phi^{(m)}(x) - \phi^{(m)}(\tau)|.$$

Let $\omega = \{\tau : |\tau| < T\}$ be a spherical domain in R^n containing the support of $\phi(\tau)$, and let $\omega' = \{\tau : |\tau| < T + 1\}$. Thus the right-hand side of the last expression is identically zero for τ outside ω'. Since $\phi \in \mathscr{D}$, $\phi^{(m)}$ is uniformly continuous on R^n. Consequently, given an $\varepsilon > 0$, we can choose δ so small that

$$\sup_{x\in\Delta_1} |D_x^m \phi(x) - D_\tau^m \phi(\tau)| < \varepsilon \exp\left[\frac{-|T+1|^2}{8+\mu}\right]$$

for all τ, in which case $|I_1(\tau)| \leqq \varepsilon$ for all τ. Fix δ this way.

Next, consider

$$\begin{aligned} I_2(\tau) = {} & \{4\pi(1-t)\}^{-n/2} \exp\left[\frac{|\tau|^2}{8+\mu}\right] \int_{R^n-\Delta_1} \exp\left[\frac{|x-\tau|^2}{4(t-1)}\right] \phi^{(m)}(x)\, dx \\ & - \{4\pi(1-t)\}^{-n/2} \exp\left[\frac{|\tau|^2}{8+\mu}\right] \phi^{(m)}(\tau) \int_{R^n-\Delta_1} \exp\left[\frac{|x-\tau|^2}{4(t-1)}\right] dx. \end{aligned} \tag{26}$$

It is readily shown that the second term on the right-hand side of (26) tends uniformly to zero as $t \to 1-$ on $\tau \in R^n$. Let $J_1(\tau)$ denote the first term on the right-hand side of (26); then

$$J_1(\tau) = \{4\pi(1-t)\}^{-n/2} \exp\left[\frac{|\tau|^2}{8+\mu}\right] \int_{\omega-\Delta_1} \exp\left[\frac{|x-\tau|^2}{4(t-1)}\right] \phi^{(m)}(x)\, dx.$$

Here $\omega - \Delta_1$ denotes the set of those points in ω that are not in Δ_1.

For $\tau \in \omega'$ and for $x \in \omega - \Delta_1$, we have $|\tau| < T + 1$ and $|x - \tau| > \delta$. Consequently,

$$|J_1(\tau)| \leqq \{4\pi(1-t)\}^{-n/2} E_1 \exp\left[\frac{\delta^2}{4(t-1)}\right],$$

where E_1 is a constant independent of $\tau \in \omega'$. Thus as $t \to 1-$, $|J_1(\tau)|$ converges uniformly to zero on $\tau \in \omega'$.

For $\tau \in R^n - \omega'$, we have $\omega - \Delta_1 = \omega$ and

$$|J_1(\tau)| \leqq \{4\pi(1-t)\}^{-n/2} \exp\left[\frac{|\tau|^2}{8+\mu}\right] \left(\sup_{x\in\omega} \exp\left[\frac{|x-\tau|^2}{4(t-1)}\right]\right) \int_\omega |\phi^{(m)}(x)|\, dx.$$

Since $\tau \notin \omega$, we may write

$$\exp\left[\frac{|\tau|^2}{8+\mu}\right]\sup_{x\in\omega}\exp\left[\frac{|x-\tau|^2}{4(t-1)}\right] \leqq \exp\left[\frac{|\tau|^2}{8+\mu}+\frac{(T-|\tau|)^2}{4(t-1)}\right]. \tag{27}$$

The right-hand side of (27) achieves its maximum value along the surface defined by

$$|\tau| = \frac{T(8+\mu)}{4(t+1)+\mu}.$$

Therefore there exists a value t_1 such that this surface lies within the region $T < |\tau| < T+1$ for all values of $t \in (t_1, 1)$. Restricting $\tau \in R^n - \omega'$ then allows us to replace $|\tau|$ by $T+1$ in (27), thereby bounding the left-hand side by

$$\exp\left[\frac{(T+1)}{8+\mu}+\frac{1}{4(t-1)}\right].$$

Consequently,

$$|J_1(\tau)| \leqq \{4\pi(1-t)\}^{-n/2}E_2\exp\left[\frac{1}{4(t-1)}\right] \to 0, \qquad t \to 1-,$$

where E_2 is a constant independent of $\tau \in R^n - \omega'$. Thus we have demonstrated that $|J_1(\tau)|$ converges uniformly to zero on $\tau \in R^n$ as $t \to 1-$. Therefore by (25) this proves that for each nonnegative integer $m \in R^n$,

$$\limsup_{t\to 1-}\gamma_m(\langle\phi(x), K(x-\tau, 1-t)\rangle - \phi(\tau)) \leqq \varepsilon.$$

Since ε is arbitrary, the proof of Theorem 4.1 is complete.

5. An application to the heat equation for an n-dimensional medium. In this section we give an application of the preceding theory. Owing to the form of the kernel function, the Weierstrass transformation arises naturally in problems involving the heat equation.

Consider the finite interval of time $0 < t \leqq T$. If we normalize t so that $0 < t \leqq 1$, then the generalized Weierstrass transform $F(x)$, $x \in R^n$, can be interpreted as the temperature at time $t = 1$ in an infinite, uniform medium whose initial temperature is a generalized function of rather rapid growth. Moreover, if we set $u(x,t) = \langle f(\tau), K(x-\tau, t)\rangle$, then $u(x,t)$ gives the temperature at any time t, $0 < t \leqq 1$, and as $t \to 0+$, $u(x,t)$ converges weakly in $\mathscr{D}'$ to f. To see this we note that $u(x,t)$ satisfies the heat equation $\nabla_x^2 u(x,t) = D_t u(x,t)$ over the interval $0 < t \leqq 1$ (see note (IV); a similar result holds true for $D_t u(x,t)$) and since $\phi(x)$ and $K(x-\tau,t)$ satisfy the hypothesis of note (V), we may write

$$\langle u(x,t), \phi(x)\rangle = \langle f(\tau), \langle\phi(x), K(x-\tau,t)\rangle_x\rangle_\tau.$$

We have already demonstrated in the last part of the proof of Theorem 4.1 (see (23)) that, as $t \to 0+$, $\langle\phi(x), K(x-\tau,t)\rangle \to \phi(\tau)$ in η_μ. This verifies that $u(x,t)$ is indeed the solution of the Cauchy problem for the heat equation.

Another representation of the fundamental solution $u(x, t)$ of the heat equation is possible as a consequence of Theorem 4.1. That is, in light of (22) we may set

$$u(x, t) = M_{x,z,1-t}F(x + z\sqrt{1 - t}), \tag{28}$$

where z is an arbitrary fixed point in R^n. Therefore if at some instant in time, which is denoted here in normalized form as $t = 1$, the local behavior of the temperature function $u(x, t)$ is known in the neighborhood of some point $y \in R^n$ (that is, $u^{(m)}(y, 1) = F^{(m)}(y)$ is known for each $m \in R^n$), then $u(x, t)$ is completely determined for all $x \in R^n$ and $t \in (0, 1)$. For if we set $z = (y - x)/(1 - t)$ in (28), we obtain

$$u(x, t) = \prod_{i=1}^{n} \left(\sum_{\rho=0}^{\infty} \frac{(1 - t)^{\rho/2}(-1)^\rho H_\rho((y_i - x_i)/2\sqrt{1 - t})}{\rho!} D_{x_i}^\rho \right) F(x) \Bigg|_{x=y}$$

where the notation $\cdots|_{x=y}$ indicates the derivatives are evaluated at the point $x = y$.

Acknowledgment. I would like to thank Professor Armen Zemanian of the State University of New York at Stony Brook for many helpful suggestions and discussions.

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SIAM J. MATH. ANAL.
Vol. 2, No. 1, February 1971

A TWO-POINT BOUNDARY PROBLEM FOR NONLINEAR SECOND ORDER DIFFERENTIAL SYSTEMS *

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Abstract. This work is concerned with nonlinear second order differential systems involving a parameter with boundary conditions specified at two points. The linear version of the two-point boundary problem is non-self-adjoint. The objective is to establish the existence of eigenvalues for the boundary problem and to determine the behavior of the associated solutions. The boundedness and oscillation of solutions is of particular interest. The results obtained in this paper extend the work of W. M. Whyburn in the sense that sign conditions on the coefficients involved in the system have been relaxed and improved conditions for the existence of eigenvalues have been established.

1. Introduction. In a series of papers [2], [3], [4], [5], W. M. Whyburn studied two-point boundary problems associated with nonlinear second order differential systems of the form

$$y' = k(x, y, z; \lambda)z, \qquad z' = g(x, y, z; \lambda)y, \tag{1}$$

where $k(x, y, z; \lambda)$ and $g(x, y, z; \lambda)$ are positive real-valued functions on X: $a \leqq x \leqq b$, $L: \lambda_0 - \delta < \lambda < \lambda_0 + \delta$, $0 < \delta \leqq \infty$, and where k and g satisfy conditions which will insure that a solution exists when appropriate initial conditions are specified. Whyburn establishes the existence of sets of characteristic numbers (eigenvalues) for the boundary problems and determines the oscillatory behavior of the associated solutions.

The purpose of this paper is to extend some of Whyburn's work by relaxing certain conditions imposed on the coefficients and by obtaining conditions insuring the existence of sets of characteristic numbers which appear to be somewhat simpler than those previously obtained.

We shall be concerned with the nonlinear system (1) together with the two-point boundary conditions

$$\alpha(\lambda)y(a, \lambda) - \beta(\lambda)z(a, \lambda) = 0, \tag{2a}$$

$$\gamma_1(\lambda)y(a, \lambda) - \delta_1(\lambda)z(a, \lambda) = \gamma_2(\lambda)y(b, \lambda) - \delta_2(\lambda)z(b, \lambda), \tag{2b}$$

where α, β, γ_i, δ_i, $i = 1, 2$, are continuous, real-valued functions on L.

We shall assume that the coefficients involved in the two-point boundary problem satisfy the following hypotheses:

(i) for each $x \in X$, each of $k(x, y, z; \lambda)$ and $g(x, y, z; \lambda)$ is continuous in (y, z, λ) for all real pairs y, z and all λ on L;

(ii) for each fixed (y, z, λ), each of $k(x, y, z; \lambda)$ and $g(x, y, z; \lambda)$ is measurable in x on X;

(iii) there exists a Lebesgue integrable function $M(x)$ on X such that $|k(x, y, z; \lambda)| \leqq M(x)$ and $|g(x, y, z; \lambda)| \leqq M(x)$ for all $x \in X$, $\lambda \in L$ and all real pairs

* Received by the editors July 21, 1970.

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(y, z);

(iv) $k(x, y, z; \lambda)$ is positive on XL for all real pairs (y, z);

(v) $\alpha^2(\lambda) + \beta^2(\lambda) > 0$ and $\gamma_i^2(\lambda) + \delta_i^2(\lambda) > 0$, $i = 1, 2$, on L;

(vi) $\beta(\lambda) \neq 0$, $\Delta(\lambda) = \alpha(\lambda)\delta_2(\lambda) - \beta(\lambda)\gamma_2(\lambda) \neq 0$ on L.

We remark here that the linear version of the two-point problem (1), (2) is non-self-adjoint. As indicated previously, Whyburn assumes that each of the functions k and g in (1) is positive. We, on the other hand, are assuming only that k is positive, making no restrictions as to sign on g. Thus, the work presented here may be viewed as more closely paralleling the linear theory. Finally, we note that with the assumptions on α, β, γ_i, δ_i in (v), we can, without loss of generality, have (v) read:

(v) $\alpha^2(\lambda) + \beta^2(\lambda) \equiv 1 \equiv \gamma_1^2(\lambda) + \delta_1^2(\lambda)$, $\gamma_2^2(\lambda) + \delta_2^2(\lambda) > 0$ on L.

2. Preliminary definitions and results. We seek to establish the existence of values of λ on L for which there corresponds a nontrivial solution of (1), (2). Such values of λ are called the *eigenvalues* for the system. By a *nontrivial solution* of the system, we shall mean a solution pair $\{y(x, \lambda), z(x, \lambda)\}$ of (1) satisfying (2) with the property that $y^2(x, \lambda) + z^2(x, \lambda) > 0$ on XL.

Our first theorem represents a slight extension of Whyburn's results [5, Theorems 1, 2.]

THEOREM 1. *There exists a solution pair* $\{y(x, \lambda), z(x, \lambda)\}$ *of* (1) *on* XL *such that*

$$y(a, \lambda) \equiv \beta(\lambda), \qquad z(a, \lambda) \equiv \alpha(\lambda) \tag{3}$$

on L. Moreover, if $\{y(x, \lambda), z(x, \lambda)\}$ *is a solution pair of* (1), (3) *on L, then the pair* $\{y(x, \lambda), z(x, \lambda)\}$ *is nontrivial,* $y(a, \lambda) \neq 0$ *on L and* $y(x, \lambda)$ *has only isolated zeros on X for each fixed* λ *on L.*

Proof. Hypotheses (i)–(iii) allow the application of a fundamental existence theorem for differential systems (see, e.g., [1, Chap. 2]).

Let $\{y(x, \lambda), z(x, \lambda)\}$ be a solution of (1), (3). Applying the polar coordinate transformation, we obtain

$$\begin{aligned} y(x, \lambda) &= r(x, \lambda) \cdot \sin v(x, \lambda), \\ z(x, \lambda) &= r(x, \lambda) \cdot \cos v(x, \lambda), \end{aligned} \tag{4}$$

where $r(x, \lambda)$ and $v(x, \lambda)$ are solutions of

$$\begin{aligned} dv/dx &= k \cos^2 v + g \sin^2 v, \\ dr/dx &= [(k - g) \sin v \cos v]r, \end{aligned} \tag{5}$$

$$\begin{aligned} &r(a, \lambda) \equiv 1, \\ &\sin v(a, \lambda) \equiv \beta(\lambda), \qquad \cos v(a, \lambda) \equiv \alpha(\lambda), \qquad 0 < v(a, \lambda) < 2\pi; \end{aligned} \tag{6}$$

in fact, either $0 < v(a, \lambda) < \pi$, or $\pi < v(a, \lambda) < 2\pi$.

As established by Whyburn [5, Theorem 2], $r(x, \lambda) > 0$ on XL. Since $r^2(x, \lambda) = y^2(x, \lambda) + z^2(x, \lambda)$ on XL, it follows that the pair $\{y, z\}$ is nontrivial.

Clearly, $y(a, \lambda) \neq 0$ on L by hypothesis (vi).

Finally, we note that, for fixed λ on L, $y(x, \lambda) = 0$ if and only if $v(x, \lambda) \equiv 0 \pmod{\pi}$. Since $dv/dx = k(x, y(x, \lambda), z(x, \lambda); \lambda) > 0$ when $v(x, \lambda) \equiv 0 \pmod{\pi}$, it follows that $y(x, \lambda)$ has only isolated zeros on x.

In the work that follows we shall let $\{y(x, \lambda), z(x, \lambda)\}$ be a fixed solution pair of (1), (3). Clearly, this pair satisfies the boundary condition (2a). By Theorem 1, $\{y, z\}$ is nontrivial and, consequently, the complex-valued function $(z(x, \lambda) - iy(x, \lambda))$ is nonzero on XL. We define $\theta(x, \lambda)$ by

$$\theta(x, \lambda) = (z + iy)/(z - iy). \tag{7}$$

THEOREM 2. *The complex-valued function $\theta(x, \lambda)$ has the following properties on X for each fixed λ on L:*

(i) $|\theta(x, \lambda)| = 1$.

(ii) *θ satisfies the first order equation $d\theta/dx = 2i\theta h(x, \lambda)$, where*

$$h(x, \lambda) = (zy' - yz')/r^2 = (kz^2 + gy^2)/r^2 = k \cos^2 v + g \sin^2 v. \tag{8}$$

(iii) $\theta(x, \lambda) = 1$ *if and only if* $y(x, \lambda) = 0$,
$\theta(x, \lambda) = -1$ *if and only if* $z(x, \lambda) = 0$.

(iv) *Let $\omega(x, \lambda) = \arg \theta(x, \lambda)$, where it is assumed that $0 \leqq \omega(a, \lambda_0) < 2\pi$ and that $\omega(x, \lambda)$ is continued as a continuous function on XL. Then*

$$\theta(x, \lambda) \equiv \cos 2v(x, \lambda) + i \sin 2v(x, \lambda), \tag{9}$$

$$\omega(x, \lambda) \equiv \begin{cases} 2v(x, \lambda) & \text{if } \beta(\lambda) > 0 \text{ on } L, \\ 2v(x, \lambda) - 2\pi & \text{if } \beta(\lambda) < 0 \text{ on } L, \end{cases} \tag{10}$$

$$2 \int_a^x h(t, \lambda)\, dt = \omega(x, \lambda) - \omega(a, \lambda). \tag{11}$$

(v) *$\theta(x, \lambda)$ moves monotonically and positively on the unit circle at the point $+1$.*

Proof. Properties (i)–(iii) are easily verified. Equation (9) follows upon using the polar coordinate transformation (4). Equations (10) result from (9) and our assumptions concerning the initial values of ω and v. Similarly, (11) is a result of solving the first order equation in θ and applying the definition of ω.

The complex-valued function $\theta(y, z)$ provides a means for determining the oscillatory behavior of the solution pair $\{y, z\}$. To develop the tools which will enable us to establish the existence of eigenvalues for the boundary problems (1), (2), we introduce the functions

$$\begin{aligned} s(x, \lambda) &= \gamma_2(\lambda) y(x, \lambda) - \delta_2(\lambda) z(x, \lambda), \\ t(x, \lambda) &= \gamma_2(\lambda) z(x, \lambda) + \delta_2(\lambda) y(x, \lambda). \end{aligned} \tag{12}$$

It is readily verified that $s^2 + t^2 > 0$ on XL and, consequently, the complex-valued function $\phi(x, \lambda)$, defined by

$$\phi(x, \lambda) = (t + is)/(t - is), \tag{13}$$

exists on XL. In addition, we have the following analogue of Theorem 2.

THEOREM 3. *The complex-valued function $\phi(s, t)$ has the following properties on X for each λ on L:*

(i) $|\phi(x, \lambda)| = 1$.

(ii) *ϕ satisfies the first order equation $d\phi/dx = 2i\phi j(x, \lambda)$, where*

(14) $$j(x, \lambda) = (ts' - st')/(s^2 + t^2) \equiv h(x, \lambda).$$

(iii) $\phi(x, \lambda) = 1$ *if and only if* $s(x, \lambda) = 0$,
$\phi(x, \lambda) = -1$ *if and only if* $t(x, \lambda) = 0$.

(iv) *Let* $\sigma(x, \lambda) = \arg \phi(x, \lambda)$, *where it is assumed that* $0 \leqq \sigma(a, \lambda_0) < 2\pi$ *and that* $\sigma(x, \lambda)$ *is continued as a continuous function on* XL. *Then, for each fixed* λ,

$$2 \int_a^x h(u, \lambda)\, du = \sigma(x, \lambda) - \sigma(a, \lambda) = \omega(x, \lambda) - \omega(a, \lambda).$$

Consider the boundary condition (2b). Using the polar coordinate transformation and the definitions of the functions s and t, (2b) may be written in the form

(15) $$\sin(v(a, \lambda) - \tau_1(\lambda)) = s(b, \lambda) = r(b, \lambda)[\gamma_2^2(\lambda) + \delta_2^2(\lambda)]^{1/2} \sin(v(b, \lambda) - \tau_2(\lambda)),$$

where $\sin \tau_1(\lambda) = \delta_1(\lambda)$, $\cos \tau_1(\lambda) = \gamma_1(\lambda)$, $\sin \tau_2(\lambda) = \delta_2(\lambda)/[\gamma_2^2(\lambda) + \delta_2^2(\lambda)]^{1/2}$ and $\cos \tau_2(\lambda) = \gamma_2(\lambda)/[\gamma_2^2(\lambda) + \delta_2^2(\lambda)]^{1/2}$.

Our assumption $\Delta(\lambda) = \alpha(\lambda)\delta_2(\lambda) - \beta(\lambda)\gamma_2(\lambda) \neq 0$ on L implies that $s(a, \lambda) \neq 0$ on L. As a result of our assumption (iv) on the initial condition of $\sigma(a, \lambda) = \arg \phi(x, \lambda)$, it follows that $0 < \sigma(a, \lambda) < 2\pi$ on L. Also, since $\beta(\lambda) \neq 0$, we have from (iv) of Theorem 2, that $0 < \omega(a, \lambda) < 2\pi$. We can now conclude that the following inequalities hold for each λ:

(16) $$\begin{aligned} 2 \int_a^b h(u, \lambda)\, du &< \omega(b, \lambda) < 2 \int_a^b h(u, \lambda)\, du + 2\pi, \\ 2 \int_a^b h(u, \lambda)\, du &< \sigma(b, \lambda) < 2 \int_a^b h(u, \lambda)\, du + 2\pi. \end{aligned}$$

The following bound on the solutions of (1), (3) in terms of the Lebesgue integrable function $M(x)$ is easily obtained.

THEOREM 4. *Let* $\{y(x, \lambda), z(x, \lambda)\}$ *be a solution of* (1), (3). *Then the following inequality holds on* X *for each fixed* λ *on* L:

(17) $$[y^2(x) + z^2(x)] \leqq \exp\left\{2 \int_a^x M(u)\, du\right\}.$$

We shall let Γ denote the collection of all pairs of functions $\{y(x, \lambda), z(x, \lambda)\}$ which are absolutely continuous in x for each fixed λ, continuous in λ for each fixed x and which satisfy the bound (17). Clearly, all solutions of (1), (3) belong to Γ.

For each pair (x, λ) on XL, define the functions $f(x, \lambda)$ and $g(x, \lambda)$ by

(18) $$\begin{aligned} f(x, \lambda) &= \inf_{y,z \in \Gamma} \{k(x, y, z; \lambda), g(x, y, z; \lambda)\}, \\ g(x, \lambda) &= \sup_{y,z \in \Gamma} \{k(x, y, z; \lambda), g(x, y, z; \lambda)\}; \end{aligned}$$

then each of $f(x, \lambda)$ and $g(x, \lambda)$ is integrable on X for each λ on L, and using (8) we have

(19) $$f(x, \lambda) \leqq h(x, \lambda) \leqq g(x, \lambda).$$

3. Existence of eigenvalues. Using the results developed in the previous section, we are in a position to specify conditions which will insure the existence of eigenvalues for the system (1), (2).

THEOREM 5. *Let* $\{y(x, \lambda), z(x, \lambda)\}$ *be a solution of* (1), (3), *and define* $q(\lambda)$ *by*

$$q(\lambda) = \int_a^b h(u, \lambda)\, du$$

($h(x, \lambda)$ *defined by* (8)). *Then* $q(\lambda) > -2\pi$ *on* L. *Let* $m \geqq -1$ *be the least integer such that* g.l.b. $q(\lambda) < (2m + 1)\pi$ *on* L, *and let* n *be an integer such that* l.u.b. $q(\lambda) > (2n + 1)\pi$ *on* L. *If* $n \geqq m + 1$ *and if any of the following conditions holds on* L:

(i) $\displaystyle\int_a^b [k(x, y, z; \lambda) - g(x, y, z; \lambda)] \sin 2v(x, \lambda)\, dx \geqq 0$ *and* $\gamma_2^2(\lambda) + \delta_2^2(\lambda) \geqq 1$,

(ii) $\displaystyle\int_a^b [k(x, y, z; \lambda) - g(x, y, z; \lambda)] \sin 2v(x, \lambda)\, dx \geqq -\ln(\gamma_2^2(\lambda) + \delta_2^2(\lambda))$

($v(x, \lambda)$ *defined by* (5)),

(iii) $\displaystyle\int_a^b |k(x, y, z; \lambda) - g(x, y, z; \lambda)|\, dx \leqq \ln(\gamma_2^2(\lambda) + \delta_2^2(\lambda))$;

then there exist at least p, $p = n - m$, *nonempty sets of eigenvalues* $T_0, T_1, \cdots, T_{p-1}$ *for the system* (1), (2).

Proof. Let $\{y(x, \lambda), z(x, \lambda)\}$ be a solution of (1), (3). As noted previously, this pair satisfies (2a). Let $\theta(x, \lambda)$, $s(x, \lambda)$, $t(x, \lambda)$ and $\phi(x, \lambda)$ be as described in § 2.

Using (11), we have $q(\lambda) = 2\int_a^b h(u, \lambda)\, du = \omega(b, \lambda) - \omega(a, \lambda)$. Since $0 < \omega(a, \lambda) < 2\pi$ on L and since $\theta(x, \lambda)$ passes through $+1$ in the positive direction only as x increases on X, it follows that $\omega(b, \lambda) > 0$ on L and, consequently, $q(\lambda) > -2\pi$ on L.

Suppose that m and n are integers with the properties described in the hypothesis. Then there exists a value of λ, say λ^*, such that $q(\lambda^*) < (2m + 1)\pi$ and a value of λ, say $\bar{\lambda}$, such that $q(\bar{\lambda}) > (2n + 1)\pi$. Clearly $\lambda^* \neq \bar{\lambda}$ so we shall assume $\lambda^* < \bar{\lambda}$. Now, from (16),

$$q(\lambda) < \sigma(b, \lambda) < q(\lambda) + 2\pi.$$

Thus $\sigma(b, \lambda^*) < (2m + 3)\pi$ and $\sigma(b, \bar{\lambda}) > (2n + 1)\pi$. Since $n = m + p$, $p \geqq 1$, there exist p values of λ, $\lambda_0, \lambda_1, \cdots, \lambda_{p-1}$ on the interval $(\lambda^*, \bar{\lambda})$ such that $\sigma(b, \lambda_j) = [2(m + j) + 3]\pi$. Moreover, since $\sigma(b, \lambda)$ is continuous in λ we may assume $\lambda_0 < \lambda_1 < \cdots < \lambda_{p-1}$. Since $\sigma(b, \lambda) = \arg \phi(b, \lambda)$, it follows that $\phi(b, \lambda_j) = -1$ for $j = 0, 1, \cdots, p - 1$, and, consequently, $t(b, \lambda_j) = 0$ for each j. Using the polar coordinate transformation, we have

$$t(b, \lambda) = r(b, \lambda)[\gamma_2^2(\lambda) + \delta_2^2(\lambda)]^{1/2} \cos[v(b, \lambda) - \tau_2(\lambda)],$$

$$s(b, \lambda) = r(b, \lambda)[\gamma_2^2(\lambda) + \delta_2^2(\lambda)]^{1/2} \sin[v(b, \lambda) - \tau_2(\lambda)].$$

Thus, for each j, $\cos[v(b, \lambda_j) - \tau_2(\lambda_j)] = 0$ which implies $\sin^2[v(b, \lambda_j) - \tau_2(\lambda_j)] = 1$. Also, as λ increases from λ_j to λ_{j+1}, $\sin[v(b, \lambda) - \tau_2(\lambda)]$ changes continuously in value from -1 to $+1$ or from $+1$ to -1.

Since $|\sin[v(a, \lambda) - \tau_1(\lambda)]| \leqq 1$, it follows that there will exist at least one value of λ on $[\lambda_j, \lambda_{j+1}]$ with the property that (2b) is satisfied provided we can show that $r(b, \lambda)[\gamma_2^2(\lambda) + \delta_2^2(\lambda)]^{1/2} \geqq 1$ on $[\lambda_j, \lambda_{j+1}]$.

Consider condition (i). From (6),

$$r(b, \lambda) = r(a, \lambda) \exp \left\{ \int_a^b (k - g) \sin v \cos v \, dx \right\}.$$

By our assumption, $r(a, \lambda) \equiv 1$ on L, so

$$r(b, \lambda) = \exp \left\{ \int_a^b (k - g) \sin v \cos v \, dx \right\}.$$

Now, the inequality

$$\int_a^b (k - g) \sin 2v \, dx \geqq 0$$

implies

$$\int_a^b (k - g) \sin v \cos v \, dx \geqq 0$$

and, consequently, $r(b, \lambda) \geqq 1$. If, moreover, $(\gamma_2^2 + \delta_2^2) \geqq 1$ on L, then, clearly, $r(b, \lambda)(\gamma_2^2 + \delta_2^2)^{1/2} \geqq 1$ on L and we can conclude that there exists at least one eigenvalue for the system on $[\lambda_j, \lambda_{j+1}]$, $j = 0, 1, \cdots, p - 1$. Let T_j be the set of all eigenvalues on $[\lambda_j, \lambda_{j+1}]$, $j = 0, 1, \cdots, p - 1$.

The inequality of condition (ii) implies

$$\exp \left\{ \int_a^b (k - g) \sin 2v \, dx \right\} \geqq \frac{1}{\gamma_2^2 + \delta_2^2}$$

or $r^2(b, \lambda)(\gamma_2^2 + \delta_2^2) \geqq 1$. Thus $r(b, \lambda)(\gamma_2^2 + \delta_2^2)^{1/2} \geqq 1$, and we proceed as above.

Finally, we consider condition (iii). Since

$$-\int_a^b (k - g) \sin 2v \, dx \leqq \int_a^b |k - g| dx,$$

we have

$$\int_a^b (k - g) \sin 2v \, dx \geqq -\int_a^b |k - g| dx.$$

Thus, if

$$\int_a^b |k - g| \, dx \leqq \ln (\gamma_2^2 + \delta_2^2),$$

then

$$-\int_a^b |k - g| \, dx \geqq \ln \frac{1}{\gamma_2^2 + \delta_2^2}$$

and, consequently, $r^2(b, \lambda) \geqq 1/(\gamma_2^2 + \delta_2^2)$ or $r^2(b, \lambda)(\gamma_2^2 + \delta_2^2) \geqq 1$. Again, we conclude that there exist at least p nonempty sets of eigenvalues $T_0, T_1, \cdots, T_{p-1}$ for the system (1), (2). This completes the proof of the theorem.

COROLLARY 1. *Under the hypotheses of Theorem* 5, *if the integer n can be chosen arbitrarily large, then there exist infinitely many nonempty sets of eigenvalues* $T_0, T_1, \cdots$ *for the system* (1), (2).

COROLLARY 2. *Under the hypotheses of Theorem* 5, *there exist p nonempty sets of eigenvalues* $J_0, J_1, \cdots, J_{p-1}$ *for the system* (1), (2) *such that if* $\rho_j \in J_j$,

$j = 0, 1, \cdots, p-1$, then $\sigma(b, \rho_j) \geqq [2(m+j)+3]\pi$. Moreover, if $j \geqq 1$, then the corresponding solution $\{y(x, \rho_j), z(x, \rho_j)\}$ has the property that $y(x, \rho_j)$ has at least $j-1$ zeros on X.

Proof. Using the continuity of $\sigma(b, \lambda)$ together with the fact that $\sigma(b, \lambda)$ increases from less than $(2m+3)\pi$ to more than $(2n+1)\pi$, select λ_j such that for $\lambda \geqq \lambda_j$, $\sigma(b, \lambda) \geqq [2(m+j)+3]\pi$, $j = 0, 1, \cdots, p-1$. Let J_j be the set of all eigenvalues on $[\lambda_j, \lambda_{j+1}]$. Using the proof of Theorem 5, we see that each J_j is nonempty.

Now, consider $\omega(b, \lambda) - \omega(a, \lambda)$. Clearly, if for fixed λ, $\omega(b, \lambda) - \omega(a, \lambda) \geqq (2q-1)\pi$, $q \geqq 1$, then $\omega(x, \lambda) \equiv 0 \pmod{2\pi}$ at least $q-1$ times on X. Since $y(x, \lambda) = 0$ if and only if $\omega(x, \lambda) \equiv 0 \pmod{2\pi}$, it follows that $y(x, \lambda)$ has at least $q-1$ zeros on X. Suppose that $\rho_j \in J_j$, $j \geqq 1$; then $\sigma(b, \rho_j) \geqq [2(m+j)+3]\pi \geqq (2j+1)\pi$. Now,

$$\sigma(b, \rho_j) - \sigma(a, \rho_j) = \omega(b, \rho_j) - \omega(a, \rho_j).$$

Thus $\omega(b, \rho_j) - \omega(a, \rho_j) \geqq (2j+1)\pi - 2\pi = (2j-1)\pi$, and we conclude that $y(x, \rho_j)$ has at least $j-1$ zeros on X.

We note the following possibilities concerning the eigenvalues of the system (1), (2): first, there may exist additional eigenvalues for the system lying outside the interval $[\lambda_0, \lambda_{p-1}]$ obtained in Theorem 5; second, the sets of eigenvalues $T_0, \cdots, T_{p-1}$ may be finite, denumerable, nondenumerable or, in fact, contain an interval; third, T_{j-1} and T_j may have an eigenvalue in common, namely λ_j.

We remark, also, that Theorem 5 has the disadvantage that the conditions insuring the existence of eigenvalues depend upon the choice of a solution pair $\{y(x, \lambda), z(x, \lambda)\}$ of (1), (3). We consider now a refinement of Theorem 5 which overcomes this disadvantage.

THEOREM 6. *If there exist integers m and n such that*

(i) $2\int_a^b g(x, \lambda^*)\,dx < (2m+1)\pi$ *for some λ^* on L,*

(ii) $2\int_a^b f(x, \bar{\lambda})\,dx > (2n+1)\pi$ *for some $\bar{\lambda}$ on L,*

(iii) $0 < p = n - m$,

and if

(iv) $\int_a^b M(x)\,dx \leqq \ln(\gamma_2^2(\lambda) + \delta_2^2(\lambda))^{1/2}$ *on L, where $M(x)$ is the Lebesgue integrable bound of the functions $k(x, y, z; \lambda)$ and $g(x, y, z; \lambda)$;*

then there exist at least p nonempty sets of eigenvalues $T_0, T_1, \cdots, T_{p-1}$ for the system (1), (2).

Proof. Let $\{y(x, \lambda), z(x, \lambda)\}$ be any solution of (1), (3). Proceeding as in the proof of Theorem 5, we have, from (19),

$$2\int_a^b f(x, \lambda)\,dx \leqq 2\int_a^b h(x, \lambda)\,dx = q(\lambda) \leqq 2\int_a^b g(x, \lambda)\,dx$$

for each λ on L. Thus, for $\lambda = \lambda^*$, we have $q(\lambda^*) < (2m+1)\pi$; and for $\lambda = \bar{\lambda}$, we have $q(\bar{\lambda}) > (2n+1)\pi$.

We can now duplicate the proof of Theorem 5 provided we can show that the inequality (iv) implies $r(b, \lambda)(\gamma_2^2 + \delta_2^2)^{1/2} \geqq 1$.

To establish the required inequality, we have

$$-\int_a^b (k-g)\sin 2v\,dx \leqq \int_a^b |k-g|\,dx \leqq 2\int_a^b M(x)\,dx,$$

and, therefore,

$$\int_a^b (k-g)\sin 2v\,dx \geqq -2\int_a^b M(x)\,dx.$$

Thus, if $\int_a^b M(x)\,dx \leqq \ln(\gamma_2^2 + \delta_2^2)^{1/2}$ on L, then

$$-2\int_a^b M(x)\,dx \geqq \ln\frac{1}{\gamma_2^2+\delta_2^2}$$

and

$$r^2(b,\lambda)(\gamma_2^2+\delta_2^2) = (\gamma_2^2+\delta_2^2)\exp\left\{2\int_a^b (k-g)\sin v\cos v\,dx\right\} \geqq 1.$$

The proof is now completed as in Theorem 5.

Again, we have two corollaries.

COROLLARY 1. *Under the hypotheses of Theorem* 6, *if the integer can be chosen arbitrarily large, then there exist infinitely many nonempty sets of eigenvalues* $T_0, T_1, \cdots$, *for the system* (1), (2).

COROLLARY 2. *Under the hypotheses of Theorem* 6, *there exist p nonempty sets of eigenvalues* $J_0, J_1, \cdots, J_{p-1}$ *for the system* (1), (2) *such that if* $\rho_j \in J_j$, $j = 0, 1, \cdots, p-1$, *then* $\sigma(b, \rho_j) \geqq [2(m+j)+3]\pi$. *Moreover, if* $j \geqq 1$, *then the corresponding solution* $\{y(x,\rho_j), z(x,\rho_j)\}$ *has the property that* $y(x,\rho_j)$ *has at least* $j-1$ *zeros on* X.

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SIAM J. MATH. ANAL.
Vol. 2, No. 1, February 1971

A CLASS OF MULTIPLE INTEGRALS *

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Abstract. The class of multiple integrals defined by

$$\iint\cdots\int_{t_1+t_2+\cdots+t_n\leqq 1} f(t_1+t_2+\cdots+t_{n-s})\phi_1(t_1)\phi_2(t_2)\cdots\phi_n(t_n)\,dt_1\,dt_2\cdots dt_n,$$

where for $t \geqq 0$, $f(t)\in\mathscr{C}$ and $\phi_i(t)\in\mathscr{K}$, $i = 1, 2, \cdots, n$, is shown to be reducible to the single integral

$$\int_0^1 \{f(u)[\phi_1(u) * \phi_2(u) * \cdots * \phi_{n-s}(u)]\} * \phi_{n-s+1}(u) * \cdots * \phi_n(u)\,du.$$

1. Introduction. Integrals of the form

$$\iint\cdots\int_{t_1+t_2+\cdots+t_n\leqq 1} f(t_1+t_2+\cdots+t_{n-s})t_1^{\alpha_1-1}t_2^{\alpha_2-1}\cdots t_n^{\alpha_n-1}\,dt_1\,dt_2\cdots dt_n, \tag{1}$$

where $f(\cdot)$ is a continuous function, $1 \leqq s \leqq n-1$, $\alpha_r > 0$, $r = 1, 2, \cdots, n$, and the integration is extended over all positive values of the variables, have been shown by the present author to be reducible to a single integral [6]. The so-called Dirichlet's integral corresponding in (1) to the case $s = 0$ is also reducible to a single integral [7, p. 258]; this last result is sometimes known as Liouville's extension of Dirichlet's theorem [2, Chap. 25]. Other variants of Dirichlet's integral have been investigated [2] and most of the results appear in [3].

In this paper we consider the more general class of multiple integrals defined by

$$I_{n-s} = \iint\cdots\int_{t_1+t_2+\cdots+t_n\leqq 1} f(t_1+t_2+\cdots+t_{n-s})\phi_1(t_1) \cdot \phi_2(t_2)\cdots\phi_n(t_n)\,dt_1\,dt_2\cdots dt_n, \tag{2}$$

where $s = 0, 1, 2, \cdots, n-1$ and the integration is extended over all positive values of the variables. It is shown that if for $t \geqq 0$, $f(t)\in\mathscr{C}$ (i.e. continuous) and the functions $\phi_i(t)\in\mathscr{K}$ (i.e., with at most a finite number of points of discontinuity in every finite interval and the integral $\int_0^t|\phi_i(u)|\,du$ has a finite value for every $t > 0$), $i = 1, 2, \cdots, n$ [4, Chap. 7], then (2) is reducible to a single integral whose integrand, involving convolution forms of the functions $f(\cdot)$ and $\phi_i(\cdot)$, is a function of class $\mathscr{K}$; the absolute convergence of the resultant single integral would thus be asserted. The relevancy and importance of the results stem from several facts. First the formal evaluation of a general class of multiple integrals given by (2) is reduced to a set of simpler analytic computations since the convolution of a large class of functions can be explicitly determined using the theory of operational calculus [1], [4]. Second, a necessary condition for the convergence of integrals of the form (2) is provided. Finally, it is easily verified that the formulas established in [7, p. 258] and [6] follow as a special case.

* Received by the editors February 9, 1970.

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2. A special case: $s = 0$. Consider first the integral

$$I_n = \iint_{t_1+t_2+\cdots+t_n \leqq 1} \cdots \int f(t_1 + t_2 + \cdots + t_n)\phi_1(t_1)\phi_2(t_2) \cdots \phi_n(t_n)\, dt_1\, dt_2 \cdots dt_n. \tag{3}$$

Let $\lambda = t_3 + t_4 + \cdots + t_n$; then

$$\begin{aligned} I_n = &\iint_{t_3+t_4+\cdots+t_n \leqq 1} \cdots \int \phi_3(t_3) \cdots \phi_n(t_n) \\ &\cdot \iint_{t_1+t_2 \leqq 1-\lambda} f(t_1 + t_2 + \lambda)\phi_1(t_1)\phi_2(t_2)\, dt_1\, dt_2 \cdots dt_n. \end{aligned} \tag{4}$$

To reduce

$$I = \iint_{t_1+t_2 \leqq 1-\lambda} f(t_1 + t_2 + \lambda)\phi_1(t_1)\phi_2(t_2)\, dt_1\, dt_2,$$

let $t_1 = \tau - \omega$ and $t_2 = \omega$; then

$$\begin{aligned} I &= \int_0^{1-\lambda} f(\tau + \lambda) \int_0^{\tau} \phi_1(\tau - \omega)\phi_2(\omega)\, d\omega\, d\tau \\ &= \int_0^{1-\lambda} f(\tau + \lambda)[\phi_1(\tau) * \phi_2(\tau)]\, d\tau, \end{aligned} \tag{5}$$

where the symbol $*$ denotes the usual operation of convolution. Using (5) in (4) yields

$$\begin{aligned} I_n &= \iint_{t_3+t_4+\cdots+t_n \leqq 1} \cdots \int \phi_3(t_3) \cdots \phi_n(t_n) \cdot \int_0^{1-\lambda} f(\tau + \lambda)[\phi_1(\tau) * \phi_2(\tau)]\, d\tau\, dt_3 \cdots dt_n \\ &= \iint_{\tau+t_3+\cdots+t_n \leqq 1} \cdots \int f(\tau + t_3 + \cdots + t_n)[\phi_1(\tau) * \phi_2(\tau)]\phi_3(t_3) \cdots \phi_n(t_n)\, d\tau\, dt_3 \cdots dt_n. \end{aligned}$$

Applying the previous reduction procedure successively to this last integral, we obtain the final result

$$I_n = \int_0^1 f(\tau)[\phi_1(\tau) * \phi_2(\tau) * \cdots * \phi_n(\tau)]\, d\tau. \tag{6}$$

Since the convolution of functions of class $\mathscr{K}$ is also a function of class $\mathscr{K}$ [5], it follows that if $f(\cdot) \in \mathscr{C}$ and $\phi_i(\cdot) \in \mathscr{K}$, $i = 1, 2, \cdots, n$, then the integrand in (6) is a function of class $\mathscr{K}$, hence (6) exists and converges absolutely. In addition, the results of Liouville's extension of Dirichlet's theorem are easily recovered from (6) if we set $\phi_i(t_i) = t_i^{\alpha_i - 1}$, $\alpha_i > 0$, $i = 1, 2, \cdots, n$.

3. The case when $s = 1, 2, \cdots, n - 1$. We now consider the multiple integral (2) and write it as

$$\begin{aligned} I_{n-s} = &\int_{t_{n-s+1}+\cdots+t_n \leqq 1} \cdots \int \phi_{n-s+1}(t_{n-s+1}) \cdots \phi_n(t_n) \\ &\cdot \mathscr{I}_{n-s}(t_{n-s+1} + \cdots + t_n)\, dt_{n-s+1} \cdots dt_n, \end{aligned} \tag{7}$$

where for $k = 1, 2, \cdots, n$, we define

$$\mathscr{I}_k(T) = \underset{t_1 + t_2 + \cdots + t_k \leqq 1 - T}{\int\int\cdots\int} f(t_1 + t_2 + \cdots + t_k) \cdot \phi_1(t_1)\phi_2(t_2)\cdots\phi_k(t_k)\,dt_1\,dt_2\cdots dt_k. \tag{8}$$

In (8) let $t_r = v_r(1 - T)$, $r = 1, 2, \cdots, k$; we obtain

$$\mathscr{I}_k(T) = \underset{v_1 + v_2 + \cdots + v_k \leqq 1}{\int\int\cdots\int} f[(1 - T)(v_1 + v_2 + \cdots + v_k)] \cdot \phi_1[v_1(1 - T)]\phi_2[v_2(1 - T)]\cdots\phi_k[v_k(1 - T)](1 - T)^k\,dv_1\,dv_2\cdots dv_k.$$

From (6), this last relation can be expressed as the single integral

$$\mathscr{I}_k(T) = (1 - T)^k \int_0^1 f[(1 - T)\tau] \cdot \{\phi_1[(1 - T)\tau] * \phi_2[(1 - T)\tau] * \cdots * \phi_k[(1 - T)\tau]\}\,d\tau.$$

Making the change in variable $(1 - T)\tau = u$, we obtain

$$\mathscr{I}_k(T) = \int_0^{1 - T} f(u)[\phi_1(u) * \phi_2(u) * \cdots * \phi_k(u)]\,du. \tag{9}$$

Substitution of (9) in (7) yields

$$\begin{aligned} I_{n-s} &= \underset{t_{n-s+1} + \cdots + t_n \leqq 1}{\int\cdots\int} \phi_{n-s+1}(t_{n-s+1})\cdots\phi_n(t_n) \\ &\quad \cdot \int_0^{1 - (t_{n-s+1} + \cdots + t_n)} f(u)[\phi_1(u) * \phi_2(u) * \cdots * \phi_{n-s}(u)] \cdot du\,dt_{n-s+1}\cdots dt_n \\ &= \int_0^1 f(u)[\phi_1(u) * \phi_2(u) * \cdots * \phi_{n-s}(u)] \\ &\quad \cdot \underset{t_{n-s+1} + \cdots + t_n \leqq 1 - u}{\int\cdots\int} \phi_{n-s+1}(t_{n-s+1})\cdots\phi_n(t_n)\,dt_{n-s+1}\cdots dt_n\,du. \end{aligned}$$

Noting that the inner multiple integral is similar to (8) with $f(\cdot) = 1$, we obtain using (9)

$$I_{n-s} = \int_0^1 f(u)[\phi_1(u) * \phi_2(u) * \cdots * \phi_{n-s}(u)] \cdot \int_0^{1-u} [\phi_{n-s+1}(v) * \cdots * \phi_n(v)]\,dv\,du. \tag{10}$$

Alternatively, using the expression form for $\mathscr{I}_k(0)$ yields

$$I_{n-s} = \int_0^1 \{f(u)[\phi_1(u) * \phi_2(u) * \cdots * \phi_{n-s}(u)]\} * \phi_{n-s+1}(u) * \cdots * \phi_n(u)\,du. \tag{11}$$

For $f(\cdot) \in \mathscr{C}$ and $\phi_i(\cdot) \in \mathscr{K}$, the integrand in (11) is clearly a function of class $\mathscr{K}$; hence the integral exists and converges absolutely. Expressions (10) and (11) are both useful. Setting $\phi_i(t) = t^{\alpha_i - 1}$, $\alpha_i > 0$, $i = 1, 2, \cdots, n$, and using the form (10), we can easily obtain the results in [6].

4. Example. Consider the integral

$$\iiint_{t_1+t_2+t_3 \leqq 1} f(t_1 + t_2) J_0(t_1) J_0(t_2) t_3^{-1/2} \, dt_1 \, dt_2 \, dt_3, \tag{12}$$

where $J_0(\cdot)$ is the Bessel function of the first kind of order zero. Following Mikusiński [4], let $\{\phi(t)\}$ denote the function $\phi(t)$ and s denote the differential operator; then

$$\{J_0(t)\}^2 = (1/\sqrt{s^2 + 1})^2 = \{\sin t\}.$$

Using the form (10), we can write (12) as

$$\int_0^1 f(u) \sin u \int_0^{1-u} t_3^{-1/2} \, dt_3 \, du = 2 \int_0^1 f(u) \sqrt{1 - u} \sin u \, du.$$

Acknowledgment. The author wishes to thank Professor Jan Mikusiński for helpful discussions.

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SIAM J. MATH. ANAL.
Vol. 2, No. 1, February 1971

ON THE METHOD OF STATIONARY PHASE FOR MULTIPLE INTEGRALS*

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Abstract. Asymptotic formulas are proved for double integrals of the type

$$\iint_G e^{iks(x,y)}t(x, y)\,dx\,dy \qquad \text{for} \quad k \to \infty,$$

in which G is a domain of the form $p_1(y) \leqq x \leqq p_2(y)$, $d_1 \leqq y \leqq d_2$, where d_1, d_2 are constants and $p_1(y)$, $p_2(y)$ are continuous. The functions $s(x, y)$ and $t(x, y)$ are real and continuously differentiable, and G contains exactly one stationary point of the integrand, that is, a point at which $\partial s/\partial x$ and $\partial s/\partial y$ both vanish.

Several applications are made, including a vibrating potential of the form

$$\iint_G u(Q)\frac{e^{ikr}}{r}\,df,$$

and a triple integral

$$\iiint_H e^{iks(x,y,z)}t(x, y, z)\,dx\,dy\,dz.$$

Introduction. The method of stationary phase can be applied to double integrals of the form

$$\text{(A)} \qquad \iint_G e^{iks(x,y)}t(x, y)\,dx\,dy.$$

A survey of the work in this field together with an extensive bibliography was given by Chako [1]. Almost at the same time the thesis of Boin [2] appeared which covers the same field. However, some of the proofs in these publications are incomplete, and in places incorrect results are given. In particular, this concerns the contribution of the boundary of the domain G to the asymptotic behavior, and the estimate of the remainder term. In the one-dimensional case van der Corput [3] has shown that a boundary point gives a significant contribution; in this paper we prove that this is also true for multiple integrals.

In § 1 we first prove a theorem on the asymptotic behavior of the integral

$$I_1 = \int_0^a e^{ikx^2}f(x)\,dx \quad \text{for} \quad k \to \infty.$$

The asymptotic behavior of this integral is well known but we give an explicit bound for the remainder, which appears to be new.

In the more general case,

$$I_2 = \int_0^a e^{ikg(x)}f(x)\,dx$$

* Received by the editors October 10, 1969.

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with $g(0) = g'(0) = 0$, $g''(x) > 0$ for $0 \leqq x \leqq a$; we can transform this integral into the type I_1 by introducing the new variable $y = \sqrt{g(x)}$. It is then necessary to introduce the inverse function $x = k(y)$ of $y = \sqrt{g(x)}$, and Theorem 2 gives bounds for the derivatives of this inverse function.

With the help of these theorems we can find an estimate for the remainder by an asymptotic formula for an integral of the type I_2, which is useful in dealing with the asymptotic behavior of integrals of the type (A) when the domain G contains a stationary point of $s(x, y)$. We discuss this in four applications. Then in Application 5 we deal with a vibrating potential and in Application 6 with a three-dimensional integral.

1. Main theorems.

THEOREM 1. *If $f(x)$ is N times continuously differentiable for $0 \leqq x \leqq a$, $N \geqq 2$, $a > 0$, then for $k > 0$,*

$$I(k) = \int_0^a e^{ikx^2} f(x)\,dx = A_N(k) + B_N(k) + R_N(k),$$

where

$$A_N(k) = \sum_{n=0}^{N-1} \frac{1}{2\,n!}\Gamma(\tfrac{1}{2}n + \tfrac{1}{2})e^{(n+1)\pi i/4} f^{(n)}(0) k^{-(n+1)/2}, \tag{1.1}$$

$$B_N(k) = \sum_{n=0}^{N-1} b_{n+1} k^{-(n+1)}, \tag{1.2}$$

$$b_{n+1} = (-i)^{n+1} e^{ika^2} \sum_{\nu=0}^{n} (-1)^\nu \frac{(2n-\nu)!}{\nu!(n-\nu)!}(2a)^{-2n+\nu-1} f^{(\nu)}(a), \tag{1.3}$$

$$\begin{aligned} |R_N(k)| \leqq {} & \frac{2^{-3/2}\Gamma(N/2 - 1/2)}{(N-1)!} M_N k^{-(N+1)/2} \\ & + b^{-2N-1} \sum_{n=0}^{N-1} \frac{(2N-n)!}{(N-n)!} \frac{|f^{(n)}(a)|}{n!} b^n \cdot k^{-N-1}, \end{aligned} \tag{1.4}$$

$$M_N = \max |f^{(N)}(x)| \quad \textit{for} \quad 0 \leqq x \leqq a, \quad b = a\sqrt{2}.$$

Proof. Following a paper by A. Erdélyi [4] we consider the function

$$G_{n+1}(x) = \frac{(-1)^{n+1}}{n!} \int_x^\infty (z - x)^n e^{ikz^2}\,dz, \tag{1.5}$$

$$n = 0, 1, 2, \cdots, \quad x \geqq 0,$$

the path of integration being the half-ray $z = x + se^{\pi i/4}$, $s \geqq 0$. Then we have

$$G_{n+1}(0) = (-1)^{n+1} \frac{1}{2\,n!}\Gamma(n/2 + 1/2)e^{(n+1)\pi i/4} k^{-(n+1)/2}, \tag{1.6}$$

$$G_{n+1}(a) = (-1)^{n+1} \frac{1}{n!} e^{(n+1)\pi i/4} e^{ika^2} \int_0^\infty s^n e^{-k(bs+s^2)} e^{ikbs}\,ds. \tag{1.7}$$

We denote the integral in (1.7) by I_n.

Now we have

$$e^{-ks^2} = \sum_{l=0}^{L} \frac{(-ks^2)^l}{l!} + (-1)^{L+1} \frac{k^{L+1} s^{2L+2}}{(L+1)!} e^t$$

for a suitable t, where $-ks^2 < t < 0$. Thus

$$\begin{aligned} I_n &= \sum_{l=0}^{L} \frac{(-k)^l}{l!} \int_0^\infty s^{n+2l} e^{-kbs(1-i)}\, ds + T_{L+1} \\ &= \sum_{l=0}^{L} \frac{(-1)^l}{l!} k^{-n-l-1} b^{-n-2l-1} \left(\frac{1+i}{2}\right)^{n+2l+1} (n+2l)! + T_{L+1}, \end{aligned} \tag{1.8}$$

$$\begin{aligned} |T_{L+1}| &\leqq \frac{k^{L+1}}{(L+1)!} \int_0^\infty s^{n+2L+2} e^{-kbs}\, ds \\ &\leqq \frac{k^{L+1}}{(L+1)!} (n+2L+2)!(kb)^{-n-2L-3}. \end{aligned} \tag{1.9}$$

From (1.8), (1.9) we deduce that

$$I_n = \sum_{l=0}^{L} c_{n,l} k^{-n-l-1} + \theta \frac{(n+2L+2)!}{(L+1)!} b^{-n-2L-3} k^{-n-L-2}, \qquad |\theta| \leqq 1, \tag{1.10}$$

with

$$c_{n,l} = (-1)^l \frac{(n+2l)!}{l!} e^{(n+2l+1)\pi i/4} (2a)^{-n-2l-1}. \tag{1.11}$$

In (1.10) we take $L = N - n - 1$, $n = 0, 1, \cdots, N-1$. Then it follows from (1.7) that

$$\begin{aligned} G_{n+1}(a) = \frac{(-1)^{n+1}}{n!} e^{(n+1)\pi i/4} e^{ika^2} \Bigg\{ &\sum_{l=0}^{N-n-1} c_{n,l} k^{-n-l-1} \\ &+ \theta \frac{(2N-n)!}{(N-n)!} b^{-2N+n-1} k^{-N-1} \Bigg\}. \end{aligned} \tag{1.12}$$

Integration by parts gives

$$\begin{aligned} I(k) = &\sum_{n=0}^{N-1} (-1)^{n+1} f^{(n)}(0) G_{n+1}(0) + \sum_{n=0}^{N-1} (-1)^n f^{(n)}(a) G_{n+1}(a) \\ &+ (-1)^N \int_0^a f^{(N)}(x) G_N(x)\, dx. \end{aligned} \tag{1.13}$$

From (1.6) it follows that the first sum in (1.13) equals $A_N(k)$. From (1.12) we deduce

$$\begin{aligned} C_N(k) &= \sum_{n=0}^{N-1} (-1)^n f^{(n)}(a) G_{n+1}(a) \\ &= -\sum_{n=0}^{N-1} \frac{e^{(n+1)\pi i/4}}{n!} f^{(n)}(a) e^{ika^2} \sum_{l=0}^{N-n-1} c_{n,l} k^{-n-l-1} \\ &\quad + \theta b^{-2N-1} \sum_{n=0}^{N-1} \frac{(2N-n)!}{(N-n)!} \frac{|f^{(n)}(a)|}{n!} b^n \cdot k^{-N-1}. \end{aligned} \tag{1.14}$$

The first sum on the right-hand side of (1.14) equals $B_N(k)$ because the coefficient of $k^{-\nu-1}$, $\nu = 0, 1, \cdots, N-1$, in this sum is

$$d_{\nu+1} = -\sum_{n=0}^{\nu} \frac{e^{(n+1)\pi i/4}}{n!} e^{ika^2} c_{n,\nu-n} f^{(n)}(a).$$

In combination with (1.11) this yields

$$d_{\nu+1} = e^{ika^2} \sum_{n=0}^{\nu} \frac{(2\nu-n)!}{n!(\nu-n)!}(-1)^{\nu-n+1} i^{\nu+1}(2a)^{-2\nu+n-1} f^{(n)}(a) = b_{\nu+1}.$$

For $N \geqq 2$ we have

$$\begin{aligned}
&\left|\int_0^a f^{(N)}(x) G_N(x)\,dx\right| \\
&= \frac{1}{(N-1)!}\left|\int_0^a f^{(N)}(x) e^{ikx^2}\left\{\int_0^\infty s^{N-1} e^{-k(\sqrt{2}xs+s^2)} e^{ik\sqrt{2}xs}\,ds\right\} dx\right| \\
&= \frac{1}{(N-1)!}\left|\int_0^\infty \left\{\int_0^a f^{(N)}(x) e^{ikx^2} e^{-k\sqrt{2}xs} e^{ik\sqrt{2}xs}\,dx\right\} e^{-ks^2} s^{N-1}\,ds\right| \\
&\leqq \frac{M_N}{(N-1)!}\int_0^\infty (k\sqrt{2})^{-1} s^{N-2} e^{-ks^2}\,ds = \frac{2^{-3/2}\Gamma(N/2-1/2)}{(N-1)!} M_N k^{-(N+1)/2}.
\end{aligned}$$

This completes the proof.

THEOREM 2. *Assume: The real function* $g(x)$ *is* $N+2$ *times continuously differentiable for* $0 \leqq x \leqq p$, $p > 0$, $N \geqq 2$;

$$g(0) = g'(0) = 0, \quad g^{(2)}(x) \geqq m \quad \textit{for } 0 \leqq x \leqq p, \quad 0 < m \leqq 1.$$

Let $x = k(y)$, $0 \leqq y \leqq a = \sqrt{g(p)}$ *be the inverse function of* $y = \sqrt{g(x)}$ *and*

$$|g^{(l)}(x)| \leqq B_l \qquad \textit{for } 0 \leqq x \leqq p, \quad l = 2, 3, \cdots, N+2.$$

Then

$$|k^{(n)}(y)| \leqq A^{n-1} P_n(B_3, \cdots, B_{n+2}) m^{-((5/2)n-2)},$$
$$0 \leqq y \leqq a, \quad n = 1, 2, \cdots, N,$$

where $P_n(B_3, \cdots, B_{n+2})$ *is a polynomial in* $B_3, \cdots, B_{n+2}$, *with coefficients depending only on* n *and* $A = \max(1, p)$.

Proof. For $0 \leqq x \leqq p$ we have

$$\begin{aligned}
g(x) &= \int_0^x g'(t)\,dt = \int_0^x \int_0^t g^{(2)}(s)\,ds\,dt \\
&= \int_0^x (x-s) g^{(2)}(s)\,ds = x^2 \int_0^1 (1-u) g^{(2)}(xu)\,du.
\end{aligned}$$

Write

$$\Psi(x) = \left\{\int_0^1 (1-u) g^{(2)}(xu)\,du\right\}^{1/2}. \tag{1.15}$$

Then

$$\{\Psi(x)\}^2 = \int_0^1 (1-u)g^{(2)}(xu)\,du, \qquad \sqrt{g(x)} = x\Psi(x). \tag{1.16}$$

From (1.15) it follows that

$$\sqrt{\frac{m}{2}} \leqq \Psi(x) \leqq \sqrt{\frac{B_2}{2}}, \tag{1.17}$$

and from (1.16),

$$2\Psi(x)\Psi'(x) = \int_0^1 u(1-u)g^{(3)}(xu)\,du.$$

Thus $|2\Psi(x)\Psi'(x)| \leqq \frac{1}{6}B_3$.

Combining this with (1.17) we see that

$$|\Psi'(x)| \leqq \frac{B_3}{6\sqrt{2m}}. \tag{1.18}$$

From (1.16) we deduce, for $n = 1, \cdots, N$,

$$\sum_{k=0}^{n} \binom{n}{k} \Psi^{(k)}(x)\Psi^{(n-k)}(x) = \int_0^1 (1-u)u^n g^{(n+2)}(xu)\,du = I_n, \tag{1.19}$$

$$|I_n| \leqq B_{n+2}\int_0^1 (1-u)u^n\,du = \frac{B_{n+2}}{(n+1)(n+2)}. \tag{1.20}$$

Next we shall prove

$$|\Psi^{(n)}(x)| \leqq \frac{Q_n(B_3, B_4, \cdots, B_{n+2})}{m^{n-1/2}}, \tag{1.21}$$

$$0 \leqq x \leqq p, \quad 1 \leqq n \leqq N,$$

where Q_n is a polynomial in $B_3, \cdots, B_{n+2}$, with coefficients depending only on n.

For $n = 1$, (1.21) is true on account of (1.18).

From (1.19) we deduce

$$2\Psi(x)\Psi^{(n)}(x) = I_n - \sum_{k=1}^{n-1} \binom{n}{k} \Psi^{(k)}(x)\Psi^{(n-k)}(x). \tag{1.22}$$

Suppose that (1.21) holds for $\Psi^{(k)}(x)$, $k = 1, \cdots, n-1$; then it follows from (1.22), (1.20) and (1.17) that

$$|\Psi^{(n)}(x)| \leqq \frac{B_{n+2}}{2(n+1)(n+2)(m/2)^{1/2}} + \frac{1}{2^{1/2}}\sum_{k=1}^{n-1}\binom{n}{k}$$
$$\cdot \frac{Q_k(B_3, \cdots, B_{k+2})Q_{n-k}(B_3, \cdots, B_{n-k+2})}{m^{n-1/2}}.$$

The validity of (1.21) then follows by induction.

We now consider $f(x) = \sqrt{g(x)}$, $0 \leqq x \leqq p$. Then

$$f'(x) = \frac{g'}{2\sqrt{g}}(x) \quad \text{for} \quad 0 < x \leqq p.$$

(We write $g'(x)/2\sqrt{g(x)} \equiv (g'/2\sqrt{g})(x)$.) Since $0 < (g'^2/g)(x) = 2g^{(2)}(t)$ for a suitable $t, 0 < t < x$, we have

$$(1.23) \qquad f'(0) = \sqrt{\frac{g^{(2)}(0)}{2}}, \quad \sqrt{\frac{m}{2}} \leqq f'(x) \leqq \sqrt{\frac{B_2}{2}} \quad \text{for} \quad 0 \leqq x \leqq p.$$

From $f(x) = x\Psi(x)$ it follows that

$$f^{(n)}(x) = n\Psi^{(n-1)}(x) + x\Psi^{(n)}(x),$$
$$0 \leqq x \leqq p, \quad n = 1, \cdots, N.$$

On account of (1.21) we have for $2 \leqq n \leqq N$,

$$|f^{(n)}(x)| \leqq n\frac{Q_{n-1}(B_3, \cdots, B_{n+1})}{m^{n-3/2}} + p\frac{Q_n(B_3, \cdots, B_{n+2})}{m^{n-1/2}}$$
$$\leqq A\frac{nQ_{n-1} + Q_n}{m^{n-1/2}}$$

with $A = \max(1, p)$. So

$$(1.24) \qquad |f^{(n)}(x)| \leqq A\frac{R_n(B_3, \cdots, B_{n+2})}{m^{n-1/2}},$$
$$0 \leqq x \leqq p, \quad 2 \leqq n \leqq N,$$

where R_n is a polynomial in $B_3, \cdots, B_{n+2}$, with coefficients depending only on n.

It is known that

$$(1.25) \qquad k^{(n)}(y) = \frac{X_n\{f'(x), \cdots, f^{(n)}(x)\}}{\{f'(x)\}^{2n-1}},$$
$$n = 1, \cdots, N, \quad 0 \leqq y \leqq a,$$

where X_n is a polynomial in $f'(x), \cdots, f^{(n)}(x)$. If

$$X_n = \sum a_{\alpha_1,\alpha_2,\cdots,\alpha_n}\{f'(x)\}^{\alpha_1}\{f^{(2)}(x)\}^{\alpha_2} \cdots \{f^{(n)}(x)\}^{\alpha_n},$$

then we have for each term of this polynomial,

$$(1.26) \qquad \sum_{\nu=1}^{n} \alpha_\nu = n - 1, \quad \sum_{\nu=1}^{n} \nu\alpha_\nu = 2(n-1).$$

We write (1.25) in the form

$$(1.27) \qquad k^{(n)}(y) = \sum a_{\alpha_1,\cdots,\alpha_n}\frac{\{f^{(2)}(x)\}^{\alpha_2} \cdots \{f^{(n)}(x)\}^{\alpha_n}}{\{f'(x)\}^{2n-1-\alpha_1}}.$$

Combining this with (1.23) and (1.24), we see that

$$|k^{(n)}(y)| \leqq A^{n-1}\frac{Z_n(B_3, \cdots, B_{n+2})}{m^{3(n-1)/2}}\frac{1}{(m/2)^{n-1/2}}, \tag{1.28}$$

because by virtue of (1.26),

$$\sum_{\nu=2}^{n} \alpha_\nu \leqq n-1, \qquad \sum_{\nu=2}^{n} (\nu - \tfrac{1}{2})\alpha_\nu \leqq \tfrac{3}{2}(n-1),$$

Z_n is a polynomial in $B_3, \cdots, B_{n+2}$ with coefficients depending only on n.

Theorem 2 now follows from (1.28).

Remark. We can find bounds for $k'(y), k^{(2)}(y). k^{(3)}(y)$ for $0 \leqq y \leqq a$ by a straightforward calculation. We write k_ν instead of $k^{(\nu)}$.

If $y = \sqrt{g(x)}$, then $dy/dx = (g_1/2\sqrt{g})(x)$, so $k_1(y) = (2\sqrt{g}/g_1)(x)$ for $0 < y \leqq a$. Since

$$0 < \frac{g}{g_1^2}(x) = \frac{g_1}{2g_1g_2}(t) \leqq \frac{1}{2m} \quad \text{for} \quad 0 < x \leqq p, \qquad 0 < t < x,$$

we have

$$\lim_{x \downarrow 0} (g/g_1^2)(x) = 1/(2g_2(0)).$$

Thus $k_1(0) = 2^{1/2}\{g_2(0)\}^{-1/2}$, $0 < k_1(y) \leqq 2^{1/2}m^{-1/2}$ for $0 \leqq y \leqq a$. Moreover,

$$y_2 = \frac{1}{4}\frac{2gg_2 - g_1^2}{g^{3/2}}(x), \qquad k_2(y) = -\frac{y_2}{y_1^3} = -2\frac{2gg_2 - g_1^2}{g_1^3}(x) \quad \text{for} \quad 0 < y \leqq a.$$

Next

$$\frac{2gg_2 - g_1^2}{g_1^3}(x) = \frac{2gg_3}{3g_1^2g_2}(t), \qquad 0 < t < x.$$

Thus

$$\lim_{x \downarrow 0}\frac{2gg_2 - g_1^2}{g_1^3}(x) = \tfrac{1}{3}g_3(0)\{g_2(0)\}^{-2},$$

$$\left|\frac{2gg_2 - g_1^2}{g_1^3}(x)\right| \leqq \tfrac{1}{3}B_3m^{-2} \quad \text{for} \quad 0 < x \leqq p.$$

Hence,

$$k_2(0) = -\tfrac{2}{3}g_3(0)\{g_2(0)\}^{-2}, \qquad |k_2(y)| \leqq \tfrac{2}{3}B_3m^{-2} \quad \text{for} \quad 0 \leqq y \leqq a.$$

Finally,

$$k_3(y) = 4\frac{g^{1/2}}{g_1}\frac{3g_2(2gg_2 - g_1^2) - 2gg_1g_3}{g_1^4}(x) \quad \text{for} \quad 0 < y \leqq a.$$

We have

$$
\begin{aligned}
A(x) &= \frac{3g_2(2gg_2 - g_1^2) - 2gg_1g_3}{g_1^4}(x) \\
&= \frac{3g_3(2gg_2 - g_1^2) + 3g_2 2gg_3 - 2g_1^2g_3 - 2gg_2g_3 - 2gg_1g_4}{4g_1^3g_2}(t) \\
&= \frac{5g_3(2gg_2 - g_1^2)}{4g_1^3g_2}(t) - \frac{1}{2}\frac{g}{g_1^2}\frac{g_4}{g_2}(t), \qquad 0 < t < x.
\end{aligned}
$$

Hence,

$$
\lim_{x\downarrow 0} A(x) = \tfrac{5}{12}\{g_3(0)\}^2\{g_2(0)\}^{-3} - \tfrac{1}{4}g_4(0)\{g_2(0)\}^{-2},
$$

$$
|A(x)| \leqq \tfrac{5}{12}B_3^2m^{-3} + \tfrac{1}{4}B_4m^{-2} \quad \text{for} \quad 0 < x \leqq p.
$$

This yields

$$
k_3(0) = \tfrac{5}{6}\sqrt{2}\{g_3(0)\}^2\{g_2(0)\}^{-7/2} - \tfrac{1}{2}\sqrt{2}g_4(0)\{g_2(0)\}^{-5/2},
$$

$$
|k_3(y)| \leqq \tfrac{5}{6}\sqrt{2}B_3^2m^{-7/2} + \tfrac{1}{2}\sqrt{2}B_4m^{-5/2}, \qquad 0 \leqq y \leqq a.
$$

2. Application 1. We suppose that $s(x, y)$ and $t(x, y)$ are real functions defined in a domain

$$
H\{p_1(y) \leqq x \leqq p_2(y), e_1 \leqq y \leqq e_2\},
$$

where $e_1 < 0$, $e_2 > 0$, $p_1(y) < 0$ and $p_1(y)$ is continuous for $e_1 \leqq y \leqq e_2$, $p_2(y) > 0$ and $p_2(y)$ is continuous for $e_1 \leqq y \leqq e_2$; $s(x, y)$ and $t(x, y)$ have as many continuous partial derivatives in the domain H as is necessary for the correctness of the proofs. Furthermore, we suppose

$$
\frac{\partial s}{\partial x}(0,0) = 0, \quad \frac{\partial^2 s}{\partial x^2} \neq 0 \quad \text{in } H. \tag{2.1}
$$

From $(\partial s/\partial x)(x, y) = 0$ it follows $x = \varphi(y)$, $\varphi(0) = 0$ in a y-interval $[\varepsilon_1, \varepsilon_2]$, $e_1 \leqq \varepsilon_1 < 0 < \varepsilon_2 \leqq e_2$. So there exist numbers d_1 and d_2, $\varepsilon_1 \leqq d_1 < 0 < d_2 \leqq \varepsilon_2$, such that

$$
p_1(y) < \varphi(y) < p_2(y) \quad \text{for} \quad d_1 \leqq y \leqq d_2.
$$

We now determine the asymptotic behavior of

$$
I_1(k) = \int_{p_1}^{p_2} e^{iks(x,y)}t(x, y)\,dx, \quad y \text{ constant}, \quad d_1 \leqq y \leqq d_2, \tag{2.2}
$$

for $k \to \infty$. We have

$$
\begin{aligned}
I_1(k) &= e^{iks(\varphi,y)}\left\{\int_{\varphi}^{p_2} e^{ik\{s(x,y)-s(\varphi,y)\}}t(x, y)\,dx + \int_{p_1}^{\varphi} e^{ik\{s(x,y)-s(\varphi,y)\}}t(x, y)\,dx\right\} \\
&= e^{iks(\varphi,y)}\{I_{11} + I_{12}\}.
\end{aligned}
$$

We write

$$I_{11} = \int_{\varphi}^{p_2} e^{ikg(x)} h(x)\,dx, \quad g(x) = s(x, y) - s(\varphi, y), \quad h(x) = t(x, y).$$

Then with $g_\nu \equiv g^{(\nu)}$, we have

$$g(\varphi) = g_1(\varphi) = 0.$$

Suppose first that $\partial^2 s/\partial x^2 > 0$ in H; then there exists a number m such that

$$\frac{\partial^2 s}{\partial x^2} \geqq m > 0 \quad \text{in } H. \tag{2.3}$$

Thus $g_2(x) \geqq m$ for $p_1 \leqq x \leqq p_2, d_1 \leqq y \leqq d_2$.

Let $x = k(\bar{y})$, $0 \leqq \bar{y} \leqq a_2 = \sqrt{g(p_2)}$ be the inverse function of $\bar{y} = \sqrt{g(x)}$, in $\varphi \leqq x \leqq p_2$. Then we have

$$I_{11} = \int_0^{a_2} e^{ik\bar{y}^2} f(\bar{y})\,d\bar{y}, \qquad f(\bar{y}) = h(k(\bar{y}))k_1(\bar{y}).$$

If we apply Theorem 1, with $N = 3$, we find

$$I_{11} = A_3(k) + B_3(k) + R_3(k), \tag{2.4}$$

$$A_3(k) = \tfrac{1}{2}\Gamma(\tfrac{1}{2})e^{\pi i/4} f(0)k^{-1/2} + \tfrac{1}{2} i f_1(0)k^{-1} + \tfrac{1}{4}\Gamma(\tfrac{3}{2})e^{3\pi i/4} f_2(0)k^{-3/2} \tag{2.5}$$

with

$$\begin{aligned} f(0) &= h(k(0))k_1(0) = h(\varphi)k_1(0) = t(\varphi, y)2^{1/2}\{g_2(\varphi)\}^{-1/2} \\ &\qquad\qquad = 2^{1/2} t(\varphi, y) \cdot \left\{\frac{\partial^2 s}{\partial x^2}(\varphi, y)\right\}^{-1/2}, \end{aligned}$$

$$\begin{aligned} f_1(0) &= h_1(k(0))k_1^2(0) + h(k(0))k_2(0) = h_1(\varphi)k_1^2(0) + h(\varphi)k_2(0) \\ &= 2\frac{\partial t}{\partial x}(\varphi, y)\left\{\frac{\partial^2 s}{\partial x^2}(\varphi, y)\right\}^{-1} - \frac{2}{3} t(\varphi, y)\frac{\partial^3 s}{\partial x^3}(\varphi, y)\left\{\frac{\partial^2 s}{\partial x^2}(\varphi, y)\right\}^{-2}, \end{aligned}$$

$$\begin{aligned} f_2(0) &= h_2(\varphi)k_1^3(0) + 3h_1(\varphi)k_1(0)k_2(0) + h(\varphi)k_3(0) \\ &= \frac{\partial^2 t}{\partial x^2}(\varphi, y)2^{3/2}\{g_2(\varphi)\}^{-3/2} + 3\frac{\partial t}{\partial x}(\varphi, y)2^{1/2}\{g_2(\varphi)\}^{-1/2} \cdot -\tfrac{2}{3} g_3(\varphi)\{g_2(\varphi)\}^{-2} \\ &\quad + t(\varphi, y)[\tfrac{5}{6}\sqrt{2}\{g_3(\varphi)\}^2\{g_2(\varphi)\}^{-7/2} - \tfrac{1}{2}\sqrt{2} g_4(\varphi)\{g_2(\varphi)\}^{-5/2}] \\ &= 2^{3/2}\frac{\partial^2 t}{\partial x^2}(\varphi, y)\left\{\frac{\partial^2 s}{\partial x^2}(\varphi, y)\right\}^{-3/2} - 2^{3/2}\frac{\partial t}{\partial x}(\varphi, y)\left\{\frac{\partial^2 s}{\partial x^2}(\varphi, y)\right\}^{-5/2}\frac{\partial^3 s}{\partial x^3}(\varphi, y) \\ &\quad + t(\varphi, y)\left[\frac{5}{6}\sqrt{2}\left\{\frac{\partial^2 s}{\partial x^2}(\varphi, y)\right\}^{-7/2}\left\{\frac{\partial^3 s}{\partial x^3}(\varphi, y)\right\}^2 \right. \\ &\qquad\qquad \left. - \frac{1}{2}\sqrt{2}\left\{\frac{\partial^2 s}{\partial x^2}(\varphi, y)\right\}^{-5/2}\frac{\partial^4 s}{\partial x^4}(\varphi, y)\right], \end{aligned}$$

$$(2.6) \qquad B_3(k) = b_1 k^{-1} + b_2 k^{-2} + b_3 k^{-3},$$

$$\begin{aligned} b_1 &= -ie^{ika_2^2}(2a_2)^{-1}f(a_2) = -ie^{ikg(p_2)}2^{-1}\{g(p_2)\}^{-1/2}h(k(a_2))k_1(a_2) \\ &= -ie^{ik\{s(p_2,y)-s(\varphi,y)\}}2^{-1}\{g(p_2)\}^{-1/2}h(p_2)2\{g(p_2)\}^{1/2}\{g_1(p_2)\}^{-1} \\ &= -ie^{ik\{s(p_2,y)-s(\varphi,y)\}}t(p_2,y)\left\{\frac{\partial s}{\partial x}(p_2,y)\right\}^{-1}, \end{aligned}$$

$$(2.7) \qquad b_2 = -e^{ika_2^2}\{2(2a_2)^{-3}f(a_2) - (2a_2)^{-2}f_1(a_2)\},$$

$$(2.8) \qquad b_3 = ie^{ika_2^2}\{12(2a_2)^{-5}f(a_2) - b(2a_2)^{-4}f_1(a_2) + (2a_2)^{-3}f_2(a_2)\}.$$

From $a_2 = \sqrt{g(p_2)}$ it follows that

$$a_2^2 = g(p_2) = s(p_2,y) - s(\varphi,y) = \frac{1}{2}\frac{\partial^2 s}{\partial x^2}(t,y)(p_2-\varphi)^2$$

for a suitable $t(=t(y))$, where $\varphi < t < p_2$.

Since $p_2 - \varphi \geqq \alpha > 0$ for $d_1 \leqq y \leqq d_2$ it follows from (2.3) that $s(p_2,y) - s(\varphi,y) \geqq \frac{1}{2}m\alpha^2$, $d_1 \leqq y \leqq d_2$. Thus

$$(2.9) \qquad a_2^{-1} = O(1) \quad \text{uniformly in } y, \qquad d_1 \leqq y \leqq d_2.$$

Further, we have

$$\begin{aligned} f(a_2) &= h(k(a_2))k_1(a_2) = h(p_2)k_1(a_2) = t(p_2,y)k_1(a_2), \\ f_1(a_2) &= h_1(p_2)k_1^2(a_2) + h(p_2)k_2(a_2) = \frac{\partial t}{\partial x}(p_2,y)k_1^2(a_2) + t(p_2,y)k_2(a_2), \\ f_2(a_2) &= h_2(p_2)k_1^3(a_2) + 3h_1(p_2)k_1(a_2)k_2(a_2) + h(p_2)k_3(a_2) \\ &= \frac{\partial^2 t}{\partial x^2}(p_2,y)k_1^3(a_2) + 3\frac{\partial t}{\partial x}(p_2,y)k_1(a_2)k_2(a_2) + t(p_2,y)k_3(a_2). \end{aligned}$$

If $A = \max(1,p)$ with $p = \max\{p_2(y) - \varphi(y)\}$, $d_1 \leqq y \leqq d_2$,

$$B_k = \max\left|\frac{\partial^k s}{\partial x^k}(x,y)\right| \quad \text{on } H,$$

then it follows from Theorem 2 that

$$(2.10) \qquad f(a_2) = O(1), \quad f_1(a_2) = O(1), \quad f_2(a_2) = O(1) \quad \text{uniformly in } y, \qquad d_1 \leqq y \leqq d_2.$$

It then follows from (2.7), (2.8), (2.9), (2.10) that

$$(2.11) \qquad b_2 = O(1), \quad b_3 = O(1) \quad \text{uniformly in } y, \quad d_1 \leqq y \leqq d_2.$$

Next we consider the remainder $R_3(k)$. By Theorem 1 we have

$$(2.12) \qquad |R_3(k)| \leqq 2^{-5/2}M_3k^{-2} + b^{-7}\left\{\sum_{n=0}^{2}\frac{(6-n)!}{(3-n)!}\frac{|f_n(a_2)|}{n!}b^n\right\}k^{-4}$$

with $b = a_2\sqrt{2}$, $M_3 = \max|f_3(\bar{y})|$, $0 \leqq \bar{y} \leqq a_2$.

We have

$$f_3(\bar y) = h_3(k(\bar y))k_1^4(\bar y) + 6h_2(k(\bar y))k_1^2(\bar y)k_2(\bar y) + 3h_1(k(\bar y))k_2^2(\bar y) + 4h_1(k(\bar y))k_1(\bar y)k_3(\bar y) + h(k(\bar y))k_4(\bar y)$$
$$= \frac{\partial^3 t}{\partial x^3}(x, y)k_1^4(\bar y) + 6\frac{\partial^2 t}{\partial x^2}(x, y)k_1^2(\bar y)k_2(\bar y) + 3\frac{\partial t}{\partial x}(x, y)k_2^2(\bar y) + 4\frac{\partial t}{\partial x}(x, y)k_1(\bar y)k_3(\bar y) + t(x, y)k_4(\bar y).$$

From this it follows by Theorem 2 that

$$f_3(\bar y) = O(1) \quad \text{for} \quad 0 \leqq \bar y \leqq a_2, \quad \text{uniformly in } y, \; d_1 \leqq y \leqq d_2.$$

Thus

$$M_3 = O(1) \quad \text{uniformly in } y, \quad d_1 \leqq y \leqq d_2. \tag{2.13}$$

From (2.9), (2.12), (2.13) we deduce

$$R_3(k) = O(k^{-2}) \quad \text{uniformly in } y, \quad d_1 \leqq y \leqq d_2, \quad k \geqq 1. \tag{2.14}$$

So we have proved

$$I_{11} = \tfrac{1}{2}\Gamma(\tfrac{1}{2})e^{\pi i/4}f(0)k^{-1/2} + \tfrac{1}{2}if_1(0)k^{-1} + \tfrac{1}{4}\Gamma(\tfrac{3}{2})e^{3\pi i/4}f_2(0)k^{-3/2} + b_1k^{-1} + O(k^{-2}) \tag{2.15}$$

uniformly in y, $d_1 \leqq y \leqq d_2$, $k > 1$.

Now we deal with

$$I_{12} = \int_{p_1}^{\varphi} e^{ik\{s(x,y)-s(\varphi,y)\}}t(x, y)\,dx = \int_{p_1}^{\varphi} e^{ikg(x)}h(x)\,dx.$$

Let $x = k(\bar y)$, $0 \leqq \bar y \leqq a_1 = \sqrt{g(p_1)}$ be the inverse function of $\bar y = \sqrt{g(x)}$, in $p_1 \leqq x \leqq \varphi$. Then

$$I_{12} = -\int_0^{a_1} e^{ik\bar y^2}{}^*f(y)\,dy, \qquad {}^*f(y) = h(k(\bar y))k_1(\bar y).$$

In this case we have

$$k_1(0) = -2^{1/2}\{g_2(\varphi)\}^{-1/2}, \quad k_2(0) = -\tfrac{2}{3}g_3(\varphi)\{g_2(\varphi)\}^{-2},$$
$$k_3(0) = -\tfrac{5}{6}\sqrt{2}\{g_3(\varphi)\}^2\{g_2(\varphi)\}^{-7/2} + \tfrac{1}{2}\sqrt{2}g_4(\varphi)\{g_2(\varphi)\}^{-5/2},$$
$${}^*f(0) = -f(0), \quad {}^*f_1(0) = f_1(0), \quad {}^*f_2(0) = -f_2(0).$$

Application of the method which led to (2.15) yields

$$I_{12} = -{}^*A_3(k) - {}^*B_3(k) - {}^*R_3(k),$$

where

$$\begin{aligned}
{}^*A_3(k) &= \tfrac{1}{2}\Gamma(\tfrac{1}{2})e^{\pi i/4}\,{}^*f(0)k^{-1/2} + \tfrac{1}{2}i\,{}^*f_1(0)k^{-1} + \tfrac{1}{4}\Gamma(\tfrac{3}{2})\,{}^*f_2(0)k^{-3/2},\\
{}^*B_3(k) &= {}^*b_1k^{-1} + {}^*b_2k^{-2} + {}^*b_3k^{-3},\\
{}^*b_1 &= -\,ie^{ika_1^2}(2a_1)^{-1}\,{}^*f(a_1) = -\,ie^{ikg(p_1)}2^{-1}\{g(p_1)\}^{-1/2}h(k(a_1))k_1(a_1)\\
&= -\,ie^{ik\{s(p_1,y)-s(\varphi,y)\}}h(p_1)\{g_1(p_1)\}^{-1}\\
&= -\,ie^{ik\{s(p_1,y)-s(\varphi,y)\}}t(p_1,y)\left\{\frac{\partial s}{\partial x}(p_1,y)\right\}^{-1}.
\end{aligned}$$

Thus,

$$I_{12} = \tfrac{1}{2}\Gamma(\tfrac{1}{2})e^{\pi i/4}f(0)k^{-1/2} - \tfrac{1}{2}if_1(0)k^{-1} + \tfrac{1}{4}\Gamma(\tfrac{3}{2})e^{3\pi i/4}f_2(0)k^{-3/2} + {}^*b_1k^{-1} + O(k^{-2}) \tag{2.16}$$

uniformly in y, $d_1 \leqq y \leqq d_2$, $k > 1$. From (2.15) and (2.16) it follows that

$$\begin{aligned}
I_1(k) = {}&\Gamma(\tfrac{1}{2})e^{\pi i/4}f(0)e^{iks(\varphi,y)}k^{-1/2} + \tfrac{1}{2}\Gamma(\tfrac{3}{2})e^{3\pi i/4}f_2(0)e^{iks(\varphi,y)}k^{-3/2}\\
&- ie^{iks(p_2,y)}t(p_2,y)\left\{\frac{\partial s}{\partial x}(p_2,y)\right\}^{-1}k^{-1}\\
&+ ie^{iks(p_1,y)}t(p_1,y)\left\{\frac{\partial s}{\partial x}(p_1,y)\right\}^{-1}k^{-1} + O(k^{-2})
\end{aligned}\tag{2.17}$$

uniformly in y, $d_1 \leqq y \leqq d_2$, $k > 1$, with

$$f(0) = 2^{1/2}t(\varphi,y)\left\{\frac{\partial^2 s}{\partial x^2}(\varphi,y)\right\}^{-1/2},$$

$$\begin{aligned}
f_2(0) = {}&2^{3/2}\frac{\partial^2 t}{\partial x^2}(\varphi,y)\left\{\frac{\partial^2 s}{\partial x^2}(\varphi,y)\right\}^{-3/2} - 2^{3/2}\frac{\partial t}{\partial x}(\varphi,y)\left\{\frac{\partial^2 s}{\partial x^2}(\varphi,y)\right\}^{-5/2}\frac{\partial^3 s}{\partial x^3}(\varphi,y)\\
&+ t(\varphi,y)\left[\frac{5}{6}\sqrt{2}\left\{\frac{\partial^2 s}{\partial x^2}(\varphi,y)\right\}^{-7/2}\left\{\frac{\partial^3 s}{\partial x^3}(\varphi,y)\right\}^{-2}\right.\\
&\left.- \frac{1}{2}\sqrt{2}\left\{\frac{\partial^2 s}{\partial x^2}(\varphi,y)\right\}^{-5/2}\frac{\partial^4 s}{\partial x^4}(\varphi,y)\right].
\end{aligned}$$

If $\partial^2 s/\partial x^2 < 0$ in H, then we find by considering the complex conjugate of $I_1(k)$:

$$\begin{aligned}
I_1(k) = {}&\Gamma(\tfrac{1}{2})e^{-\pi i/4}f(0)e^{iks(\varphi,y)}k^{-1/2} + \tfrac{1}{2}\Gamma(\tfrac{3}{2})e^{-3\pi i/4}f_2(0)e^{iks(\varphi,y)}k^{-3/2}\\
&- ie^{iks(p_2,y)}t(p_2,y)\left\{\frac{\partial s}{\partial x}(p_2,y)\right\}^{-1}k^{-1}\\
&+ ie^{iks(p_1,y)}t(p_1,y)\left\{\frac{\partial s}{\partial x}(p_1,y)\right\}^{-1}k^{-1} + O(k^{-2})
\end{aligned}\tag{2.18}$$

uniformly in y, $d_1 \leqq y \leqq d_2$, $k > 1$, with

$$f(0) = 2^{1/2} t(\varphi, y) \left\{ -\frac{\partial^2 s}{\partial x^2}(\varphi, y) \right\}^{-1/2},$$

$$\begin{aligned} f_2(0) = {} & 2^{3/2} \frac{\partial^2 t}{\partial x^2}(\varphi, y) \left\{ -\frac{\partial^2 s}{\partial x^2}(\varphi, y) \right\}^{-3/2} \\ & - 2^{3/2} \frac{\partial t}{\partial x}(\varphi, y) \left\{ -\frac{\partial^2 s}{\partial x^2}(\varphi, y) \right\}^{-5/2} \cdot -\frac{\partial^3 s}{\partial x^3}(\varphi, y) \\ & + t(\varphi, y) \left[\frac{5}{6}\sqrt{2} \left\{ -\frac{\partial^2 s}{\partial x^2}(\varphi, y) \right\}^{-7/2} \left\{ -\frac{\partial^3 s}{\partial x^3}(\varphi, y) \right\}^2 \right. \\ & \left. - \frac{1}{2}\sqrt{2} \left\{ -\frac{\partial^2 s}{\partial x^2}(\varphi, y) \right\}^{-5/2} \cdot -\frac{\partial^4 s}{\partial x^4}(\varphi, y) \right]. \end{aligned}$$

3. Application 2. Our starting point is Application 1, but in addition we suppose that

$$\frac{\partial s}{\partial y}(0, 0) = 0, \tag{3.1}$$

$$D_2(x, y) = \left(\frac{\partial^2 s}{\partial x^2} \frac{\partial^2 s}{\partial y^2} - \left(\frac{\partial^2 s}{\partial x \partial y} \right)^2 \right)(x, y) \neq 0, \tag{3.2}$$

in the domain $G\{p_1(y) \leqq x \leqq p_2(y), d_1 \leqq y \leqq d_2\}$.

We shall determine the asymptotic behavior of

$$I(k) = \iint_G e^{iks(x,y)} t(x, y)\, dx\, dy \quad \text{for} \quad k \to \infty.$$

If we write $G(y) = s(\varphi, y) - s(0, 0)$, then

$$G_1(y) = \frac{\partial s}{\partial x}(\varphi, y) \frac{d\varphi}{dy} + \frac{\partial s}{\partial y}(\varphi, y) = \frac{\partial s}{\partial y}(\varphi, y),$$

$$\begin{aligned} G_2(y) &= \frac{\partial^2 s}{\partial x \partial y}(\varphi, y) \frac{d\varphi}{dy} + \frac{\partial^2 s}{\partial y^2}(\varphi, y) \\ &= -\left\{ \frac{\partial^2 s}{\partial x \partial y}(\varphi, y) \right\}^2 \Big/ \frac{\partial^2 s}{\partial x^2}(\varphi, y) + \frac{\partial^2 s}{\partial y^2}(\varphi, y) = \left(D_2 : \frac{\partial^2 s}{\partial x^2} \right)(\varphi, y) \end{aligned}$$

so that $G(0) = G_1(0) = 0$, $G_2(y) \neq 0$ for $d_1 \leqq y \leqq d_2$. By virtue of (2.2) we have $I(k) = \int_{d_1}^{d_2} I_1(k)\, dy$.

We distinguish four cases.

Case I. $\partial^2 s/\partial x^2 > 0$ *in* G, $D_2(x, y) > 0$ *in* G. Then it follows from the fact that (2.17) holds uniformly in y, $d_1 \leqq y \leqq d_2$, that

$$I(k) = H_1(k) + H_2(k) + H_3(k) + H_4(k) + O(k^{-2}) \tag{3.3}$$

with

$$H_1(k) = 2^{1/2}\Gamma(\tfrac{1}{2})e^{\pi i/4}e^{iks(0,0)}\int_{d_1}^{d_2} e^{ikG(y)}t(\varphi, y)\left\{\frac{\partial^2 s}{\partial x^2}(\varphi, y)\right\}^{-1/2} dy\cdot k^{-1/2}, \tag{3.3.1}$$

$$H_2(k) = \tfrac{1}{2}\Gamma(\tfrac{3}{2})e^{3\pi i/4}e^{iks(0,0)}\int_{d_1}^{d_2} e^{ikG(y)}f_2(0)\, dy\cdot k^{-3/2}, \tag{3.3.2}$$

$$H_3(k) = -i\int_{d_1}^{d_2} e^{iks(p_2,y)}t(p_2, y)\left\{\frac{\partial s}{\partial x}(p_2, y)\right\}^{-1} dy\cdot k^{-1}, \tag{3.3.3}$$

$$H_4(k) = i\int_{d_1}^{d_2} e^{iks(p_1,y)}t(p_1, y)\left\{\frac{\partial s}{\partial x}(p_1, y)\right\}^{-1} dy\cdot k^{-1}. \tag{3.3.4}$$

We write

$$t(\varphi, y)\left\{\frac{\partial^2 s}{\partial x^2}(\varphi, y)\right\}^{-1/2} = h(y). \tag{3.4}$$

We have

$$\begin{aligned}\int_{d_1}^{d_2} e^{ikG(y)}h(y)\, dy &= 2^{1/2}\Gamma(\tfrac{1}{2})e^{\pi i/4}h(0)\{G_2(0)\}^{-1/2}k^{-1/2}\\ &\quad - ie^{ikG(d_2)}\frac{h(d_2)}{G_1(d_2)}k^{-1} + ie^{ikG(d_1)}\frac{h(d_1)}{G_1(d_1)}k^{-1} + O(k^{-3/2}).\end{aligned} \tag{3.5}$$

If we put

$$r(x, y) = t(x, y)\left|\frac{\partial^2 s}{\partial x^2}(x, y)\right|^{-1/2} \Big/ \frac{\partial s}{\partial y}(x, y), \tag{3.6}$$

then

$$\frac{h(d_1)}{G(d_1)} = r(\varphi(d_1), d_1), \qquad \frac{h(d_2)}{G_1(d_2)} = r(\varphi(d_2), d_2). \tag{3.7}$$

From (3.5) it follows that

$$\begin{aligned}\int_{d_1}^{d_2} e^{ikG(y)}h(y)\, dy &= 2^{1/2}\Gamma(\tfrac{1}{2})e^{\pi i/4}t(0, 0)\{D_2(0, 0)\}^{-1/2}k^{-1/2}\\ &\quad - ie^{ikG(d_2)}r(\varphi(d_2), d_2)k^{-1} + ie^{ikG(d_1)}r(\varphi(d_1), d_1)k^{-1} + O(k^{-3/2}).\end{aligned} \tag{3.8}$$

We deduce from (3.3.1) and (3.8) that

$$\begin{aligned}H_1(k) &= 2\pi i\{D_2(0, 0)\}^{-1/2}t(0, 0)e^{iks(0,0)}k^{-1}\\ &\quad - 2^{1/2}\Gamma(\tfrac{1}{2})e^{3\pi i/4}e^{iks(\varphi(d_2),d_2)}r(\varphi(d_2), d_2)k^{-3/2}\\ &\quad + 2^{1/2}\Gamma(\tfrac{1}{2})e^{3\pi i/4}e^{iks(\varphi(d_1),d_1)}r(\varphi(d_1), d_1)k^{-3/2} + O(k^{-2}).\end{aligned} \tag{3.9}$$

From (3.3.2) and (3.5) with $h(y) = f_2(0)$ (for the meaning of $f_2(0)$ see (2.17)) it follows that $H_2(k) = O(k^{-2})$.

So our result is

$$(3.10)\quad \begin{aligned} I(k) &= 2\pi i\{D_2(0,0)\}^{-1/2}t(0,0)e^{iks(0,0)}k^{-1} \\ &- 2^{1/2}\Gamma(\tfrac{1}{2})e^{3\pi i/4}e^{iks(\varphi(d_2),d_2)}r(\varphi(d_2),d_2)k^{-3/2} \\ &+ 2^{1/2}\Gamma(\tfrac{1}{2})e^{3\pi i/4}e^{iks(\varphi(d_1),d_1)}r(\varphi(d_1),d_1)k^{-3/2} \\ &+ H_3(k) + H_4(k) + O(k^{-2}). \end{aligned}$$

Case II. $\partial^2 s/\partial x^2 > 0$ *in* G, $D_2(x, y) < 0$ *in* G. Then (3.3), (3.3.1), (3.3.2), (3.3.3), (3.3.4) are valid. Instead of (3.5) we use

$$(3.11)\quad \begin{aligned} \int_{d_1}^{d_2} e^{ikG(y)}h(y)\,dy &= 2^{1/2}\Gamma(\tfrac{1}{2})e^{-\pi i/4}h(0)\{-G_2(0)\}^{-1/2}k^{-1/2} \\ &- ie^{ikG(d_2)}\frac{h(d_2)}{G_1(d_2)}k^{-1} + ie^{ikG(d_1)}\frac{h(d_1)}{G_1(d_1)}k^{-1} + O(k^{-3/2}). \end{aligned}$$

Thus

$$(3.12)\quad \begin{aligned} \int_{d_1}^{d_2} e^{ikG(y)}h(y)\,dy &= 2^{1/2}\Gamma(\tfrac{1}{2})e^{-\pi i/4}t(0,0)\{-D_2(0,0)\}^{-1/2}k^{-1/2} \\ &- ie^{ikG(d_2)}r(\varphi(d_2),d_2)k^{-1} + ie^{ikG(d_1)}r(\varphi(d_1),d_1)k^{-1} + O(k^{-3/2}). \end{aligned}$$

We deduce from (3.3) and (3.12) that

$$(3.13)\quad \begin{aligned} I(k) &= 2\pi\{-D_2(0,0)\}^{-1/2}t(0,0)e^{iks(0,0)}k^{-1} \\ &- 2^{1/2}\Gamma(\tfrac{1}{2})e^{3\pi i/4}e^{iks(\varphi(d_2),d_2)}r(\varphi(d_2),d_2)k^{-3/2} \\ &+ 2^{1/2}\Gamma(\tfrac{1}{2})e^{3\pi i/4}e^{iks(\varphi(d_1),d_1)}r(\varphi(d_1),d_1)k^{-3/2} \\ &+ H_3(k) + H_4(k) + O(k^{-2}). \end{aligned}$$

Case III. $\partial^2 s/\partial x^2 < 0$ *in* G, $D_2(x, y) > 0$ *in* G. From the fact that (2.18) holds uniformly in y, $d_1 \leqq y \leqq d_2$, it follows that

$$(3.14)\quad I(k) = L_1(k) + L_2(k) + H_3(k) + H_4(k) + O(k^{-2})$$

with

$$(3.14.1)\quad L_1(k) = 2^{1/2}\Gamma(\tfrac{1}{2})e^{-\pi i/4}e^{iks(0,0)}\int_{d_1}^{d_2} e^{ikG(y)}t(\varphi, y)\left\{-\frac{\partial^2 s}{\partial x^2}(\varphi, y)\right\}^{-1/2} dy\cdot k^{-1/2},$$

$$(3.14.2)\quad L_2(k) = \tfrac{1}{2}\Gamma(\tfrac{3}{2})e^{-3\pi i/4}e^{iks(0,0)}\int_{d_1}^{d_2} e^{ikG(y)}f_2(0)\,dy\cdot k^{-3/2},$$

and $H_3(k)$ and $H_4(k)$ as in (3.3.3) and (3.3.4).

We put

$$(3.15)\quad t(\varphi, y)\left\{-\frac{\partial^2 s}{\partial x^2}(\varphi, y)\right\}^{-1/2} = \bar{h}(y).$$

From (3.11) it follows that

$$\begin{aligned}\int_{d_1}^{d_2} e^{ikG(y)}\bar{h}(y)\,dy &= 2^{1/2}\Gamma(\tfrac{1}{2})e^{-\pi i/4}t(0,0)\{D_2(0,0)\}^{-1/2}k^{-1/2} \\ &\quad - ie^{ikG(d_2)}r(\varphi(d_2),d_2)k^{-1} + ie^{ikG(d_1)}r(\varphi(d_1),d_1)k^{-1} + O(k^{-3/2}).\end{aligned} \tag{3.16}$$

Equation (3.14) combined with (3.16) yields

$$\begin{aligned} I(k) = &- 2\pi i\{D_2(0,0)\}^{-1/2}t(0,0)e^{iks(0,0)}k^{-1} \\ &- 2^{1/2}\Gamma(\tfrac{1}{2})e^{\pi i/4}e^{iks(\varphi(d_2),d_2)}r(\varphi(d_2),d_2)k^{-3/2} \\ &+ 2^{1/2}\Gamma(\tfrac{1}{2})e^{\pi i/4}e^{iks(\varphi(d_1),d_1)}r(\varphi(d_1),d_1)k^{-3/2} \\ &+ H_3(k) + H_4(k) + O(k^{-2}).\end{aligned} \tag{3.17}$$

Case IV. $\partial^2 s/\partial x^2 < 0$ *in* G, $D_2(x,y) < 0$ *in* G. Then (3.14), (3.14.1), (3.14.2) are valid.

By virtue of (3.5) we have

$$\begin{aligned}\int_{d_1}^{d_2} e^{ikG(y)}\bar{h}(y)\,dy &= 2^{1/2}\Gamma(\tfrac{1}{2})e^{\pi i/4}t(0,0)\{-D_2(0,0)\}^{-1/2}k^{-1/2} \\ &\quad - ie^{ikG(d_2)}r(\varphi(d_2),d_2)k^{-1} + ie^{ikG(d_1)}r(\varphi(d_1),d_1)k^{-1} + O(k^{-2}).\end{aligned} \tag{3.18}$$

We deduce from (3.14) and (3.18) that

$$\begin{aligned} I(k) = &\, 2\pi\{-D_2(0,0)\}^{-1/2}t(0,0)e^{iks(0,0)}k^{-1} \\ &- 2^{1/2}\Gamma(\tfrac{1}{2})e^{\pi i/4}e^{iks(\varphi(d_2),d_2)}r(\varphi(d_2),d_2)k^{-3/2} \\ &+ 2^{1/2}\Gamma(\tfrac{1}{2})e^{\pi i/4}e^{iks(\varphi(d_1),d_1)}r(\varphi(d_1),d_1)k^{-3/2} \\ &+ H_3(k) + H_4(k) + O(k^{-2}).\end{aligned} \tag{3.19}$$

4. Application 3. Our starting point is Application 1, with the supplementary conditions

$$\frac{\partial s}{\partial y}(0,0) = 0, \tag{3.1}$$

$$D_2(0,0) = 0, \tag{4.1}$$

$$G_3(y) \neq 0 \quad \text{for} \quad d_1 \leqq y \leqq d_2. \tag{4.2}$$

It may be remarked that

$$G_3(y) = \frac{\partial^3 s}{\partial x^2 \partial y}(\varphi,y)\left(\frac{d\varphi}{dy}\right)^2 + 2\frac{\partial^3 s}{\partial x \partial y^2}(\varphi,y)\frac{d\varphi}{dy} + \frac{\partial^3 s}{\partial y^3}(\varphi,y) + \frac{\partial^2 s}{\partial x \partial y}(\varphi,y)\frac{d^2\varphi}{dy^2}$$

and $d\varphi/dy$, $d^2\varphi/dy^2$ are determinable from the equations

$$\frac{\partial^2 s}{\partial x^2}(\varphi,y)\frac{d\varphi}{dy} + \frac{\partial^2 s}{\partial x \partial y}(\varphi,y) = 0,$$

$$\frac{\partial^3 s}{\partial x^3}(\varphi,y)\left(\frac{d\varphi}{dy}\right)^2 + 2\frac{\partial^3 s}{\partial x^2 \partial y}(\varphi,y)\frac{d\varphi}{dy} + \frac{\partial^3 s}{\partial x \partial y^2}(\varphi,y) + \frac{\partial^2 s}{\partial x^2}(\varphi,y)\frac{d^2\varphi}{dy^2} = 0.$$

With these assumptions we shall determine the asymptotic behavior of $I(k)$. We discuss the case $\partial^2 s/\partial x^2 > 0$ in G and $G_3(y) > 0$ for $d_1 \leqq y \leqq d_2$. The other three cases can be treated in the same manner.

From $u^3 = G(y), 0 \leqq y \leqq d_2$, it follows that $y = k(u), 0 \leqq u \leqq a = \{G(d_2)\}^{1/3}$, and from $u = G(d_2) - G(y)$, $0 \leqq y \leqq d_2$, it follows that $y = l(u)$, $0 \leqq u \leqq G(d_2)$. We put

$$\alpha(u) = h(k(u))k_1(u), \quad \beta(u) = h(l(u))l_1(u). \tag{4.3}$$

By Erdélyi's Theorem 4 in [4], we have

$$\begin{aligned}\int_0^{d_2} e^{ikG(y)}h(y)\,dy = \sum_{n=0}^{N-1} \frac{\Gamma((n+1)/3)}{3n!} e^{(n+1)\pi i/6}\alpha_n(0)k^{-(n+1)/3} \\ - \sum_{n=0}^{N-1} e^{-(n+1)\pi i/2} e^{ikG(d_2)}\beta_n(0)k^{-(n+1)} + o(k^{-N/3}).\end{aligned} \tag{4.4}$$

From $u^3 = -G(y)$, $d_1 \leqq y \leqq 0$ it follows that $y = m(u)$, $0 \leqq u \leqq \{-G(d_1)\}^{1/3}$, and from $u = G(y) - G(d_1), d_1 \leqq y \leqq 0$, it follows that $y = n(u), 0 \leqq u \leqq -G(d_1)$. We put

$$\gamma(u) = h(m(u))m_1(u), \quad \delta(u) = h(n(u))n_1(u). \tag{4.5}$$

Then

$$\begin{aligned}\int_{d_1}^{0} e^{ikG(y)}h(y)\,dy = -\sum_{n=0}^{N-1} \frac{\Gamma((n+1)/3)}{3n!} e^{-(n+1)\pi i/6}\gamma_n(0)k^{-(n+1)/3} \\ + \sum_{n=0}^{N-1} e^{(n+1)\pi i/2} e^{ikG(d_1)}\delta_n(0)k^{-(n+1)} + o(k^{-N/3}).\end{aligned} \tag{4.6}$$

By means of (4.4), (4.6) and the equation $\alpha_n(0) = (-1)^{n+1}\gamma_n(0)$, we derive

$$\begin{aligned}\int_{d_1}^{d_2} e^{ikG(y)}h(y)\,dy = \sum_{n=0}^{N-1} \frac{\Gamma((n+1)/3)}{3n!}\{e^{(n+1)\pi i/6} + (-1)^n e^{-(n+1)\pi i/6}\}\alpha_n(0)k^{-(n+1)/3} \\ - \sum_{n=0}^{N-1} (-i)^{n+1} e^{ikG(d_2)}\beta_n(0)k^{-(n+1)} \\ + \sum_{n=0}^{N-1} i^{n+1} e^{ikG(d_1)}\delta_n(0)k^{-(n+1)} + o(k^{-N/3}).\end{aligned} \tag{4.7}$$

From this we obtain for $N = 6$,

$$\begin{aligned}\int_{d_1}^{d_2} e^{ikG(y)}h(y)\,dy = \frac{1}{3}\Gamma\left(\frac{1}{3}\right)2\cos\frac{\pi}{6}\alpha(0)k^{-1/3} + \frac{1}{3}\Gamma\left(\frac{2}{3}\right)2i\sin\frac{2\pi}{6}\alpha_1(0)k^{-2/3} \\ + \frac{1}{3}\frac{1}{3!}\Gamma\left(\frac{4}{3}\right)2i\sin\frac{4\pi}{6}\alpha_3(0)k^{-4/3} + \frac{1}{3}\frac{1}{4!}\Gamma\left(\frac{5}{3}\right)2\cos\frac{5\pi}{6}\alpha_4(0)k^{-5/3} \\ + ie^{ikG(d_2)}\beta(0)k^{-1} + ie^{ikG(d_1)}\delta(0)k^{-1} + O(k^{-2}).\end{aligned} \tag{4.8}$$

After this preparation we consider $I(k)$. Relations (3.3), (3.3.1), (3.3.2), (3.3.3), (3.3.4) are still valid, and we derive, from (3.3.1) and (4.8) with

$$h(y) = t(\varphi, y)\left\{\frac{\partial^2 s}{\partial x^2}(\varphi, y)\right\}^{-1/2},$$

the following result:

$$\begin{aligned} H_1(k) = 2^{1/2}\Gamma\left(\frac{1}{2}\right)e^{\pi i/4}e^{iks(0,0)}\Big\{&\frac{1}{3}\Gamma\left(\frac{1}{3}\right)2\cos\frac{\pi}{6}\alpha(0)k^{-5/6} + \frac{1}{3}\Gamma\left(\frac{2}{3}\right)2i\sin\frac{2\pi}{6}\alpha_1(0)k^{-7/6} \\ &+ \frac{1}{3}\frac{1}{3!}\Gamma\left(\frac{4}{3}\right)2i\sin\frac{4\pi}{6}\alpha_3(0)k^{-11/6} + ie^{ikG(d_2)}\beta(0)k^{-3/2} + ie^{ikG(d_1)}\delta(0)k^{-3/2}\Big\} \\ &+ O(k^{-13/6}). \end{aligned} \tag{4.9}$$

From (4.3) and (4.5) it follows that

$$\begin{aligned} \alpha(0) &= h(k(0))k_1(0) = h(0)k_1(0) = t(0,0)\left\{\frac{\partial^2 s}{\partial x^2}(0,0)\right\}^{-1/2} 6^{1/3}\{G_3(0)\}^{-1/3}, \\ \beta(0) &= h(l(0))l_1(0) = h(d_2)l_1(0), \\ h(d_2) &= t(\varphi(d_2), d_2)\left\{\frac{\partial^2 s}{\partial x^2}(\varphi(d_2), d_2)\right\}^{-1/2}, \\ l_1(u) &= -1\Big/\frac{dG}{dy} = -1\Big/\left\{\frac{\partial s}{\partial x}(\varphi, y)\frac{d\varphi}{dy} + \frac{\partial s}{\partial y}(\varphi, y)\right\} = -1\Big/\frac{\partial s}{\partial y}(\varphi, y), \\ l_1(0) &= -1\Big/\frac{\partial s}{\partial x}(\varphi(d_2), d_2). \end{aligned} \tag{4.10}$$

Thus

$$\beta(0) = -r(\varphi(d_2), d_2) \qquad \text{(for } r(x, y) \text{ see (3.6)).} \tag{4.11}$$

In the same way,

$$\begin{aligned} \delta(0) &= h(n(0))n_1(0) = h(d_1)n_1(0), \\ h(d_1) &= t(\varphi(d_1), d_1)\left\{\frac{\partial^2 s}{\partial x^2}(\varphi(d_1), d_1)\right\}^{-1/2}, \\ n_1(u) &= 1\Big/\frac{dG}{dy} = 1\Big/\frac{\partial s}{\partial y}(\varphi, y), \qquad n_1(0) = 1\Big/\frac{\partial s}{\partial y}(\varphi(d_1), d_1), \\ \delta(0) &= r(\varphi(d_1), d_1). \end{aligned} \tag{4.12}$$

From (3.3.2) and (4.8) with $h(y) = f_2(0)$ (for $f_2(0)$ see (2.5)) it follows that

$$H_2(k) = \frac{1}{2}\Gamma\left(\frac{3}{2}\right)e^{3\pi i/4}e^{iks(0,0)}\frac{1}{3}\Gamma\left(\frac{1}{3}\right)2\cos\frac{\pi}{6}\alpha^*(0)k^{-11/6} + O(k^{-13/6}). \tag{4.13}$$

Finally we deduce from (3.3), (4.9), (4.10), (4.11), (4.12), (4.13) the result:

$$\begin{aligned} I(k) = {} & e^{iks(0,0)}(a_1 k^{-5/6} + a_2 k^{-7/6} + a_3 k^{-11/6}) \\ & - 2^{1/2}\Gamma(\tfrac{1}{2})e^{3\pi i/4} r(\varphi(d_2), d_2) e^{iks(\varphi(d_2),d_2)} k^{-3/2} \\ & + 2^{1/2}\Gamma(\tfrac{1}{2})e^{3\pi i/4} r(\varphi(d_1), d_1) e^{iks(\varphi(d_1),d_1)} k^{-3/2} \\ & + H_3(k) + H_4(k) + O(k^{-2}). \end{aligned}$$

It is easily seen that

$$a_1 = \frac{6^{5/6}}{3}\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{3}\right)e^{\pi i/4} t(0,0)\left\{\frac{\partial^2 s}{\partial x^2}(0,0)\right\}^{-1/2}\{G_3(0)\}^{-1/3}.$$

5. Application 4. Let $p(\varphi)$ be a positive and continuous function for $0 \leqq \varphi \leqq 2\pi$ with $p(0) = p(2\pi)$. Let $r = p(\varphi)$ be the polar equation of a curve C. Then we denote by G the union of the curve C and its interior. We suppose that $s(x, y)$ and $t(x, y)$ are real functions defined in the domain G, and $s(x, y)$ and $t(x, y)$ have as many continuous partial derivatives in the domain G as is necessary for the correctness of the proofs.

Furthermore we suppose that

$$\frac{\partial s}{\partial x}(0,0) = 0, \qquad \frac{\partial s}{\partial y}(0,0) = 0, \tag{5.1}$$

$$\frac{\partial^2 s}{\partial x^2}(x, y) \neq 0 \quad \text{in } G, \tag{5.2}$$

$$D_2(x, y) = \left\{\frac{\partial^2 s}{\partial x^2}\frac{\partial^2 s}{\partial y^2} - \left(\frac{\partial^2 s}{\partial x \partial y}\right)^2\right\}(x, y) > 0 \quad \text{in } G. \tag{5.3}$$

With these assumptions we shall prove

$$\begin{aligned} I(k) &= \iint_G e^{iks(x,y)} t(x, y)\,dx\,dy \\ &= 2\pi i\sigma\{D_2(0,0)\}^{-1/2} t(0,0) e^{iks(0,0)} k^{-1} \\ &\quad - i\int_0^{2\pi} e^{iks(R)} t(R) p(\varphi)\left\{\frac{\partial s}{\partial \varphi}(R)\right\}^{-1} d\varphi \cdot k^{-1} + O(k^{-2}), \end{aligned} \tag{5.4}$$

where R is the point with polar coordinates $(p(\varphi), \varphi)$, $\sigma = 1$ if $\partial^2 s(0,0)/\partial x^2 > 0$, or $\sigma = -1$ if $\partial^2 s(0,0)/\partial x^2 < 0$. For the time being we assume that $\partial^2 s/\partial x^2 > 0$ in G and $s(0,0) = 0$. Consider

$$I_1(k) = \int_0^{p(\varphi)} e^{iks(r\cos\varphi,\, r\sin\varphi)} t(r\cos\varphi, r\sin\varphi) r\,dr, \qquad \varphi \text{ constant}, \quad 0 \leqq \varphi \leqq 2\pi, \tag{5.5}$$

and write

$$g(r) = s(r\cos\varphi, r\sin\varphi), \quad h(r) = t(r\cos\varphi, r\sin\varphi)r, \qquad 0 \leqq r \leqq p. \tag{5.6}$$

Then

$$\frac{dg}{dr} = \frac{\partial s}{\partial x}(r\cos\varphi, r\sin\varphi)\cos\varphi + \frac{\partial s}{\partial y}(r\cos\varphi, r\sin\varphi)\sin\varphi,$$

$$\frac{d^2g}{dr^2} = \frac{\partial^2 s}{\partial x^2}(r\cos\varphi, r\sin\varphi)\cos^2\varphi + 2\frac{\partial^2 s}{\partial x\partial y}(r\cos\varphi, r\sin\varphi)\cos\varphi\sin\varphi$$

$$+ \frac{\partial^2 s}{\partial y^2}(r\cos\varphi, r\sin\varphi)\sin^2\varphi, \tag{5.7}$$

$$\frac{d^3g}{dr^3} = \frac{\partial^3 s}{\partial x^3}(r\cos\varphi, r\sin\varphi)\cos^3\varphi + 3\frac{\partial^3 s}{\partial x^2\partial y}(r\cos\varphi, r\sin\varphi)\cos^2\varphi\sin\varphi$$

$$+ 3\frac{\partial^3 s}{\partial x\partial y^2}(r\cos\varphi, r\sin\varphi)\cos\varphi\sin^2\varphi + \frac{\partial^3 s}{\partial y^3}(r\cos\varphi, r\sin\varphi)\sin^3\varphi.$$

Thus

$$g(0) = g_1(0) = 0,$$

$$g_2(0) = \alpha\cos^2\varphi + 2\beta\cos\varphi\sin\varphi + \gamma\sin^2\varphi,$$

$$\alpha = \frac{\partial^2 s}{\partial x^2}(0,0), \quad \beta = \frac{\partial^2 s}{\partial x\partial y}(0,0), \quad \gamma = \frac{\partial^2 s}{\partial y^2}(0,0), \tag{5.8}$$

$$g_3(0) = a_{30}\cos^3\varphi + a_{21}\cos^2\varphi\sin\varphi + a_{12}\cos\varphi\sin^2\varphi + a_{03}\sin^3\varphi.$$

Further,

$$\frac{dh}{dr} = \left\{\frac{\partial t}{\partial x}(r\cos\varphi, r\sin\varphi)\cos\varphi + \frac{\partial t}{\partial y}(r\cos\varphi, r\sin\varphi)\right\} r + t(r\cos\varphi, r\sin\varphi),$$

$$\frac{d^2h}{dr^2} = \left\{\frac{\partial^2 t}{\partial x^2}(r\cos\varphi, r\sin\varphi)\cos^2\varphi + 2\frac{\partial^2 t}{\partial x\partial y}(r\cos\varphi, r\sin\varphi)\cos\varphi\sin\varphi\right.$$

$$\left. + \frac{\partial^2 t}{\partial y^2}(r\cos\varphi, r\sin\varphi)\sin^2\varphi\right\} r \tag{5.9}$$

$$+ 2\left\{\frac{\partial t}{\partial x}(r\cos\varphi, r\sin\varphi)\cos\varphi + \frac{\partial t}{\partial y}(r\cos\varphi, r\sin\varphi)\sin\varphi\right\}.$$

Thus

$$h(0) = 0, \quad h_1(0) = t(0,0), \quad h_2(0) = 2\left\{\frac{\partial t}{\partial x}(0,0)\cos\varphi + \frac{\partial t}{\partial y}(0,0)\sin\varphi\right\}. \tag{5.10}$$

We infer from (5.7), (5.3) and the condition $\partial^2 s/\partial x^2 > 0$ in G, that there exists a positive number m such that

$$\frac{d^2g}{dr^2} \geqq m \quad \text{in } G. \tag{5.11}$$

Let $r = k(\rho)$, $0 \leqq \rho \leqq a = \sqrt{g(p)}$ be the inverse function of $\rho = \sqrt{g(r)}$. Then

$$I_1(k) = \int_0^p e^{ikg(r)}h(r)\,dr = \int_0^a e^{ik\rho^2}f(\rho)\,d\rho \tag{5.12}$$

with $f(\rho) = h(k(\rho))k_1(\rho)$. Now

$$f_1(\rho) = h_1(k(\rho))k_1^2(\rho) + h(k(\rho))k_2(\rho),$$

$$(5.13) \qquad f_2(\rho) = h_2(k(\rho))k_1^3(\rho) + 3h_1(k(\rho))k_1(\rho)k_2(\rho) + h(k(\rho))k_3(\rho),$$

$$f_3(\rho) = h_3(k(\rho))k_1^4(\rho) + 6h_2(k(\rho))k_1^2(\rho)k_2(\rho) + 3h_1(k(\rho))k_2^2(\rho) + 4h_1(k(\rho))k_1(\rho)k_3(\rho) + h(k(\rho))k_4(\rho),$$

$$f(0) = 0, \quad f_1(0) = 2t(0,0)\{g_2(0)\}^{-1},$$

$$(5.14) \qquad f_2(0) = 2^{5/2}\left\{\frac{\partial t}{\partial x}(0,0)\cos\varphi + \frac{\partial t}{\partial y}(0,0)\sin\varphi\right\}\{g_2(0)\}^{-3/2} - 2^{3/2}t(0,0)\{g_2(0)\}^{-5/2}g_3(0).$$

If we apply Theorem 1 with $N = 3$ we find that

$$(5.15) \qquad I_1(k) = A_3(k) + B_3(k) + R_3(k),$$

$$(5.15.1) \quad A_3(k) = \tfrac{1}{2}\Gamma(\tfrac{1}{2})e^{\pi i/4}f(0)k^{-1/2} + \tfrac{1}{2}if_1(0)k^{-1} + \tfrac{1}{4}\Gamma(\tfrac{3}{2})e^{3\pi i/4}f_2(0)k^{-3/2},$$

$$(5.15.2) \quad B_3(k) = b_1k^{-1} + b_2k^{-2} + b_3k^{-3},$$

with

$$b_1 = -ie^{ika^2}(2a)^{-1}f(a) = -ie^{ikg(p)}h(p)\{g_1(p)\}^{-1} = -ie^{iks(R)}t(R)p(\varphi)\left\{\frac{\partial s}{\partial\varphi}(R)\right\}^{-1},$$

$$b_2 = -e^{ika^2}\{2(2a)^{-3}f(a) - (2a)^{-2}f_1(a)\},$$

$$b_3 = ie^{ika^2}\{12(2a)^{-5}f(a) - 6(2a)^{-4}f_1(a) + (2a)^{-3}f_2(a)\}.$$

We infer from (5.11), (5.13) and Theorem 2 that $f(\rho), f_1(\rho), f_2(\rho), f_3(\rho)$ are bounded for $0 \leqq \rho \leqq a$, uniformly with respect to φ, $0 \leqq \varphi \leqq 2\pi$. From (5.11) we deduce that $g(p) = s(p\cos\varphi, p\sin\varphi)$ is positive for $0 \leqq \varphi \leqq 2\pi$; hence, $a^{-1} = \{g(p)\}^{-1/2} = O(1)$ for $0 \leqq \varphi \leqq 2\pi$.

It follows that $b_2 = O(1)$, $b_3 = O(1)$ uniformly in φ, $0 \leqq \varphi \leqq 2\pi$, and by Theorem 1, $R_3(k) = O(k^{-2})$ uniformly in φ, $0 \leqq \varphi \leqq 2\pi$.

We now deduce from (5.14), (5.15) that

$$(5.16) \qquad I(k) = \int_0^{2\pi} I_1(k)\,d\varphi = it(0,0)\int_0^{2\pi}\{g_2(0)\}^{-1}\,d\varphi\cdot k^{-1} + \tfrac{1}{4}\Gamma(\tfrac{3}{2})e^{3\pi i/4}\int_0^{2\pi} f_2(0)\,d\varphi\cdot k^{-3/2} + \int_0^{2\pi} b_1\,d\varphi\cdot k^{-1} + O(k^{-2}).$$

Since

$$\int_0^{2\pi}\{g_2(0)\}^{-1}\,d\varphi = \int_0^{2\pi}\frac{d\varphi}{\alpha\cos^2\varphi + 2\beta\cos\varphi\sin\varphi + \gamma\sin^2\varphi} = 2\pi\{D_2(0,0)\}^{-1/2},$$

$$\int_0^{2\pi} f_2(0)\,d\varphi = 0,$$

we obtain the following equation from (5.16), (5.15.2):

$$I(k) = 2\pi i\{D_2(0,0)\}^{-1/2}t(0,0)k^{-1} - i\int_0^{2\pi} e^{iks(R)}t(R)p(\varphi)\left\{\frac{\partial s}{\partial\varphi}(R)\right\}^{-1} d\varphi\cdot k^{-1} + O(k^{-2}).$$

If $\partial^2 s/\partial x^2 < 0$ in G, then we find by considering the complex conjugate of $I(k)$,

$$I(k) = -2\pi i\{D_2(0,0)\}^{-1/2}t(0,0)k^{-1} - i\int_0^{2\pi} e^{iks(R)}p(\varphi)\left\{\frac{\partial s}{\partial\varphi}(R)\right\}^{-1} t(R)\,d\varphi\cdot k^{-1} + O(k^{-2}).$$

If $s(0,0) \neq 0$, then we write

$$I(k) = e^{iks(0,0)}\iint_G e^{ik\{s(x,y)-s(0,0)\}}t(x,y)\,dx\,dy.$$

This completes the proof of (5.4).

6. Application 5. Let G be a convex closed domain, bounded by a continuous curve C, in the coordinate plane $x0y$. Let P be the point (x, y, z) with $z > 0$, Q a point of G, r the distance PQ, R a point of C.

We suppose that the function $u(Q)$ has as many continuous partial derivatives in the domain G as is necessary for the correctness of the proofs.

We shall determine the asymptotic behavior of the vibrating potential

$$I(k) = \iint_G u(Q)\frac{e^{ikr}}{r}\,df \quad \text{for} \quad k \to \infty \tag{6.1}$$

if $P_1(x, y, 0)$ lies in the interior of G.

We introduce polar coordinates (ρ, φ) with pole P_1. Then, if we denote the distance P_1R by $g(\varphi)$, we have

$$I(k) = \int_0^{2\pi}\left\{\int_0^{g(\varphi)} u(Q)\frac{e^{ikr}}{r}\rho\,d\rho\right\} d\varphi\cdot \tag{6.2}$$

Next

$$\begin{aligned}\int_0^{g(\varphi)} u(Q)\frac{e^{ikr}}{r}\rho\,d\rho &= -ik^{-1}\int_0^{g(\varphi)} u(Q)\frac{de^{ikr}}{d\rho}\,d\rho \\ &= ik^{-1}\left\{e^{ik(z^2+g^2)^{1/2}}u(R) - e^{ikz}u(P_1) - \int_0^{g(\varphi)} e^{ikr}\frac{\partial u}{\partial\rho}\,d\rho\right\}.\end{aligned} \tag{6.3}$$

Put

$$I_1 = \int_0^g e^{ikr}\frac{\partial u}{\partial\rho}\,d\rho = e^{ikz}\int_0^g e^{ik(z^2+\rho^2)^{1/2}-z)}\frac{\partial u}{\partial\rho}\,d\rho = e^{ikz}I_2. \tag{6.4}$$

If we make in I_2 the change of variable

$$x = \{(z^2 + \rho^2)^{1/2} - z\}^{1/2},$$

then we obtain

$$I_2 = \int_0^a e^{ikx^2} f(x)\,dx \tag{6.5}$$

with $a = ((z^2 + g^2)^{1/2} - z)^{1/2}$, $f(x) = (\partial u/\partial\rho)(d\rho/dx)$. By Theorem 1 with $N = 2$ we have

$$\begin{aligned} I_2 &= A_2(k) + B_2(k) + R_2(k), \\ A_2(k) &= \tfrac{1}{2}\Gamma(\tfrac{1}{2})e^{\pi i/4} f(0)k^{-1/2} + \tfrac{1}{2}if_1(0)k^{-1}, \\ B_2(k) &= b_1k^{-1} + b_2k^{-2}, \\ b_1 &= -ie^{ika^2}(2a)^{-1}f(a), \\ b_2 &= -e^{ika^2}\{2(2a)^{-3}f(a) - (2a)^{-2}f_1(a)\}, \\ |R_2(k)| &\leqq 2^{-3/2}M_2k^{-3/2} + \tfrac{1}{8}\sqrt{2}a^{-5}\{12|f(a)| + 6|f_1(a)|a\sqrt{2}\}k^{-3}, \end{aligned} \tag{6.6}$$

where $M_2 = \max|f_2(x)|$ for $0 \leqq x \leqq a$. It is easily seen that

$$\rho = x(x^2 + 2z)^{1/2}, \quad f(0) = \frac{\partial u}{\partial\rho}(P_1)(2z)^{1/2}, \quad f_1(0) = \frac{\partial^2 u}{\partial\rho^2}(P_1)2z,$$

$$b_1 = -ie^{ika^2}\frac{\partial u}{\partial\rho}(R)\frac{(z^2 + g^2)^{1/2}}{g}, \quad b_2 = -e^{ika^2}\left\{\frac{z^2}{g^3}\frac{\partial u}{\partial\rho}(R) - \frac{z^2 + g^2}{g^2}\frac{\partial^2 u}{\partial\rho^2}(R)\right\}.$$

Next we deduce from (6.4) and (6.6) that

$$\begin{aligned} I_1 = {}& e^{ikz}\tfrac{1}{2}\Gamma(\tfrac{1}{2})e^{\pi i/4}(2z)^{1/2}\frac{\partial u}{\partial\rho}(P_1)k^{-1/2} + \tfrac{1}{2}ie^{ikz}2z\frac{\partial^2 u}{\partial\rho^2}(P_1)k^{-1} \\ & - ie^{ik(z^2+g^2)^{1/2}}\frac{(z^2 + g^2)^{1/2}}{g}\frac{\partial u}{\partial\rho}(R)k^{-1} + e^{ikz}(b_2k^{-2} + R_2(k)), \end{aligned} \tag{6.7}$$

and further from (6.2), (6.3) and (6.7) that

$$\begin{aligned} I = {}& 2\pi ie^{ikz}u(P_1)k^{-1} - i\int_0^{2\pi} e^{ik(z^2+g^2)^{1/2}}u(R)\,d\varphi\cdot k^{-1} \\ & + ie^{ikz}\tfrac{1}{2}\Gamma(\tfrac{1}{2})e^{\pi i/4}(2z)^{1/2}\int_0^{2\pi}\frac{\partial u}{\partial\rho}(P_1)\,d\varphi\cdot k^{-3/2} - e^{ikz}z\int_0^{2\pi}\frac{\partial^2 u}{\partial\rho^2}(P_1)\,d\varphi\cdot k^{-2} \\ & + \int_0^{2\pi} e^{ik(z^2+g^2)^{1/2}}\frac{(z^2 + g^2)^{1/2}}{g}\frac{\partial u}{\partial\rho}(R)\,d\varphi\cdot k^{-2} \\ & + ie^{ikz}\int_0^{2\pi}\{b_2k^{-3} + R_2(k)k^{-1}\}\,d\varphi. \end{aligned} \tag{6.8}$$

Since

$$\frac{\partial u}{\partial\rho} = \frac{\partial u}{\partial x}\cos\varphi + \frac{\partial u}{\partial y}\sin\varphi,$$

$$\frac{\partial^2 u}{\partial\rho^2} = \frac{\partial^2 u}{\partial x^2}\cos^2\varphi + 2\frac{\partial^2 u}{\partial x\partial y}\cos\varphi\sin\varphi + \frac{\partial^2 u}{\partial y^2}\sin^2\varphi,$$

we have

$$\int_0^{2\pi} \frac{\partial u}{\partial \rho}(P_1)\,d\varphi = 0, \qquad \int_0^{2\pi} \frac{\partial^2 u}{\partial \rho^2}(P_1)\,d\varphi = \pi\left\{\frac{\partial^2 u}{\partial x^2}(P_1) + \frac{\partial^2 u}{\partial y^2}(P_1)\right\}.$$

Thus by (6.8),

$$\begin{aligned} I &= 2\pi i e^{ikz} u(P_1) k^{-1} - i\int_0^{2\pi} e^{ik(z^2+g^2)^{1/2}} u(R)\,d\varphi\cdot k^{-1} \\ &\quad - \pi z e^{ikz}\left\{\frac{\partial^2 u}{\partial x^2}(P_1) + \frac{\partial^2 u}{\partial y^2}(P_1)\right\} k^{-2} \\ &\quad + \int_0^{2\pi} e^{ik(z^2+g^2)^{1/2}} \frac{(z^2+g^2)^{1/2}}{g}\frac{\partial u}{\partial \rho}(R)\,d\varphi\cdot k^{-2} \\ &\quad + i e^{ikz}\int_0^{2\pi}\{b_2 k^{-3} + R_2(k)k^{-1}\}\,d\varphi. \end{aligned} \tag{6.9}$$

We estimate the last integral in (6.9). Let D be a closed subdomain of G that has no point in common with the curve C; let the distance of C and D be denoted by d_1 and the diameter of G by d_2. We suppose that P_1 belongs to D and that $z \geqq p$, where p is a fixed positive number.

From the equations

$$\frac{d^2\rho}{dx^2} = 2x\frac{x^2+3z}{(x^2+2z)^{3/2}}, \qquad \frac{d^3\rho}{dx^3} = \frac{12z^2}{(x^2+2z)^{5/2}},$$

it follows that

$$0 < \frac{d^3\rho}{dx^3} \leqq \frac{3}{2}\sqrt{2}z^{-1/2},$$

$$\begin{aligned} 0 < \frac{d^2\rho}{dx^2} &\leqq \left(\frac{d^2\rho}{dx^2}\right)_{x=a} = 2a\frac{a^2+3z}{(a^2+2z)^{3/2}} = 2a\frac{(z^2+g^2)^{1/2}+2z}{((z^2+g^2)^{1/2}+z)^{3/2}} \\ &= 2\frac{((z^2+g^2)^{1/2}-z)^{1/2}}{((z^2+g^2)^{1/2}+z)^{1/2}}\frac{(z^2+g^2)^{1/2}+2z}{(z^2+g^2)^{1/2}+z} \\ &= 2\frac{g}{(z^2+g^2)^{1/2}+z}\left(1+\frac{z}{(z^2+g^2)^{1/2}+z}\right) < 3. \end{aligned}$$

Further we have

$$0 < d_1 \leqq g(\varphi) \leqq d_2 \quad \text{for} \quad 0 \leqq \varphi \leqq 2\pi,$$

$$\begin{aligned} 0 < \frac{d\rho}{dx} &= \frac{2(x^2+z)}{(x^2+2z)^{1/2}} \leqq 2\frac{a^2+z}{(a^2+2z)^{1/2}} \\ &= 2\frac{(z^2+g^2)^{1/2}}{((z^2+g^2)^{1/2}+z)^{1/2}} < 2\frac{(z^2+g^2)^{1/2}}{(z+g)^{1/2}} < 2(z+g)^{1/2} \\ &\leqq 2(z+d_2)^{1/2} \leqq 2\left(1+\frac{d_2}{p}\right)^{1/2} z^{1/2}, \end{aligned}$$

$$a^{-1} = \frac{((z^2+g^2)^{1/2}+z)^{1/2}}{g} \leqq \frac{(2z+d_2)^{1/2}}{d_1} \leqq \frac{(2+d_2/p)^{1/2}}{d_1} z^{1/2}.$$

Thus

$$f(a) = O(z^{1/2}), \quad f_1(a) = O(z), \quad f_2(x) = O(z^{3/2}) \quad \text{for} \quad 0 \leqq x \leqq a, \quad b_2 = O(z^2),$$

and these estimates hold uniformly with respect to φ, $0 \leqq \varphi \leqq 2\pi$.

We can now state the result

$$\begin{aligned} I(k) = {} & 2\pi i e^{ikz} u(P_1) k^{-1} - i \int_0^{2\pi} e^{ik(z^2+g^2)^{1/2}} u(R)\, d\varphi \cdot k^{-1} \\ & - \pi z e^{ikz} \left\{ \frac{\partial^2 u}{\partial x^2}(P_1) + \frac{\partial^2 u}{\partial y^2}(P_1) \right\} k^{-2} + \int_0^{2\pi} e^{ik(z^2+g^2)^{1/2}} \frac{(z^2+g^2)^{1/2}}{g} \frac{\partial u}{\partial \rho}(R)\, d\varphi \cdot k^{-2} \\ & + O(z^{3/2} k^{-5/2} + z^2 k^{-3} + z^3 k^{-4}), \end{aligned}$$

and the estimate of the remainder holds uniformly if P_1 belongs to D and $z \geqq p > 0$.

7. Application 6. We suppose that the real functions $s(x, y, z)$ and $t(x, y, z)$, defined in a domain $H\{|x| \leqq a, |y| \leqq b, |z| \leqq c, a > 0, b > 0, c > 0\}$, have as many continuous partial derivatives in the domain H as is necessary for the correctness of the proofs.

Furthermore we suppose that

$$\frac{\partial s}{\partial x}(0,0,0) = \frac{\partial s}{\partial y}(0,0,0) = \frac{\partial s}{\partial z}(0,0,0) = 0, \tag{7.1}$$

$$D_1 = \frac{\partial^2 s}{\partial z^2} \neq 0 \quad \text{in } H,$$

$$D_2 = \begin{vmatrix} \dfrac{\partial^2 s}{\partial x^2} & \dfrac{\partial^2 s}{\partial x \partial z} \\ \dfrac{\partial^2 s}{\partial x \partial z} & \dfrac{\partial^2 s}{\partial z^2} \end{vmatrix} \neq 0 \quad \text{in } H,$$

$$D_3 = \begin{vmatrix} \dfrac{\partial^2 s}{\partial x^2} & \dfrac{\partial^2 s}{\partial x \partial y} & \dfrac{\partial^2 s}{\partial x \partial z} \\ \dfrac{\partial^2 s}{\partial x \partial y} & \dfrac{\partial^2 s}{\partial y^2} & \dfrac{\partial^2 s}{\partial y \partial z} \\ \dfrac{\partial^2 s}{\partial x \partial z} & \dfrac{\partial^2 s}{\partial y \partial z} & \dfrac{\partial^2 s}{\partial z^2} \end{vmatrix} \neq 0 \quad \text{in } H. \tag{7.2}$$

(A) The equation $\partial s / \partial z = 0$ has a solution $z = \Psi(x, y)$ for $|x| \leqq a$, $|y| \leqq b$ with $\Psi(0,0) = 0$ and $|\Psi(x, y)| < c$ for $|x| \leqq a$, $|y| \leqq b$.

(B) The equation $(\partial s / \partial x)(x, y, \Psi(x, y)) = 0$ has a solution $x = \varphi(y)$ for $|y| \leqq b$ with $\varphi(0) = 0$ and $|\varphi(y)| < a$ for $|y| \leqq b$.

With these assumptions we shall determine the asymptotic behavior of

$$I(k) = \iiint_H e^{iks(x,y,z)} t(x, y, z)\, dx\, dy\, dz$$

for $k \to \infty$.

We start with

$$I_3(k) = \int_{-c}^{c} e^{iks(x,y,z)} t(x, y, z)\,dz, \quad x \text{ and } y \text{ constant}, \quad |x| \leqq a, \quad |y| \leqq b.$$

We first suppose that $\partial^2 s/\partial z^2 > 0$ in H, and write

$$I_3(k) = e^{iks(x,y,\Psi)} \left\{ \int_{\Psi}^{c} e^{ikg(z)} t(x, y, z)\,dz + \int_{-c}^{\Psi} e^{ikg(z)} t(x, y, z)\,dz \right\}$$

with $g(z) = s(x, y, z) - s(x, y, \Psi)$. Then we have $g(\Psi) = g_1(\Psi) = 0$. It follows from (7.2) that there exists a number m such that

$$g_2(z) \geqq m > 0 \quad \text{in } H.$$

On account of (A) we can apply to $I_3(k)$ the method used in Application 1 and obtain

$$\begin{aligned} I_3(k) = {} & \Gamma(\tfrac{1}{2}) e^{\pi i/4} f(0) e^{iks(x,y,\Psi)} k^{-1/2} + \tfrac{1}{2}\Gamma(\tfrac{3}{2}) e^{3\pi i/4} f_2(0) e^{iks(x,y,\Psi)} k^{-3/2} \\ & - i e^{iks(x,y,c)} t(x, y, c) \left\{ \frac{\partial s}{\partial z}(x, y, c) \right\}^{-1} k^{-1} \\ & + i e^{iks(x,y,-c)} t(x, y, -c) \left\{ \frac{\partial s}{\partial z}(x, y, -c) \right\}^{-1} k^{-1} + O(k^{-2}) \end{aligned} \tag{7.3}$$

with

$$f(0) = 2^{1/2} t(x, y, \Psi) \left\{ \frac{\partial^2 s}{\partial z^2}(x, y, \Psi) \right\}^{-1/2}, \tag{7.3.1}$$

$$\begin{aligned} f_2(0) = {} & 2^{3/2} \frac{\partial^2 t}{\partial z^2} \left(\frac{\partial^2 s}{\partial z^2} \right)^{-3/2} - 2^{3/2} \frac{\partial t}{\partial z} \left(\frac{\partial^2 s}{\partial z^2} \right)^{-5/2} \frac{\partial^3 s}{\partial z^3} \\ & + t \left\{ \frac{5}{6}\sqrt{2} \left(\frac{\partial^2 s}{\partial z^2} \right)^{-7/2} \left(\frac{\partial^3 s}{\partial z^3} \right)^2 - \frac{1}{2}\sqrt{2} \left(\frac{\partial^2 s}{\partial z^2} \right)^{-5/2} \frac{\partial^4 s}{\partial z^4} \right\} (x, y, \Psi). \end{aligned} \tag{7.3.2}$$

The estimate of the remainder in (7.3) holds uniformly with respect to x and y, $|x| \leqq a, |y| \leqq b$.

If $\partial^2 s/\partial z^2 < 0$ in H, then

$$\begin{aligned} I_3(k) = {} & \Gamma(\tfrac{1}{2}) e^{-\pi i/4} f(0) e^{iks(x,y,\Psi)} k^{-1/2} + \tfrac{1}{2}\Gamma(\tfrac{3}{2}) e^{-3\pi i/4} f_2(0) e^{iks(x,y,\Psi)} k^{-3/2} \\ & - i e^{iks(x,y,c)} t(x, y, c) \left\{ \frac{\partial s}{\partial z}(x, y, c) \right\}^{-1} k^{-1} \\ & + i e^{iks(x,y,-c)} t(x, y, -c) \left\{ \frac{\partial s}{\partial z}(x, y, -c) \right\}^{-1} k^{-1} + O(k^{-2}) \end{aligned} \tag{7.4}$$

with

$$f(0) = 2^{1/2} t(x, y, \Psi) \left\{ -\frac{\partial^2 s}{\partial z^2}(x, y, \Psi) \right\}^{-1/2}, \tag{7.4.1}$$

$$\begin{aligned} f_2(0) = {} & 2^{3/2} \frac{\partial^2 t}{\partial z^2} \left(-\frac{\partial^2 s}{\partial z^2} \right)^{-3/2} - 2^{3/2} \frac{\partial t}{\partial z} \left(-\frac{\partial^2 s}{\partial z^2} \right)^{-5/2} \left(-\frac{\partial^3 s}{\partial z^3} \right) \\ & + t \left\{ \frac{5}{6}\sqrt{2} \left(-\frac{\partial^2 s}{\partial z^2} \right)^{-7/2} \left(-\frac{\partial^3 s}{\partial z^3} \right)^2 - \frac{1}{2}\sqrt{2} \left(-\frac{\partial^2 s}{\partial z^2} \right)^{-5/2} \left(-\frac{\partial^4 s}{\partial z^4} \right) \right\} (x, y, \Psi), \end{aligned} \tag{7.4.2}$$

and the estimate of the remainder in (7.4) holds uniformly with respect to x and y, $|x| \leqq a, |y| \leqq b$.

Let

$$S(x, y) = s(x, y, \Psi(x, y)).$$

Then from condition (A) we see that

$$\begin{aligned} \frac{\partial S}{\partial x} &= \frac{\partial s}{\partial x}(x, y, \Psi) + \frac{\partial s}{\partial z}(x, y, \Psi)\frac{\partial \Psi}{\partial x} = \frac{\partial s}{\partial x}(x, y, \Psi), \\ \frac{\partial S}{\partial y} &= \frac{\partial s}{\partial y}(x, y, \Psi) + \frac{\partial s}{\partial z}(x, y, \Psi)\frac{\partial \Psi}{\partial y} = \frac{\partial s}{\partial y}(x, y, \Psi), \end{aligned} \tag{7.5}$$

and thence on account of (7.1),

$$\frac{\partial S}{\partial x}(0,0) = \frac{\partial S}{\partial y}(0,0) = 0. \tag{7.6}$$

From $(\partial s/\partial z)\,(x, y, \Psi) = 0$ it follows that

$$\begin{aligned} \frac{\partial^2 s}{\partial x \partial z}(x, y, \Psi) + \frac{\partial^2 s}{\partial z^2}(x, y, \Psi)\frac{\partial \Psi}{\partial x} &= 0, \\ \frac{\partial^2 s}{\partial y \partial z}(x, y, \Psi) + \frac{\partial^2 s}{\partial z^2}(x, y, \Psi)\frac{\partial \Psi}{\partial y} &= 0. \end{aligned} \tag{7.7}$$

From (7.5) we deduce that

$$\frac{\partial^2 S}{\partial x^2} = \frac{\partial^2 s}{\partial x^2}(x, y, \Psi) + \frac{\partial^2 s}{\partial x \partial z}(x, y, \Psi)\frac{\partial \Psi}{\partial x},$$

and so together with (7.7) we arrive at

$$\frac{\partial^2 S}{\partial x^2} = \frac{(\partial^2 s/\partial x^2)(\partial^2 s/\partial z^2) - (\partial^2 s/\partial x \partial z)^2}{\partial^2 s/\partial z^2}(x, y, \Psi) = \frac{D_2}{\partial^2 s/\partial z^2}(x, y, \Psi). \tag{7.8}$$

Furthermore,

$$\begin{aligned} \frac{\partial^2 S}{\partial x \partial y} &= \frac{\partial^2 s}{\partial x \partial y}(x, y, \Psi) + \frac{\partial^2 s}{\partial x \partial z}(x, y, \Psi)\frac{\partial \Psi}{\partial y} \\ &= \frac{(\partial^2 s/\partial x \partial y)(\partial^2 s/\partial z^2) - (\partial^2 s/\partial x \partial z)(\partial^2 s/\partial y \partial z)}{\partial^2 s/\partial z^2}(x, y, \Psi), \\ \frac{\partial^2 S}{\partial y^2} &= \frac{\partial^2 s}{\partial y^2}(x, y, \Psi) + \frac{\partial^2 s}{\partial y \partial z}(x, y, \Psi)\frac{\partial \Psi}{\partial y} \\ &= \frac{(\partial^2 s/\partial y^2)(\partial^2 s/\partial z^2) - (\partial^2 s/\partial y \partial z)^2}{\partial^2 s/\partial z^2}(x, y, \Psi), \end{aligned}$$

giving

$$\left\{\frac{\partial^2 S}{\partial x^2}\frac{\partial^2 S}{\partial y^2} - \left(\frac{\partial^2 S}{\partial x \partial y}\right)^2\right\}\frac{\partial^2 s}{\partial z^2}(x, y, \Psi) = D_3(x, y, \Psi). \tag{7.9}$$

Now we turn to $I(k) = \int_{-a}^{a} \int_{-b}^{b} I_3(k)\, dx\, dy$. If $\partial^2 s/\partial z^2 > 0$ in H, then we have, by (7.3),

$$I(k) = T_1(k) + T_2(k) + T_3(k) + T_4(k) + O(k^{-2}) \tag{7.10}$$

with

$$\begin{aligned} T_1(k) = {} & 2^{1/2}\Gamma(\tfrac{1}{2})e^{\pi i/4} \int_{-a}^{a} \int_{-b}^{b} e^{ikS(x,y)} t(x, y, \Psi) \\ & \cdot \left\{\frac{\partial^2 s}{\partial z^2}(x, y, \Psi)\right\}^{-1/2} dx\, dy \cdot k^{-1/2}, \end{aligned} \tag{7.10.1}$$

$$T_2(k) = \tfrac{1}{2}\Gamma(\tfrac{3}{2})e^{3\pi i/4} \int_{-a}^{a} \int_{-b}^{b} e^{ikS(x,y)} f_2(0)\, dx\, dy \cdot k^{-3/2} \tag{7.10.2}$$

(for $f_2(0)$ see (7.3.2)),

$$T_3(k) = -i \int_{-a}^{a} \int_{-b}^{b} e^{iks(x,y,c)} t(x, y, c) \left\{\frac{\partial s}{\partial z}(x, y, c)\right\}^{-1} dx\, dy \cdot k^{-1}, \tag{7.10.3}$$

$$T_4(k) = i \int_{-a}^{a} \int_{-b}^{b} e^{iks(x,y,-c)} t(x, y, -c) \left\{\frac{\partial s}{\partial z}(x, y, -c)\right\}^{-1} dx\, dy \cdot k^{-1}. \tag{7.10.4}$$

If $\partial^2 s/\partial z^2 < 0$ in H, then we have, by (7.4),

$$I(k) = T_1(k) + T_2(k) + T_3(k) + T_4(k) + O(k^{-2}) \tag{7.11}$$

with

$$\begin{aligned} T_1(k) = {} & 2^{1/2}\Gamma(\tfrac{1}{2})e^{-\pi i/4} \int_{-a}^{a} \int_{-b}^{b} e^{ikS(x,y)} t(x, y, \Psi) \\ & \cdot \left\{-\frac{\partial^2 s}{\partial z^2}(x, y, \Psi)\right\}^{-1/2} dx\, dy \cdot k^{-1/2}, \end{aligned} \tag{7.11.1}$$

$$T_2(k) = \tfrac{1}{2}\Gamma(\tfrac{3}{2})e^{-3\pi i/4} \int_{-a}^{a} \int_{-b}^{b} e^{ikS(x,y)} f_2(0)\, dx dy \cdot k^{-3/2} \tag{7.11.2}$$

(for $f_2(0)$ see (7.4.2)),

and $T_3(k)$ and $T_4(k)$ as in (7.10.3) and (7.10.4).

We distinguish eight cases:

I. $D_1 > 0, D_2 > 0, D_3 > 0$.
II. $D_1 > 0, D_2 > 0, D_3 < 0$.
III. $D_1 > 0, D_2 < 0, D_3 > 0$.
IV. $D_1 > 0, D_2 < 0, D_3 < 0$.
V. $D_1 < 0, D_2 > 0, D_3 > 0$.
VI. $D_1 < 0, D_2 > 0, D_3 < 0$.
VII. $D_1 < 0, D_2 < 0, D_3 > 0$.
VIII. $D_1 < 0, D_2 < 0, D_3 < 0$.

Case I. By virtue of (7.6), (7.8), (7.9) and condition (B) we may apply the result (3.10) of Application 2 to $T_1(k)$ of (7.10.1). We obtain

$$\begin{aligned} T_1(k) = {} & 2^{1/2}\Gamma(\tfrac{1}{2})e^{\pi i/4}2\pi i\{D_3(0,0,0)\}^{-1/2}t(0,0,0)e^{iks(0,0,0)}k^{-3/2} \\ & + 2^{1/2}\Gamma(\tfrac{1}{2})e^{\pi i/4}\cdot -2^{1/2}\Gamma(\tfrac{1}{2})e^{3\pi i/4}e^{ikS(\varphi(b),b)}r(\varphi(b),b)k^{-2} \\ & + 2^{1/2}\Gamma(\tfrac{1}{2})^{\pi i/4}2^{1/2}\Gamma(\tfrac{1}{2})e^{3\pi i/4}e^{ikS(\varphi(-b),-b)}r(\varphi(-b),-b)k^{-2} \\ & + T_{11}(k) + T_{12}(k) + O(k^{-5/2}) \end{aligned}$$

with

$$T_{11}(k) = -2^{1/2}\Gamma(\tfrac{1}{2})e^{3\pi i/4}\int_{-b}^{b} e^{iks}t\left(\frac{\partial^2 s}{\partial z^2}\right)^{-1/2}\frac{\partial s}{\partial x}(a,y,\Psi(a,y))\,dy\cdot k^{-3/2},$$

$$T_{12}(k) = 2^{1/2}\Gamma(\tfrac{1}{2})e^{3\pi i/4}\int_{-b}^{b} e^{iks}t\left(\frac{\partial^2 s}{\partial z^2}\right)^{-1/2}\frac{\partial s}{\partial x}(-a,y,\Psi(-a,y))\,dy\cdot k^{-3/2}.$$

From (7.10.2) and the result (3.10) it follows that $T_2(k) = O(k^{-5/2})$. So we obtain from (7.10) the result

$$\begin{aligned} I(k) = {} & (2\pi)^{3/2}e^{3\pi i/4}\{D_3(0,0,0)\}^{-1/2}t(0,0,0)e^{iks(0,0,0)}k^{-3/2} \\ & + T_{11}(k) + T_{12}(k) + T_3(k) + T_4(k) + O(k^{-2}). \end{aligned}$$

The other cases can be treated in the same manner. Only in Case I do we deduce a general formula for the main contribution of the stationary point of $s(x, y, z)$, given by

$$t_1 = -(2\pi)^{3/2}e^{-\pi i/4}\{D_3(0,0,0)\}^{-1/2}t(0,0,0)e^{iks(0,0,0)}k^{-3/2}.$$

With the help of the results of Application 2 and (7.10) and (7.11) we find that in all cases

$$t_1 = (2\pi)^{3/2}\sigma_1 e^{\sigma_2\pi i/4}|D_3(0,0,0)|^{-1/2}t(0,0,0)e^{iks(0,0,0)}k^{-3/2},$$

where

$$\sigma_1 = \begin{cases} -1 & \text{if } D_1,\ D_1D_2,\ D_3 \text{ all have the same sign,} \\ 1 & \text{if } D_1,\ D_1D_2,\ D_3 \text{ have differing signs,} \end{cases}$$

$$\sigma_2 = \begin{cases} 1 & \text{if } D_3 < 0, \\ -1 & \text{if } D_3 > 0. \end{cases}$$

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SIAM J. MATH. ANAL.
Vol. 2, No. 1, February 1971

SOME INFINITE SUMS*

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1. Introduction. We are interested in deriving properties of

$$S_m(k) = \sum_{n=0}^{\infty} \alpha_n^{-2m-2}, \tag{1}$$

where the α's are the nonzero roots of $\tan \alpha = k\alpha$, for real nonzero constant k. It is clear that such roots occur in pairs $\pm\alpha$, and we take only one of each pair. The numbers α_n arise from the following Sturm–Liouville system:

$$\begin{gathered} u'' + \alpha^2 u = 0, \\ u(0) = 0, \qquad u(1) = ku'(1). \end{gathered} \tag{2}$$

We know from general theory that there exists an infinite, strictly increasing sequence of eigenvalues, $\alpha_0^2 < \alpha_1^2 < \cdots$. Moreover, the numbers α_n^2 are all real and positive with two exceptions: $\alpha_0^2 < 0$ if $0 < k < 1$, and vanishes for $k = 1$. Corresponding to each eigenvalue α_n^2 of (2) we have the eigenfunction

$$u_n(x) = \frac{\sin \alpha_n x}{\sin \alpha_n}. \tag{3}$$

These satisfy

$$\int_0^1 u_r(x)u_s(x)\,dx = \frac{1}{2}\delta_{rs}\frac{1 - k\cos^2\alpha_r}{\sin^2\alpha_r}. \tag{4}$$

In the special case $k = 1$, we have $\alpha_0 = 0$ and $u_0(y) = y$.

2. Generating functions. We define the generating function

$$G_k(t) = \sum_{l=0}^{\infty} S_l(k)t^{2l} = \sum_{n=0}^{\infty} \frac{1}{\alpha_n^2 - t^2} \tag{5}$$

and shall show that

$$G_k(t) = \frac{1}{2kt^2} + \left[1 + \frac{1-k}{k^2t^2}\right]\frac{k\sin t}{2(kt\cos t - \sin t)} \quad \text{for } k \neq 1 \tag{6}$$

and

$$G_1(t) = \frac{3}{2t^2} + \frac{\sin t}{2(t\cos t - \sin t)}. \tag{7}$$

We shall prove this by a number of stages, beginning with the case $k \neq 1$.

LEMMA 1.

$$x^3 + \frac{1-3k}{k-1}x = \sum_{n=0}^{\infty} A_n u_n(x), \tag{8}$$

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* Received by the editors November 3, 1969, and in revised form June 4, 1970.

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where

$$A_n = \frac{12(1-k)}{k} \cdot \frac{\sin^2 \alpha_n}{1 - k\cos^2 \alpha_n} \frac{1}{\alpha_n^4}, \qquad n = 0, 1, \cdots. \tag{9}$$

Proof. The left-hand side is twice continuously differentiable and satisfies the boundary conditions in (2). The possibility of an expansion of the form (8) follows from Sturm–Liouville theory. We determine the coefficients by use of the orthogonality relations (4).

LEMMA 2.

$$x = \frac{2(k-1)}{k} \sum_{n=0}^{\infty} \frac{\sin^2 \alpha_n}{1 - k\cos^2 \alpha_n} \cdot \frac{1}{\alpha_n^2} u_n(x). \tag{10}$$

Proof. Since $\alpha_n \sim (n + \frac{1}{2})\pi$ for large n, it follows that the coefficients in (9) satisfy $A_n = O(n^{-4})$. We may therefore differentiate (8) twice, term by term, and the result follows.

LEMMA 3. *If* $v(x)$ *is such that* $v(0) = 0$ *and* $v''(x)$ *is continuous in* $(0, 1)$, *then* $v(x)$ *can be expanded in the form* $\sum_{n=0}^{\infty} B_n u_n(x)$.

Proof. Define

$$u(x) = v(x) + \frac{v(1) - kv'(1)}{k-1} x. \tag{11}$$

Then $u(x)$ satisfies the boundary conditions in (2), while the last term on the right-hand side of (11) can be expanded by Lemma 2.

LEMMA 4. *For each fixed* t,

$$\sin xt = \sum_{n=0}^{\infty} C_n(t) u_n(x), \tag{12}$$

where

$$C_n(t) = \frac{2}{k}(\sin t - kt\cos t)(t^2 - \alpha_n^2)^{-1} \frac{\sin^2 \alpha_n}{1 - k\cos^2 \alpha_n}. \tag{13}$$

Proof. The possibility of the expansion (12) follows from Lemma 3, and the coefficients $C_n(t)$ are easily obtained in the usual way, using (4).

LEMMA 5.

$$G_k(t) = \left[1 + \frac{1-k}{k^2 t^2}\right] \sum_{l=0}^{\infty} h_l t^{2l} - \frac{(1-k)}{k^2 t^2} h_0, \tag{14}$$

where

$$h_l = \sum_{n=0}^{\infty} \frac{\sin^2 \alpha_n}{1 - k\cos^2 \alpha_n} \alpha_n^{-2l-2}, \qquad l = 0, 1, 2, \cdots. \tag{15}$$

Proof.

$$\begin{aligned} h_{l+1} &= \sum_{n=0}^{\infty} \frac{\sin^2 \alpha_n}{1 - k\cos^2 \alpha_n} \alpha_n^{-2l-4} \\ &= k^2 \sum_{n=0}^{\infty} \frac{\cos^2 \alpha_n}{1 - k\cos^2 \alpha_n} \alpha_n^{-2l-2} \qquad \text{(since } \tan \alpha_n = k\alpha_n\text{)} \\ &= \frac{k^2}{k-1} \sum_{n=0}^{\infty} \frac{\sin^2 \alpha_n - (1 - k\cos^2 \alpha_n)}{1 - k\cos^2 \alpha_n} \cdot \alpha_n^{-2l-2} \\ &= \frac{k^2}{k-1}(h_l - S_l). \end{aligned} \tag{16}$$

If we now define

$$H(t) = \sum_{l=0}^{\infty} h_l t^{2l}, \tag{17}$$

then it follows from (16) that

$$\frac{1}{t^2}[H(t) - h_0] = \frac{k^2}{k-1}[H(t) - G_k(t)] \tag{18}$$

and (14) follows immediately.

LEMMA 6.

$$H(t) = \frac{k \sin t}{2(kt \cos t - \sin t)}. \tag{19}$$

Proof. Set $x = 1$ in (12). Then

$$\begin{aligned} \sin t &= \sum_{n=0}^{\infty} C_n(t) \\ &= \frac{2}{k}(\sin t - kt\cos t) \sum_{n=0}^{\infty} \frac{\sin^2 \alpha_n}{1 - k\cos^2 \alpha_n} \frac{1}{t^2 - \alpha_n^2} \qquad \text{(by (13))} \\ &= -\frac{2}{k}(\sin t - kt\cos t) \sum_{l=0}^{\infty} h_l t^{2l}, \qquad \text{by (15) and hence (19).} \end{aligned}$$

LEMMA 7.

$$h_0 = \frac{k}{2(k-1)}. \tag{20}$$

Proof. This follows immediately from (19) since

$$h_0 = \lim_{t \to 0} H(t).$$

The formula (6) now follows immediately from (14), (17), (19), (20). To derive (7), we recall that for $k = 1$, $\alpha_0 = 0$, $u_0(y) = y$. It follows that

$$G_1(t) = \sum_{n=0}^{\infty} \frac{1}{\alpha_n^2 - t^2} + \frac{1}{t^2} = \lim_{k \to 1} G_k(t) + \frac{1}{t^2}.$$

3. Recursion relations.

Case 1. $k \neq 1$. If we clear the expression (6) of fractions and then expand both sides in powers of t, we get the relation

$$\sum_{m=0}^{n} \frac{(-1)^m[k(2n-2m+1)-1]}{(2n-2m+1)!} S_m = \frac{n+1}{(2n+3)!}[k(2n+3)-1]. \tag{21}$$

As special cases we obtain

$$\begin{aligned} S_0 &= \sum_{l=0}^{\infty} \alpha_l^{-2} = \frac{3k-1}{6(k-1)}, \\ S_1 &= \sum_{l=0}^{\infty} \alpha_l^{-4} = \frac{15k^2-6k+1}{90(k-1)^2}, \\ S_2 &= \sum_{l=0}^{\infty} \alpha_l^{-6} = \frac{63k^3-36k^2+9k-1}{945(k-1)^3}. \end{aligned} \tag{22}$$

Case 2. $k = 1$. By using the above technique on (7) we obtain

$$\sum_{m=0}^{n} \frac{(-1)^m(n+1-m)}{(2n-2m+3)!} S_m = \frac{(n+1)(n+2)}{(2n+5)!}; \tag{23}$$

and we now get $S_0 = 1/10$, $S_1 = 1/350$, and $S_2 = 1/7875$ (see also § 7).

4. Limiting values of k.

Case 1. $k \to 0$. We consider k tending to 0 through positive values. For small positive k, α_0 is purely imaginary and $|\alpha_0|$ tends to infinity as $k \to 0$. The term α_0^{-2l-2} therefore drops out of S_l in the limit. On the other hand $\lim_{k\to 0} \alpha_n = n\pi$ and hence

$$\lim_{k\to 0} S_l = \sum_{n=1}^{\infty} (n\pi)^{-2l-2} = \pi^{-2l-2}\zeta(2l+2). \tag{24}$$

The limiting process in (24) is immediate since the convergence of the series for S_l is uniform with respect to k.

By letting $k \to 0$ in (6), we get the well-known formula

$$\pi t \cdot \cot \pi t = 1 + 2t^2 \cdot \sum_{n=1}^{\infty} \frac{1}{t^2 - n^2} \tag{25}$$

(see [1, p. 207]). Again it is known that

$$\zeta(2p) = (-1)^{p-1} \frac{B_{2p}(2\pi)^{2p}}{2 \cdot (2p)!}, \tag{26}$$

where the B_r's are Bernoulli numbers (see [1, p. 237]). By combining (24), (26) and (21), we get, after a little manipulation,

$$\sum_{r=0}^{2n+1} \binom{2n+1}{r} 2^r B_r = 0,$$

or, in the usual symbolic form,

$$(1 + 2B)^{2n+1} = 0, \qquad n = 0, 1, 2, \cdots, \tag{27}$$

which is a well-known formula.

Case 2. $k \to \infty$. Clearly $\lim_{k\to\infty} \alpha_n = (n + \frac{1}{2})\pi$ and we get

$$\lim_{k\to\infty} S_l = \sum_{n=1}^{\infty} [(n + \tfrac{1}{2})\pi]^{-2l-2} = \pi^{-2l-2}(2^{2l+2} - 1)\zeta(2l + 2), \tag{28}$$

while the limiting form of $G_k(t)$ yields

$$\tan\frac{\pi t}{2} = \sum_{n=0}^{\infty} \frac{4t}{(2n + 1)^2 - t^2} \tag{29}$$

(see [1, p. 208]). Substitution of (28) and (26) in the recurrence relation yields another known symbolic formula:

$$(1 + 2B)^{2n} - (1 + 4B)^{2n} = 0, \qquad n = 0, 1, \cdots. \tag{30}$$

5. General form of S_m for $k \neq 1$.

THEOREM.

$$S_m(k) = (k - 1)^{-m-1}P_{m+1}(k), \tag{31}$$

where P_{m+1} is a polynomial of degree $m + 1$ in k, with rational coefficients, and

$$P_{m+1}(1) = 3^{-m-1}. \tag{32}$$

In particular $P_{m+1}(1) \neq 0$.

Proof. The theorem is true by (22) for $m = 0, 1, 2$. Suppose it is true for $0 \leqq m \leqq n$. By (21) we have

$$\begin{aligned}(-1)^{n+1}&(k - 1)S_{n+1}\\ &= \frac{n + 2}{(2n + 5)!}[k(2n + 5) - 1] - \sum_{m=0}^{n} \frac{(-1)^m[k(2n - 2m + 3) - 1]}{(2n - 2m + 3)!}S_m \\ &= \frac{n + 2}{(2n + 5)!}[k(2n + 5) - 1] - \sum_{m=0}^{n} \frac{(-1)^m[k(2n - 2m + 3) - 1]}{(2n - 2m + 3)!}\,\frac{P_{m+1}(k)}{(k - 1)^{m+1}},\end{aligned} \tag{33}$$

by the inductive hypothesis.

The right-hand side of (33) is clearly of the form $Q(k)/(k - 1)^{n+1}$, where Q is a polynomial of degree $\leqq n + 2$ in k, with rational coefficients, and

$$S_{n+1} = \frac{(-1)^{n+1}Q(k)}{(k - 1)^{n+2}}.$$

However we have seen above that $\lim_{k\to\infty} S_l(k) \neq 0$ for each l, and so $Q(k)$ must be of the degree exactly $n + 2$. It follows then from (33) that

$$(-1)^{n+1}P_{n+2}(k) = (-1)^{n+1}\frac{3k - 1}{6}P_{n+1}(k) + O(k - 1),$$

and hence

$$P_{n+2}(1) = \tfrac{1}{3}P_{n+1}(1).$$

This completes the induction. We may note further that the leading term in $P_{n+1}(k)$ is

$$\frac{(2^{2n+2}-1)}{\pi^{2n+2}}\zeta(2n+2)k^{n+1}.$$

This follows immediately from (31) and (28). Again it follows from (24) that

$$P_{n+1}(0) = \frac{(-1)^{n+1}}{\pi^{2n+2}}\zeta(2n+2).$$

An explicit form of S_m. By using (21), one can give a determinant representation of $S_m(k)$. On taking the upper limit of the summation in (21) to be $0, 1, 2, \cdots, n$, we obtain a set of $n+1$ linear equations in $S_0, \cdots, S_n$. The determinant Δ of this system is triangular, and hence,

$$\Delta = (-1)^{n(n+1)/2}(k-1)^{n+1}.$$

Then, by Cramer's rule,

$$S_n(k) = \frac{(-1)^n}{(k-1)^{n+1}}\begin{vmatrix} \frac{(k-1)}{1!} & 0 & 0 & \cdots & 0 & 1\cdot\frac{3k-1}{3!} \\ \frac{3k-1}{3!} & \frac{k-1}{1!} & 0 & \cdots & 0 & 2\cdot\frac{5k-1}{5!} \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \frac{(2n+1)k-1}{(2n+1)!} & \cdot & \cdot & \cdots & \frac{3k-1}{3!} & (n+1)\cdot\frac{(2n+3)k-1}{(2n+3)!} \end{vmatrix}. \tag{34}$$

Letting $k \to 0$ and using (24) and (26), we have

$$B_{2n} = \frac{(2n)!}{2^{2n-1}}\begin{vmatrix} \frac{1}{1!} & 0 & 0 & \cdots & 0 & \frac{1}{3!} \\ \frac{1}{3!} & \frac{1}{1!} & 0 & \cdots & 0 & \frac{2}{5!} \\ \cdot & \cdot & & \cdots & \cdot & \cdot \\ \frac{1}{(2n-1)!} & \frac{1}{(2n-3)!} & & \cdots & \frac{1}{3!} & \frac{n}{(2n+1)!} \end{vmatrix},$$

which after some manipulation coincides with the expression given by Kishore [4].

Letting $k \to \infty$ and using (26) and (28), we obtain

$$G_n = 2(1 - 2^{2n})B_{2n} = \frac{-(2n)!}{2^{2n-2}} \begin{vmatrix} \frac{1}{1!} & 0 & 0 & \cdots & 0 & 1 \cdot \frac{1}{2!} \\ \frac{1}{2!} & \frac{1}{1!} & 0 & \cdots & 0 & 2 \cdot \frac{1}{4!} \\ \cdot & & & \cdots & \cdot & \cdot \\ \frac{1}{(2n-2)!} & & & \cdots & \frac{1}{2!} & n \cdot \frac{1}{(2n)!} \end{vmatrix},$$

where G_n are the Genocchi numbers. Again this coincides with the expression given by Kishore [4].

6. An analogous problem. If we start from the numbers β_n, defined as the roots of $k \cot \beta + \beta = 0$, we can construct

$$T_l(k) = \sum_{n=0}^{\infty} \beta_n^{-2l-2}, \qquad l = 0, 1, \cdots.$$

The generating function now turns out to be

$$(35) \qquad \begin{aligned} A_k(t) &= \sum_{l=0}^{\infty} T_l(k)t^{2l} = \sum_{n=0}^{\infty} \frac{1}{\beta_n^2 - t^2} \\ &= \frac{k-1}{2t^2} - \left[1 + \frac{k(k-1)}{t^2}\right] \frac{\cos t}{2(k \cos t + t \sin t)}, \end{aligned}$$

and hence the recurrence relation is

$$(36) \qquad \sum_{m=0}^{n} \frac{(-1)^m(2n-2m-k)}{(2n-2m)!} T_m(k) = \frac{2n+2-k}{2[(2n+1)!]}.$$

Again we have

$$(37) \qquad \lim_{k \to \infty} T_l(k) = \sum_{k=1}^{\infty} [(n + \tfrac{1}{2})\pi]^{-2l-2}.$$

We see at once that $\lim_{k \to 0} \beta_0 = 0$, and so the sums for $T_l(0)$ must start with $n = 1$ with the appropriate modification to the generating function. We then deduce that $\lim_{k \to \infty} T_l(k) = \pi^{-2l-2}\zeta(2l + 2)$.

We can verify that

$$T_0(k) = \frac{1}{2} - \frac{1}{k}, \qquad T_1(k) = \frac{1}{6} - \frac{2}{3k} + \frac{1}{k^2},$$

and, by induction using (36), that $T_l(k)$ is a polynomial of degree $l + 1$ in k^{-1} with rational coefficients. Again, by using the same method as was used in deriving the explicit form of S_m in § 5, one gets an explicit determinant representation for $T_n(k)$:

$$(38)\qquad T_n(k) = \frac{(-1)^n}{(-k)^{n+1}} \begin{vmatrix} \frac{(-k)}{0!} & 0 & 0 & \cdots & 0 & \frac{2-k}{2\cdot 1!} \\ \frac{2-k}{2!} & \frac{(-k)}{0!} & 0 & \cdots & 0 & \frac{4-k}{2\cdot 3!} \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \frac{2n-k}{(2n)!} & & & & \frac{2-k}{2!} & \frac{2n+2-k}{2(2n+1)!} \end{vmatrix} .$$

7. Related work. When $k = 1$,

$$S_m(1) = \sigma_{2m+2}(\tfrac{3}{2})$$

and

$$T_m(1) = \sigma_{2m+2}(-\tfrac{3}{2}),$$

where $\sigma_{2n}(v)$, $n = 1, 2, \cdots$, are the well-known Rayleigh functions, i.e.,

$$\sigma_{2n}(v) \equiv \sum_{m=1}^{\infty} \frac{1}{j_{v,m}^{2n}}, \qquad n = 1, 2, \cdots,$$

where $j_{v,m}$ is the mth positive zero of the Bessel function $J_v(z)$. These functions are used to evaluate the first zeros of $J_v(z)$. The values for $S_m(1)$ and $T_m(1)$ agree with the values quoted by Watson [2, p. 502] for $v = \frac{3}{2}$ and $v = -\frac{3}{2}$ respectively.

Lorch [3] has also used $S_m(1)$ to evaluate certain estimates connected with the Riemann summation method $(R, 2l)$, $l = 1, 2, \cdots$. Kishore has discussed properties of the Rayleigh functions in [4] and [8], and of the related Rayleigh polynomial in [5], [6], [7] and [9]. The Rayleigh functions have also been discussed by Carlitz [10].

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SIAM J. MATH. ANAL.
Vol. 2, No. 1, February 1971

AN INTEGRAL OPERATOR APPROACH TO CAUCHY'S PROBLEM FOR $\Delta_{p+2}u(\mathbf{x}) + F(\mathbf{x})u(\mathbf{x}) = 0$*

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Abstract. In this paper an integral operator approach is developed for the Cauchy problem associated with the elliptic equation

$$\Delta_{p+2}u(\mathbf{x}) + F(\mathbf{x})u(\mathbf{x}) = 0, \quad \mathbf{x} = (x_1, x_2, \cdots, x_{p+2}), \quad p = 1, 2.$$

For the case $p = 1$, use is made of an integral representation due to Tjong, which reduces to the Bergman–Whittaker representation when $F(\mathbf{x}) = 0$. For $p = 2$, we develop a new representation of the Tjong type, which reduces to Gilbert's operator for harmonic functions in four variables when $F(\mathbf{x}) \equiv 0$. We also treat the case of Cauchy's problem for the given equation when $p \geqq 1$ and $F(\mathbf{x}) \equiv B(r^2)$, $r = |\mathbf{x}|$. Here the appropriate integral representations are found by using the method of ascent.

1. Introduction. In this paper we shall develop a constructive method for obtaining solutions to Cauchy's problem for a class of linear second order elliptic partial differential equations in $p + 2$ variables with analytic coefficients. Such problems arise frequently in the use of inverse methods to solve free boundary problems in mathematical physics (cf. [6], [7], [12]), and our approach is important in the sense that it can be used to obtain sequences of analytic approximations which converge to the desired solution. The difficulties in trying to develop such approximation methods arise from the fact that Cauchy's problem for elliptic equations is one of the classical examples of an improperly posed problem in the sense of Hadamard. We illustrate this by two examples for the case of Laplace's equation.

Example 1 (Hadamard). Consider

$$\Delta_2 u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1.1}$$

with Cauchy data

$$u(x, 0) = 0, \qquad \frac{\partial u}{\partial y}(x, 0) = n^{-1} \sin nx. \tag{1.2}$$

It is easily verified that $u(x, y) = n^{-2} \sin nx \sinh ny$ is the unique solution of this particular Cauchy problem. However as $n \to \infty$ the Cauchy data tends to zero whereas the solution does not. Since $u(x, y) \equiv 0$ is the only solution of Cauchy's problem with zero Cauchy data, it is seen that the solution does not depend continuously on the initial data. (However in this regard see [14] and the references cited there.)

Example 2 (Schwarz). Suppose there exists a solution $u(x, y)$ for $y > 0$ to (1.1) with Cauchy data

$$u(x, 0) = 0, \quad \frac{\partial u}{\partial y}(x, 0) = f(x). \tag{1.3}$$

* Received by the editors February 26, 1970, and in revised form July 16, 1970.

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By the Schwarz reflection principle [3, p. 254] it is possible to analytically continue $u(x, y)$ across the x-axis, which implies that $f(x)$ must be an analytic function of x. Therefore in general no solution of Cauchy's problem exists unless the Cauchy data is analytic.

In view of Example 2 we shall only consider problems with analytic Cauchy data, which is in fact what occurs in the abovementioned physical applications. The Cauchy–Kowalewski theorem [3] will in this case assure us of the existence of a unique solution. However such a power series solution is not satisfactory for approximations and we look for more constructive techniques. Several approaches have been given in recent years, and indeed the theory of two-dimensional problems is now essentially complete [2], [7], [13]. For higher dimensions the situation is not so pleasant however, since the only available procedures thus far consist either in converting an elliptic Cauchy problem in only $p + 2$ variables to a hyperbolic Cauchy problem in no less than $2p + 3$ variables [5], or in evaluating intricate contour integrals in the space of several complex variables [4]. Except for the special case of the $(p + 2)$-dimensional Laplace equation, prodigious difficulties arise in employing either of these methods for analytic approximation. This is due in the first case to the singular nature of the Riemann matrix in spaces of high dimension and in the second case to problems in explicitly evaluating the resulting contour integral representations of the solution.

In view of these difficulties in approximating solutions to Cauchy's problem in higher dimensional space we present in this paper a new approach to the problem through the use of integral operators. The theory of integral operators for equations in more than two independent variables, as created by Bergman [1], has gained wide renown for its elegant development of essential parts of the theory of partial differential equations on the basis of the theory of functions of several complex variables. The basic analytic nature of Cauchy's problem for elliptic equations suggests the possible fruitfulness of an integral operator approach to the problem, and this indeed turns out to be the case. Several significant modifications and extensions of the original ideas of Bergman must be made however in order to achieve this success. To be specific, let

$$\mathbf{L}u = 0, \tag{1.4}$$

where $\mathbf{L}$ is a linear second order elliptic operator in $p + 2$ variables with analytic coefficients, and suppose we wish to find a solution u of (1.4) such that u satisfies the Cauchy data

$$\begin{aligned} u(\mathbf{x})|_{x_{p+2}=0} &= f(x_1, \cdots, x_{p+1}), \\ \frac{\partial u}{\partial x_{p+2}}(\mathbf{x})\bigg|_{x_{p+2}=0} &= g(x_1, \cdots, x_{p+1}) \end{aligned} \tag{1.5}$$

on the plane $x_{p+2} = 0$, where $\mathbf{x} = (x_1, \cdots, x_{p+2})$ and f and g are analytic functions of their independent variables. We now note that if $\mathbf{L}$ is the Laplacian, then a closed form solution of the Cauchy problem (1.4), (1.5) can be found which is suitable for analytic approximations (see § 2). This suggests trying to construct an operator which maps *harmonic functions* onto solutions u of (1.4) in a manner which preserves Cauchy data. To this end we begin in the spirit of Bergman by finding an integral operator $\mathbf{T}_1$ which maps *analytic functions* h of $p + 1$ complex variables onto solutions u of (1.4):

$$u = \mathbf{T}_1 h. \tag{1.6}$$

In many cases one can construct a great variety of such operators. Hence the problem is not to introduce *any* such operator but rather ones which possess certain properties pertinent to our problem; in particular, $\mathbf{T}_1$ should have an *inverse* such that the analytic function corresponding to a given solution u of (1.4) can be determined. To accomplish this we require that when $\mathbf{L}$ reduces to the Laplacian, $\mathbf{T}_1$ remains invertible and in fact reduces to a known representation of harmonic functions in terms of analytic functions of several complex variables. Let $\mathbf{B}_{p+2}$ be such an operator for Laplace's equation and regard $\mathbf{T}_1$ as a perturbation of $\mathbf{B}_{p+2}$, i.e.,

$$u = \mathbf{T}_1 h = (\mathbf{B}_{p+2} + \mathbf{T}_2)h, \tag{1.7}$$

where $\mathbf{T}_2 \equiv \mathbf{T}_1 - \mathbf{B}_{p+2}$. Since $\mathbf{B}_{p+2}$ is invertible, we can express (1.7) as

$$\begin{aligned} u &= H + \mathbf{T}_2\mathbf{B}_{p+2}^{-1}H \\ &= H + \mathbf{T}H \\ &= (\mathbf{I} + \mathbf{T})H, \end{aligned} \tag{1.8}$$

where $H = \mathbf{B}_{p+2}h$ is a harmonic function and $\mathbf{T} \equiv (\mathbf{T}_1 + \mathbf{B}_{p+2})\mathbf{B}_{p+2}^{-1}$. If $\|\mathbf{T}\| < 1$, then the operator $\mathbf{I} + \mathbf{T}$ is invertible. The problem with which we are now faced is to select $\mathbf{T}_1$ and $\mathbf{B}_{p+2}$ (and hence $\mathbf{T}$) in such a way that $\mathbf{T}_1 = \mathbf{B}_{p+2}$ for $\mathbf{L} = \mathbf{\Delta}_{p+2}$ and such that Cauchy data for H can be constructed in a simple manner from Cauchy data for u. We can then construct H from this data and represent the solution by (1.8). We also want $\mathbf{T}$ to have a sufficiently "nice" kernel such that $(\mathbf{I} + \mathbf{T})H$ can be readily approximated.

In this paper we carry out the above program for the differential equations

$$\Delta_3 u + F_1(x_1, x_2)u = 0, \tag{1.9}$$

$$\Delta_4 u + F_2(x_1, x_2, x_3)u = 0, \tag{1.10}$$

$$\Delta_{p+2} u + B(r^2)u = 0, \tag{1.11}$$

where F_1, F_2 and B are entire functions of their independent variables, $r = |\mathbf{x}|$. Our results are also valid when F_1, F_2 and B are only assumed to be analytic in a sufficiently large ball about the origin. However for the sake of clarity of presentation we make the assumption that these functions are in fact entire. For (1.9) we can select $\mathbf{T}_1$ to be an operator recently discovered by Tjong [15] and $\mathbf{B}_3$ to be the well-known Bergman–Whittaker operator [1]. Equation (1.10) requires the construction of a new integral operator $\mathbf{T}_1$, the first such operator found for equations in more than three variables which do not have spherically symmetric coefficients. The operator $\mathbf{B}_4$ in this case is Gilbert's generalization of the Bergman–Whittaker operator to four dimensions [8]. For more than four variables there is no convenient generalization of the Bergman–Whittaker operator and hence no natural way to motivate the construction of $\mathbf{T}_1$. However we are able to handle equations of the form (1.11) by making use of the *method of ascent*, which has recently been developed by one of us [9], [10]. In each case the kernel of the transformation $\mathbf{T}$ is an entire function of its independent variables and is expressed in terms of an infinite series which can be easily approximated.

We finally would like to emphasize that this paper should be viewed only as an introduction to the use of integral operator techniques in the investigation of improperly posed problems of mathematical physics. Our hope is that this paper will encourage and motivate further developments in this area by other mathematicians and physicists now working in the field.

2. Cauchy's problem for Laplace's equation. Consider the $(p+2)$-dimensional Laplace equation

$$u_{x_{p+2}x_{p+2}} = -u_{x_1x_1} - u_{x_2x_2} - \cdots - u_{x_{p+1}x_{p+1}} \tag{2.1}$$

subject to the initial conditions

$$\begin{aligned} u(x_1, \cdots, x_{p+1}, 0) &= f(x_1, \cdots, x_{p+1}), \\ \frac{\partial u}{\partial x_{p+2}}(x_1, \cdots, x_{p+1}, 0) &= g(x_1, \cdots, x_{p+1}), \end{aligned} \tag{2.2}$$

where f and g are analytic functions of the variables $x_1, \cdots, x_{p+1}$ in some neighborhood of the origin. We first note that if u is a solution of (2.1), (2.2) for $f \equiv 0$, then $u_1 = \partial u/\partial x_{p+2}$ is a solution of (2.1) with Cauchy data $u_1 = g$, $\partial u_1/\partial x_{p+2} = 0$. Hence $U = u + \tilde{u}_1$ is the solution of the complete Cauchy problem if $\tilde{u}$ is a solution of (2.1) such that $\tilde{u} = 0$, $\partial \tilde{u}/\partial x_{p+2} = f$. Therefore without loss of generality we set $f = 0$ in (2.2). From the Cauchy–Kowalewski theorem we know that there exists a convergent power series solution u in some neighborhood of the origin to (2.1) such that $u = 0$, $\partial u/\partial x_{p+2} = g$. Following Garabedian [5] we extend this power series solution to complex values of its independent variables; in particular, we keep the coordinate x_{p+2} real but replace x_j by corresponding complex variables $z_j = x_j + iy_j$, $j = 1, \cdots, p+1$. Since u is an analytic function, the Cauchy–Riemann equations

$$\frac{\partial u}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial u}{\partial x_j} + i\frac{\partial u}{\partial y_j}\right) = 0, \tag{2.3}$$

and hence Laplace's equation

$$u_{x_jx_j} + u_{y_jy_j} = 0 \tag{2.4}$$

is satisfied. Equations (2.1) and (2.4) now give

$$\begin{aligned} u_{x_{p+2}x_{p+2}} &= \sum_{j=1}^{p+1} 2(u_{x_jx_j} + u_{y_jy_j}) - u_{x_1x_1} - \cdots - u_{x_{p+1}x_{p+1}} \\ &= u_{x_1x_1} + \cdots + u_{x_{p+1}x_{p+1}} + 2u_{y_1y_1} + \cdots + 2u_{y_{p+1}y_{p+1}}, \end{aligned} \tag{2.5}$$

i.e.,

$$\tilde{u}(x_1, \cdots, x_{p+2}, y_1, \cdots, y_{p+1}) \equiv u(x_1 + iy_1, \cdots, x_{p+1} + iy_{p+1}, x_{p+2})$$

satisfies a linear hyperbolic equation. Now make the change of variables

$$\tilde{y}_j = \frac{1}{\sqrt{2}} y_j, \qquad j = 1, 2, \cdots, p+1. \tag{2.6}$$

Then $v(x_1, \cdots, x_{p+2}, \tilde{y}_1, \cdots, \tilde{y}_{p+1}) \equiv \tilde{u}(x_1, \cdots, x_{p+2}, \sqrt{2}\tilde{y}_1, \cdots, \sqrt{2}\tilde{y}_{p+1})$ satisfies

$$v_{x_{p+2}x_{p+2}} = v_{x_1x_1} + \cdots + v_{x_{p+1}x_{p+1}} + v_{\tilde{y}_1\tilde{y}_1} + \cdots + v_{\tilde{y}_{p+1}\tilde{y}_{p+1}} \tag{2.7}$$

and the initial conditions

$$\begin{aligned} v(x_1, \cdots, x_{p+1}, 0, \tilde{y}_1, \cdots, \tilde{y}_{p+1}) &= 0, \\ v_{x_{p+2}}(x_1, \cdots, x_{p+1}, 0, \tilde{y}_1, \cdots, \tilde{y}_{p+1}) &= \tilde{g}(x_1, \cdots, x_{p+1}, \tilde{y}_1, \cdots, \tilde{y}_{p+1}) \\ &\equiv g(x_1 + i\sqrt{2}\tilde{y}_1, \cdots, x_{p+1} + i\sqrt{2}\tilde{y}_{p+1}). \end{aligned} \tag{2.8}$$

Equations (2.7) and (2.8) constitute a well-posed Cauchy problem for the wave equation in $2p + 3$ variables and hence we can immediately write down the solution in terms of spherical means (cf. [3]):

$$\begin{aligned} &v(x_1, \cdots, x_{p+2}, \tilde{y}_1, \cdots, \tilde{y}_{p+1}) \\ &= \frac{1}{(2p)!} \frac{\partial^{2p}}{\partial x_{p+2}^{2p}} \int_0^{x_{p+2}} (x_{p+2}^2 - \rho^2)^{(2p-1)/2} \rho Q_p(\mathbf{x}, \rho)\, d\rho, \end{aligned} \tag{2.9}$$

where

$$\begin{aligned} Q_p(\mathbf{x}, \rho) &= \frac{1}{\omega_{2p+2}} \int \cdots \int \tilde{g}(\mathbf{x} + \boldsymbol{\alpha}\rho)\, d\omega_{2p+2}, \\ \mathbf{x} &= (x_1, \cdots, x_{p+1}, \tilde{y}_1, \cdots, \tilde{y}_{p+1}), \\ \boldsymbol{\alpha} &= (\alpha_1, \cdots, \alpha_{2p+2}), \qquad \boldsymbol{\alpha} \cdot \boldsymbol{\alpha} = 1 \end{aligned} \tag{2.10}$$

and ω_{2p+2} is the surface area of the unit sphere in $(2p + 2)$-dimensional space.

We are interested in the restriction of (2.9) to real $(p + 2)$-dimensional space. Therefore setting $y_j = 0, j = 1, \cdots, p + 1$, we obtain the solution to the Cauchy problem (2.1), (2.2) ($f \equiv 0$) as

$$\begin{aligned} &u(x_1, \cdots, x_{p+2}) \\ &= \frac{1}{(2p)!} \frac{\partial^{2p}}{\partial x_{p+2}^{2p}} \int_0^{x_{p+2}} (x_{p+2}^2 - \rho^2)^{(2p-1)/2} \rho Q_p(x_1, \cdots, x_{p+1}, \rho)\, d\rho, \end{aligned} \tag{2.11}$$

where

$$\begin{aligned} &Q_p(x_1, \cdots, x_{p+1}, \rho) \\ &= \frac{1}{\omega_{2p+2}} \int \cdots \int g(x_1 + (\alpha_1 + i\sqrt{2}\alpha_{p+1})\rho, \\ &\qquad \cdots, x_{p+1} + (\alpha_{p+1} + i\sqrt{2}\alpha_{2p+2})\rho)\, d\omega_{2p+2}. \end{aligned} \tag{2.12}$$

If $\tilde{u}$ denotes the solution (2.11), (2.12) with g replaced by f, then the solution U to the complete Cauchy problem (2.1), (2.2) can be written as

$$U = u + \frac{\partial \tilde{u}}{\partial x_{p+2}}. \tag{2.13}$$

Equations (2.11), (2.12), (2.13) can now be used to obtain analytic approximations to solutions of (2.1), (2.2). Note that in order to accomplish this it is necessary in view of (2.12) to approximate $g(x_1, \cdots, x_{p+1})$ uniformly for complex values of its independent variables.

3. Cauchy's problem for $\Delta_3 u + F(x_1, x_2)u = 0$. In [15] Tjong showed that it was possible to generate solutions of

$$\Delta_3 u + F(x_1, x_2, x_3)u = 0 \tag{3.1}$$

when $F(x_1, x_2, x_3)$ was an entire function in $\mathcal{C}^3$ where $\mathcal{C}^3$ denotes the space of three complex variables. In order to present her result we first introduce the following notations:

$$\begin{aligned} X &= x_3, \quad Z = \tfrac{1}{2}(x_1 + ix_2), \quad Z^* = \tfrac{1}{2}(-x_1 + ix_2), \\ w &= (1 - t^2)v, \quad v = X + \zeta Z + \zeta^{-1}Z^*, \\ \xi_1 &= X, \quad \xi_2 = X + 2\zeta Z, \quad \xi_3 = X + 2\zeta^{-1}Z^*. \end{aligned} \tag{3.2}$$

Then (3.1) can be written in the form

$$\frac{\partial^2 \psi}{\partial X^2} - \frac{d^2 \psi}{\partial Z \partial Z^*} + \hat{F}(X, Z, Z^*)\psi = 0, \tag{3.3}$$

where $\psi(X, Z, Z^*) \equiv u(x_1, x_2, x_3)$, $\hat{F}(X, Z, Z^*) \equiv F(x_1, x_2, x_3)$, and there exist solutions of (3.1) of the form

$$\begin{aligned} u(x_1, x_2, x_3) &\equiv \psi(X, Z, Z^*) = \mathbf{T}_1 f \\ &= \frac{1}{2\pi i}\int_{|\zeta|=1}\int_\gamma E(X, Z, Z^*, \zeta, t) f(w, \zeta)\frac{dt}{\sqrt{1-t^2}}\frac{d\zeta}{\zeta}, \end{aligned} \tag{3.4}$$

where γ is a rectifiable curve joining $t = -1$ to $t = 1$, $f(w, \zeta)$ is an analytic function of two complex variables, and

$$\begin{aligned} E(X, Z, Z^*, \zeta, t) &= \hat{E}(\xi_1, \xi_2, \xi_3, \zeta, t) \\ &= 1 + \sum_{n \geqq 1} t^{2n} v^n p^{(n)}(\xi_1, \xi_2, \xi_3, \zeta). \end{aligned} \tag{3.5}$$

The $p^{(n)}$ are defined by

$$\begin{aligned} &\frac{\partial p^{(n+1)}}{\partial \xi_1} \equiv p_1^{(n+1)} = -\frac{1}{2n+1}\{p_{11}^{(n)} + p_{22}^{(n)} + p_{33}^{(n)} + 2p_{12}^{(n)} + 2p_{13}^{(n)} - 2p_{23}^{(n)} + \hat{F}p^{(n)}\}, \\ &p^{(0)}(\xi_1, \xi_2, \xi_3, \zeta) = 1, \quad p^{(n+1)}(0, \xi_2, \xi_3, \zeta) = 0, \\ &\frac{\partial^2 p^{(n)}}{\partial \xi_i \partial \xi_j} = p_{ij}^{(n)}, \qquad i = 1, 2, 3, \quad j = 1, 2, 3. \end{aligned} \tag{3.6}$$

Let us introduce the further notation

$$\mathbf{X} = (X, Z, Z^*), \quad \boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3), \quad \mathbf{X}' = (X', Z', Z^{*\prime}),$$

$$\hat{v} = X' + \zeta\left(1 - \frac{1}{\alpha}\right)Z' + \zeta^{-1}\left(1 - \frac{1}{\alpha}\right)^{-1} Z^{*\prime}.$$

In addition, we introduce the function

$$M(\mathbf{X}; \mathbf{X}') \equiv \frac{-a}{4\pi^3 i} \int_{|\zeta|=1} \frac{d\zeta}{\zeta} \int_0^u P(\boldsymbol{\xi}; v - s, \zeta) A_a(s, \hat{v})\, ds,$$

where

$$P(\boldsymbol{\xi}; v - s, \zeta) \equiv \sum_{n \geqq 1} \frac{1}{B(n, \frac{1}{2})} p^{(n)}(\boldsymbol{\xi}; \zeta)(v - s)^{n-1} \tag{3.7}$$

and

$$A_a(s, \hat{v}) \equiv \int_0^1 d\alpha \int_0^1 d\beta \sqrt{\frac{\beta}{1 - \beta}} \cdot \frac{12 s\alpha\beta(1 - \alpha)\hat{v} + a}{[4s\alpha\beta(1 - \alpha)\hat{v} - a]^3}.$$

In (3.7) $B(p, q)$ denotes the beta function. We remark that the parameter $a > 0$ is chosen to be sufficiently large so that the domain of definition of $\psi(\mathbf{X})$ (which we assume is bounded) is contained in the sphere $S_a \equiv \{|\mathbf{X}| \leqq a\}$.

By using the representation (3.21) given in [11] for $\psi(\mathbf{X})$ it is possible to rewrite (3.4) as

$$u(x_1, x_2, x_3) \equiv \psi(\mathbf{X}) = H(\mathbf{X}) + \underset{|\mathbf{Y}'| = R'}{\int_0^{2\pi} \int_0^{\pi}} H(\mathbf{Y}') M(\mathbf{X}; \mathbf{Y}' R^2) \sin\theta'\, d\theta'\, d\varphi', \tag{3.8}$$

where $H(\mathbf{X}) \equiv H(X, Z, Z^*)$ is a harmonic function defined in a neighborhood of the origin by the Bergman–Whittaker representation

$$H(X, Z, Z^*) = \mathbf{B}_3 g \equiv \frac{1}{2\pi i} \int_{|\zeta|=1} g(v, \zeta) \frac{d\zeta}{\zeta}, \tag{3.9}$$

where $g(v, \zeta)$ is related to $f(w, \zeta)$ by

$$g(v, \zeta) = \int_\gamma f(w, \zeta) \frac{dt}{\sqrt{1 - t^2}}. \tag{3.10}$$

In (3.8) the radius of the integration sphere is given by $R' = a^2/R$, $R = |\mathbf{X}|$.

Now consider the equation

$$\Delta_3 u + F(x_1, x_2) u = 0, \tag{3.11}$$

i.e., $F(x_1, x_2, x_3) = F(x_1, x_2)$ is independent of x_3. Then $u(x_1, x_2, -x_3)$ is also a solution of (3.11) and hence so is the *even part* of u with respect to x_3 (denoted by

$u_E(x_1, x_2, x_3))$. $u_E(x_1, x_2, x_3)$ can be represented as

$$
\begin{aligned}
u_E(x_1, x_2, x_3) &\equiv \frac{1}{2}[\psi(-X, Z, Z^*) + \psi(X, Z, Z^*)] \\
&= H_E(\mathbf{X}) + \underset{|\mathbf{Y}'|=R'}{\int_0^{2\pi}\int_0^{\pi}} H(\mathbf{Y}')M_E(\mathbf{X}; \mathbf{Y}'R^2)\sin\theta'\,d\theta'\,d\varphi',
\end{aligned}
\tag{3.12}
$$

where

$$
\begin{aligned}
H_E(\mathbf{X}) &= \tfrac{1}{2}[H(-X, Z, Z^*) + H(X, Z, Z^*)], \\
M_E(\mathbf{X}; \mathbf{X}') &= \tfrac{1}{2}[M(-X, Z, Z^*; \mathbf{X}') + M(X, Z, Z^*; \mathbf{X}')].
\end{aligned}
\tag{3.13}
$$

Note that as a consequence of (3.6) we have $M_E(\mathbf{X}_j\mathbf{X}') = 0$ when $x_3 = 0$. Now suppose we wish to find a solution of (3.11) satisfying the Cauchy data

$$
\begin{aligned}
u(x_1, x_2, 0) &= f(x_1, x_2), \\
\frac{\partial u}{\partial x_3}(x_1, x_2, 0) &= g(x_1, x_2),
\end{aligned}
\tag{3.14}
$$

where $f(x_1, x_2)$ and $g(x_1, x_2)$ are holomorphic in some neighborhood of the origin. First consider the case where $g(x_1, x_2) \equiv 0$ and let $H_E(\mathbf{X})$ be the (unique) harmonic function of the variables x_1, x_2, x_3 constructed from § 2 satisfying

$$
\begin{aligned}
H_E(\mathbf{X})\Big|_{x_3=0} &= f(x_1, x_2), \\
\frac{\partial H_E}{\partial x_3}(\mathbf{X})\Big|_{x_3=0} &= 0.
\end{aligned}
\tag{3.15}
$$

Then from (3.15), (3.12), and the fact that $M_E(\mathbf{X}; \mathbf{X}')$ is an even function of $X = x_3$ we have that, if we set $H(\mathbf{X}) = H_E(\mathbf{X})$, (3.12) defines a solution $u_E(x_1, x_2, x_3)$ satisfying the Cauchy data (3.14) with $g(x_1, x_2) \equiv 0$. With this construction in mind let $\tilde{u}_E(x_1, x_2, x_3)$ be the solution of (3.11) with Cauchy data

$$
\begin{aligned}
\tilde{u}_E(x_1, x_2, 0) &= g(x_1, x_2), \\
\frac{\partial \tilde{u}_E}{\partial x_3}(x_1, x_2, 0) &= 0.
\end{aligned}
\tag{3.16}
$$

Then, since $F(x_1, x_2)$ is independent of x_3,

$$
u_0(x_1, x_2, x_3) = \int_0^{x_3} \tilde{u}_E(x_1, x_2, x_3)\,dx_3
\tag{3.17}
$$

is a solution of (3.11) which satisfies

$$
\begin{aligned}
u_0(x_1, x_2, 0) &= 0, \\
\frac{\partial u_0}{\partial x_3}(x_1, x_2, 0) &= g(x_1, x_2).
\end{aligned}
\tag{3.18}
$$

Hence the solution of our original Cauchy problem (3.11), (3.14) is

$$
\begin{aligned}
u(x_1, x_2, x_3) &= u_E(x_1, x_2, x_3) + u_0(x_1, x_2, x_3) \\
&= H_E(\mathbf{X}) + \underset{|\mathbf{Y}'|=R'}{\int_0^{2\pi}\int_0^{\pi}} H_E(\mathbf{Y}')M_E(\mathbf{X}; \mathbf{Y}'R^2)\sin\theta'\, d\theta'\, d\varphi' \\
&\quad + \int_0^{x_3} \tilde{H}_E(\mathbf{X})\, dx_3 + \underset{|\mathbf{Y}'|=R'}{\int_0^{2\pi}\int_0^{\pi}\int_0^{x_3}} \tilde{H}_E(\mathbf{Y}')M_E(\mathbf{X}; \mathbf{Y}'R^2)\sin\theta'\, d\theta'\, d\varphi'\, dx_3,
\end{aligned}
\tag{3.19}
$$

where $\tilde{H}_E(\mathbf{X})$ is the (unique) harmonic function of the variables x_1, x_2, x_3 constructed by means of § 2 which satisfies the Cauchy data

$$
\begin{aligned}
\tilde{H}_E(\mathbf{X})\Big|_{x_3=0} &= g(x_1, x_2), \\
\frac{\partial \tilde{H}_E}{\partial x_3}(\mathbf{X})\Big|_{x_3=0} &= 0,
\end{aligned}
\tag{3.20}
$$

and $H_E(\mathbf{X})$ is the harmonic function satisfying the Cauchy data (3.15).

A word should be said about the domain of regularity of $u(x_1, x_2, x_3)$ as given by (3.19). As pointed out by Gilbert and Lo in [11], $u(x_1, x_2, x_3)$ has the same domain of regularity as (in our case) the largest domain in which $H_E(\mathbf{X})$ and $\tilde{H}_E(\mathbf{X})$ are both regular. Equations (2.11) and (2.12) show that this domain is in turn determined by the domain of regularity of $f(x_1, x_2)$ and $g(x_1, x_2)$ in the space $\mathbb{C}^2$ of two complex variables. Hence (3.19) gives a solution of Cauchy's problem "in the large" and can be used for analytic continuation if so desired. Equations (3.7), (3.13) and (3.19) in conjunction with the results of § 2 give a means for analytically approximating this solution. Note that $M_E(\mathbf{X}; \mathbf{X}')$ is an entire function of its independent variables.

4. An integral operator for $\Delta_4 u + F(x_1, x_2, x_3, x_4)u = 0$. Consider the partial differential equation

$$\Delta_4 u + F(x_1, x_2, x_3, x_4)u = 0, \tag{4.1}$$

where $F(x_1, x_2, x_3, x_4)$ is an entire function of its independent variables. Equation (4.1) also takes the form

$$\psi_{YY^*} - \psi_{ZZ^*} + \hat{F}(Y, Y^*, Z^*) = 0, \tag{4.2}$$

where

$$
\begin{aligned}
\hat{F}(Y, Y^*, Z, Z^*) &\equiv F(x_1, x_2, x_3, x_4), \\
\psi(Y, Y^*, Z, Z^*) &\equiv u(x_1, x_2, x_3, x_4), \\
Z = x_4 + ix_3, &\quad Z^* = -(x_4 - ix_3), \\
Y = x_1 + ix_2, &\quad Y^* = x_1 - ix_2.
\end{aligned}
\tag{4.3}
$$

Throughout this section we shall use the following notations:

$$
\begin{aligned}
&\xi_1 = Z - Z^*, \\
&\xi_2 = \zeta^{-1}Z + Y, \\
&\xi_3 = \eta^{-1}Z^* + Y, \\
&\xi_4 = \eta^{-1}Z^* + \eta^{-1}\zeta^{-1}Y^*, \\
&\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3, \xi_4), \\
&\mathbf{X} = (Y, Y^*, Z, Z^*), \\
&v = \xi_2 + \xi_4 = Y + \zeta^{-1}Z + \eta^{-1}Z^* + \eta^{-1}\zeta^{-1}Y^*, \\
&w = (1 - t^2)v, \qquad |t| \leqq 1, \\
&\tilde{F}(\xi_1, \xi_2, \xi_3, \xi_4, \zeta, \eta) \equiv \hat{F}(Y, Y^*, Z, Z^*), \\
&E(\xi_1, \xi_2, \xi_3, \xi_4, \zeta, \eta, t) \equiv \hat{E}(Y, Y^*, Z, Z^*, \zeta, \eta, t), \\
&E_i = \frac{\partial E}{\partial \xi_i}, \qquad E_{ij} = \frac{\partial^2 E}{\partial \xi_i \partial \xi_j}, \qquad E_{it} = \frac{\partial^2 E}{\partial \xi_i \partial t}.
\end{aligned}
\tag{4.4}
$$

THEOREM 4.1. *Let D be a domain in the w-plane containing the origin $w = 0$,*

$$B = \{(\zeta, \eta) | \tfrac{1}{2} - \varepsilon < |\zeta| < \tfrac{1}{2} + \varepsilon, 1 - \varepsilon < |\eta| < 1 + \varepsilon\}, \quad 0 < \varepsilon < \tfrac{1}{8},$$

G be a neighborhood of the origin in the $(\xi_1, \xi_2, \xi_3, \xi_4)$-space, and $T = \{t | |t| \leqq 1\}$. Let $f(w, \zeta, \eta)$ be an analytic function of three complex variables w, ζ, η in the product domain $D \times B$ and $E(\boldsymbol{\xi}, \zeta, \eta, t) \equiv E(\xi_1, \xi_2, \xi_3, \xi_4, \zeta, \eta, t)$ be a solution of the equation

$$
\begin{aligned}
&2vt[-\eta\zeta E_{11} - \eta E_{12} + \zeta E_{13} + E_{23} + \zeta E_{14} - E_{34} - \eta\zeta E\tilde{F}] \\
&\qquad + (1 - t^2)(\zeta - \eta)E_{1t} - \frac{1}{t}(\zeta - \eta)E_1 = 0,
\end{aligned}
\tag{4.5}
$$

which is regular in the product domain $G \times B \times T$. Then

$$
\begin{aligned}
&\psi(Y, Y^*, Z, Z^*) \equiv \mathbf{P}\{f\} \\
&= \frac{-1}{4\pi^2} \int_{|\zeta| = 1/2} \int_{|\eta| = 1} \int_\gamma \hat{E}(Y, Y^*, Z, Z^*, \zeta, \eta, t) f(w, \zeta, \eta) \frac{dt}{\sqrt{1 - t^2}} \frac{d\eta}{\eta} \frac{d\zeta}{\zeta},
\end{aligned}
\tag{4.6}
$$

where γ is a path in T joining $t = -1$ and $t = +1$, is a solution of (4.2) *which is regular in a neighborhood of the origin in (Y, Y^*, Z, Z^*)-space.*

Proof. $E(\boldsymbol{\xi}, \zeta, \eta, t)$ is regular in $G \times B \times T$, $f(w, \zeta, \eta)$ is regular in $D \times B$ and the Jacobian of the transformation from $(\xi_1, \xi_2, \xi_3, \xi_4, \zeta, \eta, t)$ to $(Y, Y^*, Z, Z^*, \zeta, \eta, t)$ is

$$\frac{1}{\eta\zeta}\left(\frac{1}{\eta} + \frac{1}{\zeta}\right).$$

Hence $\hat{E}(Y, Y^*, Z, Z^*, \zeta, \eta, t)f(w, \zeta, \eta)$ is regular in the product domain $H \times B \times T$, where H is a neighborhood of the origin in the (Y, Y^*, Z, Z^*)-space. Thus $\psi(Y, Y^*, Z, Z^*) \equiv \mathbf{P}\{f\}$ is regular in a neighborhood of the origin in the (Y, Y^*, Z, Z^*)-space. Straightforward differentiation (using the fact that $f_{ZZ^*} - f_{YY^*} = 0$ due to analyticity in w) gives

$$\begin{aligned} &\psi_{ZZ^*} - \psi_{YY^*} - \hat{F}\psi \\ (4.7)\quad &= \frac{-1}{4\pi^2}\int_{|\zeta|=1/2}\int_{|\eta|=1}\int_{\gamma}\Bigg[(\hat{E}_{ZZ^*} - \hat{E}_{YY^*} - \hat{E}\hat{F})f \\ &\quad + (1 - t^2)f_w\left(\frac{\hat{E}_{Z^*}}{\zeta} + \frac{\hat{E}_Z}{\eta} - \hat{E}_{Y^*} - \frac{\hat{E}_Y}{\eta\zeta}\right)\Bigg]\frac{dt}{\sqrt{1-t^2}}\frac{d\eta}{\eta}\frac{d\zeta}{\zeta}. \end{aligned}$$

Using the relation

$$f_w = -\frac{1}{2tv}f_t$$

and integrating by parts we find

$$\begin{aligned} \psi_{ZZ^*} - \psi_{YY^*} - \hat{F}\psi &= \frac{-1}{4\pi^2}\int_{|\zeta|=1/2}\int_{|\eta|=1}\int_{\gamma}\Bigg[(\hat{E}_{ZZ^*} - \hat{E}_{YY^*} - \hat{E}\hat{F}) \\ &\quad + \frac{1}{2v}\Bigg(\frac{\hat{E}_{Z^*t}}{t\zeta}(1 - t^2) - \frac{\hat{E}_{Z^*}}{t^2\zeta} + \frac{\hat{E}_{Zt}}{t\eta}(1 - t^2) - \frac{\hat{E}_Z}{t^2\eta} - \frac{\hat{E}_{Y^*t}}{t}(1 - t^2) \\ (4.8)\quad &\quad + \frac{\hat{E}_{Y^*}}{t^2} - \frac{\hat{E}_{Yt}}{t\eta\zeta}(1 - t^2) + \frac{\hat{E}_Y}{t^2\eta\zeta}\Bigg)\Bigg]f\frac{dt}{\sqrt{1-t^2}}\frac{d\eta}{\eta}\frac{d\zeta}{\zeta}. \end{aligned}$$

Changing to the variables $\xi_1, \xi_2, \xi_3, \xi_4$ gives

$$\begin{aligned} \psi_{ZZ^*} - \psi_{YY^*} - \hat{F}\psi &= -\frac{1}{4\pi^2}\int_{|\zeta|=1/2}\int_{|\eta|=1}\int_{\gamma}\Bigg[2vt(-\eta\zeta E_{11} \\ (4.9)\quad &\quad - \eta E_{12} + \zeta E_{13} + E_{23} + \zeta E_{14} - E_{34} - \eta\zeta E\tilde{F}) \\ &\quad + (1 - t^2)(\zeta - \eta)E_{1_t} - \frac{1}{t}(\zeta - \eta)E_1\Bigg]\frac{f}{2vt\eta\zeta}\frac{dt}{\sqrt{1-t^2}}\frac{d\eta}{\eta}\frac{d\zeta}{\zeta}. \end{aligned}$$

Hence if E satisfies (4.5), then ψ as defined by (4.6) is a solution of (4.2).

We now need to show that the operator $\mathbf{P}$ exists; i.e., we need to show the existence of a *generating function* E satisfying the hypothesis of Theorem 4.1. In order to do this we need to make use of the idea of a *dominant* which we define below.

DEFINITION 4.1. A function

$$g(z_1, \cdots, z_k) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots\sum_{n_k=0}^{\infty} a_{n_1 n_1 \cdots n_k} z_1^{n_1} z_2^{n_2} \cdots z_k^{n_k}$$

is said to be a *dominant* of a function

$$f(z_1, \cdots, z_k) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_k=0}^{\infty} b_{n_1 n_2 \cdots n_k} z_1^{n_1} z_2^{n_2} \cdots z_k^{n_k}$$

if and only if $a_{n_1 n_2 \cdots n_k} \geqq 0$ and $|b_{n_1 n_2 \cdots n_k}| \leqq a_{n_1 n_2 \cdots n_k}$ for all $n_i = 0, 1, 2, \cdots, i = 1, 2, \cdots, k$. If g is a dominant of f, we shall write

$$f(z_1, z_2, \cdots, z_k) \ll g(z_1, z_2, \cdots, z_k)$$

or more concisely $f \ll g$.

The use of dominants is a standard tool in the theory of several complex variables and the reader is referred to [1], [8] and [15] for further details. In particular it is easy to verify that if $f(z_1, z_2, \cdots, z_k)$ is regular in a polydisc

$$\bar{D} = \{(z_1, z_2, \cdots, z_k) | |z_i| \leqq r_i, i = 1, 2, \cdots, k\},$$

then

$$f(z_1, z_2, \cdots, z_k) \ll C\left(1 - \frac{z_1}{r_1}\right)^{-1}\left(1 - \frac{z_2}{r_2}\right)^{-1} \cdots \left(1 - \frac{z_k}{r_k}\right)^{-1} \tag{4.10}$$

holds for some $C > 0$ in

$$D = \{(z_1, z_2, \cdots, z_k) | |z_i| < r_i, i = 1, \cdots, k\}.$$

THEOREM 4.2. *Let $D_r = \{(\xi_1, \xi_2, \xi_3, \xi_4) | |\xi_i| < r,\ i = 1, 2, 3, 4\}$, where r is an arbitrary positive number, and*

$$B_{2\varepsilon} = \{(\eta, \zeta) | |\eta - \eta_0| < 2\varepsilon, |\zeta - \zeta_0| < 2\varepsilon\}, \quad 0 < \varepsilon < \tfrac{1}{8},$$

where ζ_0, η_0 are arbitrary with $|\eta_0| = 1$, $|\zeta_0| = \frac{1}{2}$. Then for each n, $n = 0, 1, 2, \cdots$, there exists a unique function $p^{(n)}(\boldsymbol{\xi}, \zeta, \eta) \equiv p^{(n)}(\xi_1, \xi_2, \xi_3, \xi_4, \zeta, \eta)$ which is regular in $\bar{D}_r \times \bar{B}_{2\varepsilon}$ and satisfies

$$\begin{aligned} p_1^{(n+1)} = \frac{2}{(\zeta - \eta)(2n + 1)} \{ & \eta\zeta p_{11}^{(n)} + \eta p_{12}^{(n)} - \zeta p_{13}^{(n)} - p_{23}^{(n)} - \zeta p_{23}^{(n)} \\ & - \zeta p_{14}^{(n)} + p_{34}^{(n)} + \eta\zeta p^{(n)} \tilde{F}\}, \end{aligned} \tag{4.11}$$

where

$$\begin{aligned} p^{(n+1)}(0, \xi_2, \xi_3, \xi_4, \zeta, \eta) &= 0, \qquad n = 0, 1, 2, \cdots, \\ p^{(0)}(\xi_1, \xi_2, \xi_3, \xi_4, \zeta, \eta) &\equiv 1 \end{aligned} \tag{4.12}$$

and

$$p_i^{(n)} = \frac{\partial p^{(n)}}{\partial \xi_i}, \qquad p_{ij}^{(n)} = \frac{\partial p^{(n)}}{\partial \xi_i \partial \xi_j}, \qquad i, j = 1, 2, 3, 4.$$

Furthermore the function

$$\begin{aligned} E(\boldsymbol{\xi}, \zeta, \eta, t) &\equiv E(\xi_1, \xi_2, \xi_3, \xi_4, \zeta, \eta, t) \\ &= 1 + \sum_{n=1}^{\infty} t^{2n} v^n p^{(n)}(\boldsymbol{\xi}, \zeta, \eta) \end{aligned} \tag{4.13}$$

is a solution of (4.5), *which is regular in the product domain* $G_R \times B \times T$, *where* R *is an arbitrary positive number and*

$$G_R = \{(\xi_1, \xi_2, \xi_3, \xi_4) \,|\, |\xi_i| < R, i = 1, 2, 3, 4\},$$
$$(4.14) \qquad B = \{(\zeta, \eta) \,|\, \tfrac{1}{2} - \varepsilon < |\zeta| < \tfrac{1}{2} + \varepsilon, 1 - \varepsilon < |\eta| < 1 + \varepsilon\},$$
$$T = \{t \,|\, |t| \leqq 1\}.$$

The function defined in (4.13) *satisfies*

$$(4.15) \qquad E(0, \xi_2, \xi_3, \xi_4, \zeta, \eta, t) = 1.$$

Proof. For $n = 0$, (4.11) and (4.12) become

$$(4.16) \qquad \begin{aligned} p_1^{(1)} &= \eta\zeta\tilde{F}, \\ p^{(1)}(0, \xi_2, \xi_3, \xi_4, \zeta, \eta) &= 0; \end{aligned}$$

and hence

$$(4.17) \qquad p^{(1)}(\boldsymbol{\xi}, \zeta, \eta) = \int_0^{\xi_1} \eta\zeta\tilde{F}(\xi_1, \xi_2, \xi_3, \xi_4, \zeta, \eta)\, d\xi$$

is unique and regular in $\bar{D}_r \times \bar{B}_{2\varepsilon}$. By induction it follows that each $p^{(n)}(\boldsymbol{\xi}, \zeta, \eta)$ exists, is unique, and is regular in $\bar{D}_r \times \bar{B}_{2\varepsilon}$. Now consider the formal series defined by (4.13). By straightforward differentiation and collection of terms it is easily verified that if the $p^{(n)}(\boldsymbol{\xi}, \zeta, \eta)$ are defined by (4.11) and (4.12), then $E(\boldsymbol{\xi}, \zeta, \eta, t)$ formally satisfies (4.5). It remains to be shown that $E(\boldsymbol{\xi}, \zeta, \eta, t)$ is regular in $G_R \times B \times T$, i.e., the series (4.13) converges uniformly in this region. Since $\bar{B}$ is a compact subset of the (ζ, η)-space, there are finitely many points (ζ_j, η_j) with $|\zeta_j| = \frac{1}{2}$, $|\eta_j| = 1$, $j = 1, 2, \cdots, N$, such that B is covered by the union of sets

$$(4.18) \qquad U_j = \{(\zeta, \eta) \,|\, |\zeta - \zeta_j| < \tfrac{3}{2}\varepsilon, |\eta - \eta_j| < \tfrac{3}{2}\varepsilon\}, \quad j = 1, 2, \cdots, N.$$

Thus it is sufficient to show that the series (4.13) converges uniformly in $\bar{G}_R \times \bar{U}_j \times T$. To this end we proceed to majorize the $p^{(n)}$. Since $F(\boldsymbol{\xi}, \zeta, \eta)$ is regular in $\bar{D}_r \times \bar{B}_{2\varepsilon}$, we have

$$(4.19) \qquad \begin{aligned} F(\boldsymbol{\xi}, \zeta, \eta) \ll C\left(1 - \frac{\xi_1}{r}\right)^{-1}\left(1 - \frac{\xi_2}{r}\right)^{-1}\left(1 - \frac{\xi_3}{r}\right)^{-1}\left(1 - \frac{\xi_4}{r}\right)^{-1} \\ \cdot \left(1 - \frac{\zeta - \zeta_0}{2\varepsilon}\right)^{-1}\left(1 - \frac{\eta - \eta_0}{2\varepsilon}\right)^{-1} \end{aligned}$$

for some $C > 0$ and $(\xi_1, \xi_2, \xi_3, \xi_4, \zeta, \eta) \in D_r \times B_{2\varepsilon}$. Furthermore in $\bar{U}_j$ we have

$$(4.20) \qquad \begin{aligned} |\zeta| &< \tfrac{11}{16} < 1, \\ |\eta| &< \tfrac{19}{16} < 2, \\ |\eta\zeta| &< \tfrac{209}{256} < 1, \\ |\eta - \zeta| &> \tfrac{1}{8}. \end{aligned}$$

We shall now show by induction that, in $\bar{D}_r \times \bar{U}_j \times T$,

$$\begin{aligned} p_1^{(n)} &\ll M(112+\delta)^n(2n-1)^{-1}\left(1-\frac{\xi_1}{r}\right)^{-(2n-1)}\left(1-\frac{\xi_2}{r}\right)^{-(2n-1)} \\ &\quad \cdot\left(1-\frac{\xi_3}{r}\right)^{-(2n-1)}\left(1-\frac{\xi_4}{r}\right)^{-(2n-1)}\left(1-\frac{\zeta-\zeta_0}{2\varepsilon}\right)^{-n}\left(1-\frac{\eta-\eta_0}{2\varepsilon}\right)^{-n} r^{-n} \\ &= M(112+\delta)^n(2n-1)^{-1}\square_n, \end{aligned} \tag{4.21}$$

where

$$\square_n = \left[\prod_{i=1}^{4}\left(1-\frac{\xi_i}{r}\right)^{-(2n-1)}\right]\left(1-\frac{\zeta-\zeta_0}{2\varepsilon}\right)^{-n}\left(1-\frac{\eta-\eta_0}{2\varepsilon}\right)^{-n} r^{-n} \tag{4.22}$$

and M, δ are positive constants independent of n. Equations (4.16), (4.19) show that (4.21) is true for $n = 1$. Suppose now that (4.21) is true for $n = k$. Then using the facts that dominance is maintained under the operations of differentiation and integration and $f \ll g$ implies

$$f \ll g \cdot \left(1 - \frac{z_i}{r_i}\right)^{-1}, \qquad i = 1, 2, \cdots, k,$$

we arrive at

$$\begin{aligned} p_{11}^{(k)} &\ll M(112+\delta)^k\,\square_{k+1}, \\ p_{12}^{(k)} &\ll M(112+\delta)^k\,\square_{k+1}, \\ p_{13}^{(k)} &\ll M(112+\delta)^k\,\square_{k+1}, \\ p_{14}^{(k)} &\ll M(112+\delta)^k\,\square_{k+1}, \\ p_{23}^{(k)} &\ll M(112+\delta)^k(2k-1)(2k)^{-1}\,\square_{k+1}, \\ p_{24}^{(k)} &\ll M(112+\delta)^k(2k-1)(2k)^{-1}\,\square_{k+1}, \\ \tilde{F}p^{(k)} &\ll M(112+\delta)^k C(2k-1)^{-1}(2k)^{-1} r^2\,\square_{k+1}. \end{aligned} \tag{4.23}$$

Equations (4.11), (4.20) and (4.23) now show that

$$p_1^{(k+1)} \ll \frac{M}{2k+1}\left(80 + 32\frac{2k-1}{2k} + \frac{8Cr^2}{2k(2k-1)}\right)(112+\delta)^k\,\square_{k+1}. \tag{4.24}$$

For k sufficiently large,

$$\left(80 + 32\frac{2k-1}{2k} + \frac{8Cr^2}{2k(2k-1)}\right) < 112 + \delta; \tag{4.25}$$

and hence if M is chosen sufficiently large to begin with, we have shown (4.21) is true for $n = k + 1$, thus completing the induction proof. Equation (4.21) now implies that, in $\bar{D}_r \times \bar{U}_j \times T$,

$$
(4.26)\qquad \begin{aligned}
|p^{(n)}| \ll M(112+\delta)^n(2n-2)(2n-1)\left(1-\frac{|\xi_1|}{r}\right)^{-(2n-2)} \\
\cdot\left(1-\frac{|\xi_2|}{r}\right)^{-(2n-1)}\left(1-\frac{|\xi_3|}{r}\right)^{-(2n-1)}\left(1-\frac{|\xi_4|}{r}\right)^{-(2n-1)} \\
\cdot\left(1-\frac{|\zeta-\zeta_j|}{2\varepsilon}\right)^{-n}\left(1-\frac{|\eta-\eta_j|}{2\varepsilon}\right)^{-n} r^{-n+1}.
\end{aligned}
$$

Now consider $|t^n v^n p^{(n)}(\xi, \zeta, \eta)|$ in $\bar{D}_{\alpha r} \times \bar{U}_j \times T$ where

$$
(4.27)\qquad D_{\alpha r} = \{(\xi_1, \xi_2, \xi_3, \xi_4) \,|\, |\xi_i| < r/\alpha, \alpha > 1, i = 1, 2, 3, 4\}.
$$

Then in $\bar{D}_{\alpha r} \times \bar{U}_j \times T$ we have

$$
(4.28)\qquad \begin{aligned}
1-\frac{|\xi_i|}{r} &\geqq \frac{\alpha-1}{\alpha}, \qquad i = 1,2,3,4,\\
1-\frac{|\zeta-\zeta_j|}{2\varepsilon} &\geqq \frac{1}{4},\\
1-\frac{|\eta-\eta_j|}{2\varepsilon} &\geqq \frac{1}{4},\\
|v| = |\xi_2+\xi_4| &\leqq \frac{2r}{\alpha},\\
|t| &\leqq 1.
\end{aligned}
$$

Thus from (4.26) and (4.28) we have

$$
(4.29)\qquad \begin{aligned}
|t^{2n}v^n p^{(n)}| \leqq Mr(\alpha-1)^5(2n-2)^{-1}(2n-1)^{-1}\alpha^5 \\
\cdot[32\alpha^7(112+\delta)(\alpha-1)^{-8}]^n.
\end{aligned}
$$

Choose α such that

$$
(4.30)\qquad 32\alpha^7(112+\delta)(\alpha-1)^{-8} < 1.
$$

Then the series for $E(\xi, \zeta, \eta, t)$ converges absolutely and uniformly in $\bar{D}_{\alpha r} \times \bar{U}_j \times T$. By taking $r = \alpha R$ we have that $E(\xi, \zeta, \eta, t)$ is regular in $\bar{G}_R \times \bar{U}_j \times T$ and hence in $G_R \times B \times T$. Equation (4.15) follows from (4.12).

5. Cauchy's problem for $\Delta_4 + F(x_1, x_2, x_3)u = 0$. We now put the operator **P** into a different form with the aim of converting it into an operator mapping harmonic functions instead of analytic functions onto solutions ψ of (4.2). Equation (4.6) can be written as

$$
(5.1)\qquad \begin{aligned}
\psi(\mathbf{X}) &\equiv \mathbf{P}\{f\}\\
&= \frac{-1}{4\pi^2}\int_{|\zeta|=1/2}\int_{|\eta|=1}\int_{\gamma}\left\{1+\sum_{n=1}^{\infty} t^{2n}v^n p^{(n)}(\xi, \zeta, \eta)\right\} f(w, \zeta, \eta)\frac{dt}{\sqrt{1-t^2}}\frac{d\eta}{\eta}\frac{d\zeta}{\zeta}.
\end{aligned}
$$

Let

$$f(w,\zeta,\eta)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}a_{klm}v^k\zeta^l\eta^m(1-t^2)^k. \tag{5.2}$$

Then

$$\int_\gamma t^{2n}f(w,\zeta,\eta)\frac{dt}{\sqrt{1-t^2}}=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}a_{klm}v^k\zeta^l\eta^m\frac{\Gamma(k+\frac{1}{2})\Gamma(n+\frac{1}{2})}{\Gamma(n+k+1)}. \tag{5.3}$$

Define

$$g(v,\zeta,\eta)=\int_\gamma f(w,\zeta,\eta)\frac{dt}{\sqrt{1-t^2}}. \tag{5.4}$$

Then straightforward calculation using well-known properties of the beta function $B(m, n)$ gives

$$\int_0^v(v-\alpha)^{n-1}g(\alpha,\zeta,\eta)\,d\alpha=\frac{\Gamma(\frac{1}{2})\Gamma(n)}{\Gamma(n+\frac{1}{2})}v^n\int_\gamma t^{2n}f(w,\zeta,\eta)\frac{dt}{\sqrt{1-t^2}}. \tag{5.5}$$

Hence

$$\begin{aligned}\psi(\mathbf{X})&=\mathbf{P}\{f\}\equiv\mathbf{P}^*\{g\}\\&=\frac{-1}{4\pi^2}\int_{|\zeta|=1/2}\int_{|\eta|=1}g(v,\zeta,\eta)\frac{d\eta}{\eta}\frac{d\zeta}{\zeta}\\&\quad-\sum_{n=1}^{\infty}\frac{1}{4\pi^2B(n,\frac{1}{2})}\int_{(\zeta|=1/2}\int_{|\eta|=1}\\&\quad\cdot\left\{p^{(n)}(\xi,\zeta,\eta)\int_0^v(v-\alpha)^{n-1}g(\alpha,\zeta,\eta)\,d\alpha\right\}\frac{d\eta}{\eta}\frac{d\zeta}{\zeta}.\end{aligned} \tag{5.6}$$

We now make use of Gilbert's generalization of the Bergman–Whittaker operator[1] [8, p. 75] to observe that if $g(v,\zeta,\eta)$ is regular in $N\times B$, where N is some domain in the v-plane containing the origin and B is defined as in Theorem 4.1, then

$$H(\mathbf{X})=\mathbf{G}_4g\equiv\frac{-1}{4\pi^2}\int_{|\zeta|=1/2}\int_{|\eta|=1}g(v,\zeta,\eta)\frac{d\eta}{\eta}\frac{d\zeta}{\zeta} \tag{5.7}$$

is a harmonic function of the variables x_1, x_2, x_3, x_4 regular in some neighborhood of the origin. We note that, since $f(w,\zeta,\eta)$ is regular in $D\times B$ (D being defined in Theorem 4.1), (5.4) in conjunction with Gilbert's "envelope method" [8] shows that $g(v,\zeta,\eta)$ is regular in $N\times B$ for some domain N as defined above. If we express $H(\mathbf{X})$ in terms of hyperspherical coordinates

$$H(\mathbf{X})=V(\rho,\theta_1,\theta_2,\varphi) \tag{5.8}$$

[1] We use the notation $\mathbf{G}_4$ instead of $\mathbf{B}_4$ in order to distinguish this operator from Bergman's operator for $p+2$ variables [8, p. 82].

and define

$$\delta(\mathbf{X}) = \eta\zeta Y^* + \eta Z, \tag{5.9}$$
$$\beta(\mathbf{X}) = Y - Y^*\eta\zeta - Z\eta + Z^*\zeta,$$

then for ρ sufficiently small we can invert the operator $\mathbf{G}_4$ via the formula [8, p. 82]

$$g(\alpha, \zeta, \eta) = \mathbf{G}_4^{-1}H \equiv \frac{1}{2\pi^2}\int_\Omega \frac{\overline{V(\rho, \theta_1, \theta_2, \varphi)}(2\alpha^{-1} - \beta - 2\delta)\delta}{\alpha^2(\alpha^{-1} - \delta)^2(\alpha^{-1} - \delta - \beta)^2}\, d\Omega, \tag{5.10}$$

where the integration is over the sphere of radius ρ and the bar denotes complex conjugation. Substituting (5.7), (5.8) and (5.10) into the formula given by (5.6) and interchanging orders of integration by Fubini's theorem gives

$$\psi(\mathbf{X}) = H(\mathbf{X}) + \int_\Omega \overline{H(\mathbf{Y})}M(\mathbf{X}; \mathbf{Y})\, d\Omega_Y, \tag{5.11}$$

where

$$\begin{aligned} M(\mathbf{X}; \mathbf{Y}) = -\frac{1}{8\pi^3}\int_{|\zeta|=1/2}\int_{|\eta|=1}\int_0^v & p(\xi, v - \alpha, \zeta, \eta) \\ & \cdot \frac{[2\alpha^{-1} - \beta(\mathbf{Y}) - 2\delta(\mathbf{Y})]\delta(\mathbf{Y})}{\alpha^2[\alpha^{-1} - \delta(\mathbf{Y})]^2[\alpha^{-1} - \delta(\mathbf{Y}) - \beta(\mathbf{Y})]^2}\, d\alpha \frac{d\eta}{\eta}\frac{d\zeta}{\zeta}, \\ p(\xi, v - \alpha, \zeta, \eta) = \sum_{n=1}^{\infty} \frac{1}{B(n, \frac{1}{2})} & p^{(n)}(\xi, \zeta, \eta)(v - \alpha)^{n-1} \end{aligned} \tag{5.12}$$

and $\mathbf{Y}$ is a point on the sphere Ω of radius ρ.

Now suppose we wish to find a solution of

$$\Delta_4 u + F(x_1, x_2, x_3)u = 0 \tag{5.13}$$

(i.e., $F(x_1, x_2, x_3, x_4) = F(x_1, x_2, x_3)$ is independent of x_4) satisfying the Cauchy data

$$\begin{aligned} u(x_1, x_2, x_3, 0) &= f(x_1, x_2, x_3), \\ \frac{\partial u}{\partial x_4}(x_1, x_2, x_3, 0) &= g(x_1, x_2, x_3), \end{aligned} \tag{5.14}$$

where $f(x_1, x_2, x_3)$ and $g(x_1, x_2, x_3)$ are holomorphic functions of x_1, x_2, x_3 in some neighborhood of the origin. Using (5.11), (5.12), (4.12), (4.3), (4.4), and following the same analysis as in § 3, we can express the solution to (5.13), (5.14) as

$$\begin{aligned} u(x_1, x_2, x_3, x_4) = H_E(\mathbf{X}) &+ \int_\Omega H_E(\mathbf{Y})M_E(\mathbf{X}; \mathbf{Y})\, d\Omega_{\mathbf{Y}} \\ &+ \int_0^{x_4} \tilde{H}_E(\mathbf{X})\, dx_4 + \int_\Omega \int_0^{x_4} \tilde{H}_E(\mathbf{Y})M_E(\mathbf{X}; \mathbf{Y})\, dx_4\, d\Omega, \end{aligned} \tag{5.15}$$

where $M_E(\mathbf{X};\mathbf{Y})$ denotes the even part of $M(\mathbf{X};\mathbf{Y})$ with respect to x_4 and $H_E(\mathbf{X})$, $\tilde{H}_E(\mathbf{X})$ are the (unique) harmonic functions of the variables x_1, x_2, x_3, x_4 constructed by the methods of § 2 which satisfy the Cauchy data

$$\begin{aligned} H_E(\mathbf{X})\Big|_{x_4=0} &= f(x_1, x_2, x_3), \\ \frac{\partial H_E}{\partial x_4}(\mathbf{X})\Big|_{x_4=0} &= 0, \end{aligned} \tag{5.16}$$

$$\begin{aligned} \tilde{H}_E(\mathbf{X})\Big|_{x_4=0} &= g(x_1, x_2, x_3), \\ \frac{\partial \tilde{H}_E}{\partial x_4}(\mathbf{X})\Big|_{x_4=0} &= 0, \end{aligned} \tag{5.17}$$

respectively.

As we discussed in § 3, the domain of regularity of $u(x_1, x_2, x_3, x_4)$ as given by (5.15) is determined by the domain of regularity of $f(x_1, x_2, x_3)$ and $g(x_1, x_2, x_3)$ in the space $\mathcal{C}^3$ of three complex variables. Approximations can be readily made through the use of (5.12) and (5.15) in conjunction with the results of § 2.

6. Cauchy's problem for $\Delta_{p+2}u + B(r^2)u = 0$. In this section we solve Cauchy's problem for the equation

$$\mathbf{L}[u] \equiv \Delta_{p+2}u + B(r^2)u = 0, \tag{6.1}$$

where $B(r^2)$ is an entire function of r^2, $r = |\mathbf{X}|$, $\mathbf{X} = (x_1, \cdots, x_{p+2})$. To accomplish this we use the *method of ascent* [9], [10] to represent $u(\mathbf{X})$ in terms of a harmonic function of $p + 2$ variables $H(\mathbf{X})$:

$$u(\mathbf{X}) = (\mathbf{I} + \mathbf{G})H(\mathbf{X}) \equiv H(\mathbf{X}) + \int_0^1 \sigma^{p+1} G(r, 1 - \sigma^2) H(\mathbf{X}\sigma^2)\, d\sigma, \tag{6.2}$$

where

$$G(r, 1 - \sigma^2) = -2rR_1(r\sigma^2, 0; r, r) \tag{6.3}$$

and $R(z, z^*; \zeta, \zeta^*)$ is the Riemann function for $\Delta_2 u + B(r^2)u = 0$, the subscript in (6.3) denoting differentiation with respect to the first variable. If r, $\theta \equiv (\theta_1, \theta_2, \cdots, \theta_p)$, φ are hyperspherical coordinates, the harmonic function $H(\mathbf{X}) \equiv \tilde{H}(r, \theta, \varphi)$ in (6.2) is given by

$$\tilde{H}(r, \theta, \varphi) = \tilde{u}(r, \theta, \varphi) + r^{-p/2} \int_0^r \Gamma(\rho, r) \rho^{p/2} \tilde{u}(\rho, \theta, \varphi)\, d\rho, \tag{6.4}$$

where

$$\begin{aligned} \Gamma(\rho, r) &= \sum_{l=1}^{\infty} K^{(l)}(\rho, r), \\ K^{(l+1)}(\rho, r) &= \int_\rho^r K^{(1)}(t, r) K^{(l)}(\rho, t)\, dt, \\ K^{(1)}(\rho, r) &= R_1(\rho, 0; r, r), \\ u(\mathbf{X}) &\equiv \tilde{u}(r, \theta, \varphi). \end{aligned} \tag{6.5}$$

Equation (6.4) can be written in Cartesian coordinates as

$$H(\mathbf{X}) = u(\mathbf{X}) + \int_0^1 \Gamma(tr, r)t^{p/2}u(\mathbf{X}t)\,dt. \tag{6.6}$$

Now suppose we wish to find a solution to (6.1) such that

$$\begin{aligned} u(x_1, x_2, \cdots, x_{p+1}, 0) &= f(x_1, \cdots, x_{p+1}), \\ \frac{\partial u}{\partial x_{p+2}}(x_1, x_2, \cdots, x_{p+1}, 0) &= g(x_1, \cdots, x_{p+1}), \end{aligned} \tag{6.7}$$

where f and g are regular in some neighborhood of the origin in the space $\mathbb{C}^{p+1}$ of $p + 1$ complex variables. From (6.6) we have

$$H(\mathbf{X})\Big|_{x_{p+2}=0} = f(\mathbf{Y}) + \int_0^1 \Gamma(tR, R)t^{p/2}f(t\mathbf{Y})\,dt, \tag{6.8}$$

where

$$\begin{aligned} \mathbf{Y} &= (x_1, \cdots, x_{p+1}), \\ |\mathbf{Y}| &= R \end{aligned} \tag{6.9}$$

and

$$\begin{aligned} \frac{\partial H(\mathbf{X})}{\partial x_{p+2}}\bigg|_{x_{p+2}=0} &= g(\mathbf{Y}) + \int_0^1 \left(\frac{\partial}{\partial x_{p+2}}\Gamma(tr, r)\right)_{x_{p+2}=0} t^{p/2}f(t\mathbf{Y})\,dt \\ &\quad + \int_0^1 \Gamma(tR, R)t^{p/2+1}g(t\mathbf{Y})\,dt. \end{aligned} \tag{6.10}$$

We can now use the results of § 2 to construct $H(\mathbf{X})$. By using this harmonic function the solution to the Cauchy problem (6.1), (6.7) is given by (6.2). From its definition in terms of the Riemann function it is seen that $G(r, 1 - \sigma^2)$ is an entire function of its variables [8], [9]. Hence (6.2) in conjunction with the formulas of § 2 give an efficient method of approximating solutions to the Cauchy problem (6.1), (6.7).

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SIAM J. MATH. ANAL.
Vol. 2, No. 1, February 1971

STABILITY AND ENTERING OF THE ORIGIN FOR REAL, NONLINEAR, AUTONOMOUS DIFFERENTIAL EQUATIONS OF THIRD ORDER*

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Abstract. This paper is concerned with the asymptotic behavior of solutions to third order differential equations in the neighborhood of a stable critical point. In particular, the question of whether or not solution trajectories enter the critical point is investigated. Two new theorems on entering (extensions of known theorems for the two-dimensional case) are proved, and these, as well as known results on asymptotic behavior, are applied to a case-by-case analysis of the possibilities for third order equations.

1. Introduction. Although not of such universal importance as second order equations, third order autonomous differential equations do arise in a number of applications, for example, in the study of some types of electronic oscillator circuits (see [2], [8], [14]). It is therefore of importance that stability and qualitative behavior of solution curves of such equations be investigated.

Most of the literature on the problem deals with special cases. However, in 1964 Reyn [15] classified the critical points of the third order linear autonomous system

$$\dot{z}(t) = Az(t), \tag{1}$$

where A is a real, constant, third order matrix and $z(t) = (z_1(t), z_2(t), z_3(t))$. He also described the behavior of the integral curves in a neighborhood of the critical points. Reyn suggested that his work be the starting point for the study of nonlinear third order autonomous systems of the form

$$\dot{z} = Az + R(z), \tag{n}$$

where $R(z) = o(\|z\|)$ as $\|z\| \to 0$. Recently Bihari [1] has carried out such a study, but restricted himself to the case where A is nonsingular. We have studied (n) with the restriction that it be equivalent to some differential equation $\dddot{x} = f(x, \dot{x}, \ddot{x})$. In § 3 of this paper we summarize our more interesting findings; more details are to be found in [16]. Before presenting these results, in § 2 we establish necessary and sufficient conditions for an integral curve of (n) to "enter" a simple critical point at the origin with direction cosines n_1, n_2, n_3. We also give a set of equations which the quantities n_1, n_2, n_3 must satisfy. These theorems, extensions to three dimensions of analogous theorems given by Hurewicz [5, p. 87 ff.] for the two-dimensional case, are useful for obtaining some of the results of § 3.

2. General qualitative behavior. Throughout this paper we assume that conditions are such that the local existence of solutions of (n) for $\|z\|$ sufficiently small is guaranteed. (We shall be using the usual Euclidean norm for $\|z\|$.) Requiring $R(z)$ to be continuous is sufficient. We then expect that at least in some cases

* Received by the editors December 30, 1969, and in revised form July 10, 1970.

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the behavior of the integral curves of a nonlinear system (n) will be similar in some sense to that of the integral curves of the related linear system (l) for small $\|z\|$. We present some results to this effect in this section. Our definitions and methods in general follow Hurewicz [5].

First, we require some terminology. A point p is called a *critical point* of (n) if $Ap + R(p) \equiv 0$. Thus the origin ($z = 0$) is a critical point of both (n) and (l). A critical point p is referred to as *isolated* if there exists a sphere centered at p containing no other critical point, and the critical point $z = 0$ of the nonlinear system (n) is said to be *simple* if the determinant of A is nonzero.

Now suppose that (l) and (n) are defined in a domain D containing the origin and that the origin is a simple critical point. Then with each point $z \neq 0$ in D there are associated two vectors $N(z) \equiv Az + R(z)$ and $X(z) \equiv Az$. Let $\rho(z)$ be the vector $-z$. Finally, define angles $\alpha(z)$, $\beta(z)$, and $\gamma(z)$, all lying in $[0, \pi]$ and such that α is the angle between N and ρ, β the angle between X and ρ, and γ the angle between N and X. The geometry of this situation is pictured in Fig. 1.

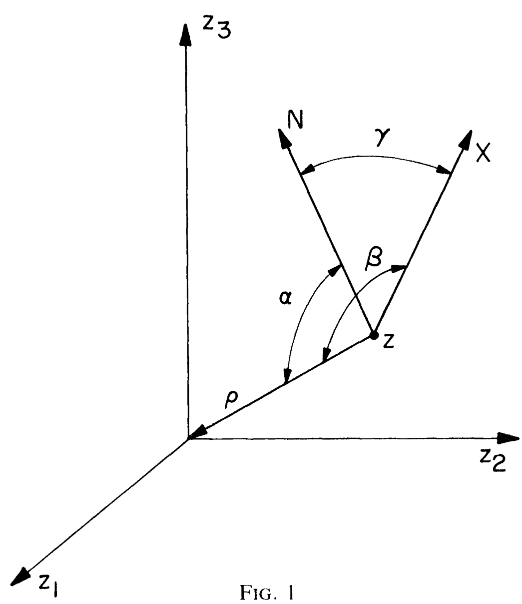

FIG. 1

THEOREM 1. *If the origin is a simple critical point of* (n), *it is also isolated. Furthermore, as* $\|z\|$ *approaches zero we have*

(i) $\lim_{\|z\| \to 0} \|N(z)\|/\|X(z)\| = 1$,

(ii) $\lim_{\|z\| \to 0} \gamma(z) = 0$,

(iii) $\lim_{\|z\| \to 0} (\alpha(z) - \beta(z)) = 0$.

Proof. For all $z \neq 0$, $\|X(z)\| > 0$ since the critical point at the origin is simple. Let $k = \min_{\|z\|=1} \|X(z)\|$. Then $k > 0$, and for all $z \neq 0$,

$$\|X(z)\|/\|z\| = \|X(z/\|z\|)\| \geqq k.$$

Therefore $\|X(z)\| \geqq k\|z\|$ for all $z \neq 0$, and

$$\lim_{\|z\| \to 0} \left\| \frac{N(z) - X(z)}{\|X(z)\|} \right\| \leqq \lim_{\|z\| \to 0} \frac{\|R(z)\|}{k\|z\|} = 0. \tag{2.1}$$

Now if $N(z) = 0$, then $\|(N(z) - X(z))/\|X(z)\|\| = 1$. But by (2.1) we see that this quantity cannot equal 1 arbitrarily close to the origin. Therefore a simple critical point at the origin is isolated. Now we proceed to the proofs of statements (i)–(iii).

(i) We have $\|(N(z) - X(z))/\|X(z)\|\| \geqq \|N(z)\|/\|X(z)\| - 1$. Taking the limit as $\|z\|$ goes to zero we conclude that $\lim_{\|z\|\to 0} \|N(z)\|/\|X(z)\| \leqq 1$. Similarly from $\|(X(z) - N(z))/\|X(z)\|\| \geqq 1 - \|N(z)\|/\|X(z)\|$, we find that

$$\lim_{\|z\|\to 0} \|N(z)\|/\|X(z)\| \geqq 1.$$

Hence (i) holds.

(ii) Suppose $\lim_{\|z\|\to 0} \gamma(z) \neq 0$. Then $N(z)/\|z\| - X(z)/\|z\|$ is a vector of length bounded away from zero as $\|z\| \to 0$. Hence $\lim_{\|z\|\to 0} \|(N(z) - X(z))/\|z\|\| \neq 0$. But this contradicts (2.1).

(iii) Suppose $\lim_{\|z\|\to 0} (\alpha(z) - \beta(z)) \neq 0$. Then $\lim_{\|z\|\to 0} (\cos\alpha(z) - \cos\beta(z)) \neq 0$. Hence

$$\lim_{\|z\|\to 0} \left[\frac{\rho(z)\cdot N(z)}{\|\rho(z)\|\,\|N(z)\|} - \frac{\rho(z)\cdot X(z)}{\|\rho(z)\|\,\|X(z)\|}\right]$$

$$= \lim_{\|z\|\to 0} \frac{\rho(z)}{\|\rho(z)\|}\cdot\left[\frac{N(z)}{\|N(z)\|} - \frac{X(z)}{\|X(z)\|}\right] \neq 0,$$

where the dot $(\cdot)$ indicates the inner product of two vectors. Therefore $\lim_{\|z\|\to 0} \gamma(z) \neq 0$. But this contradicts (ii), so the proof of the theorem is complete.

Now for the next theorem, let the determinant of A be nonzero, and let system (n) be defined in a domain D (containing the origin) and have unique solutions there. For any $p \in D$ let $z_p(t)$ be the unique trajectory of (n) passing through p, and suppose t_p is a time such that $z_p(t_p) = p$. Then the half-trajectory $z_p^+(t) = \{z_p(t) | t \geqq t_p\}$ is said to *approach* the critical point at the origin if and only if $\lim_{t\to+\infty} z_p^+(t) = 0$. A half-trajectory $z_p^+(t)$ which approaches the critical point at the origin is said to *enter* the critical point if and only if the radius vector from the origin to the point $z_p^+(t)$ has a limiting direction as $t \to +\infty$.

THEOREM 2. *Let $z_p^+(t)$ be a half-trajectory of system* (n) *which approaches the simple critical point at the origin. A necessary and sufficient condition that $z_p^+(t)$ enter the origin with direction cosines $n = (n_1, n_2, n_3)$, where $n_i = \lim_{t\to+\infty} [z_i(t)\|z(t)\|^{-1}]$, is that for some k ($k = 1, 2, 3$) and for all $j \neq k$, $\lim_{t\to+\infty} \dot z_j(t)/\dot z_k(t)$ exists and equals n_j/n_k. The only values that n_1, n_2, n_3 may have are normalized* ($\|n\| = 1$) *solutions of*

$$n_j/n_k = (a_j\cdot n)/(a_k\cdot n),$$

where $n_k \neq 0$ and a_j denotes the j-th row of A.

Proof. Suppose that for some k ($k = 1, 2, 3$), $\lim_{t\to+\infty} \dot z_j(t)/\dot z_k(t) = n_j/n_k$ for $j \neq k$. Without loss of generality we may take $k = 1$. Then by l'Hospital's rule, $\lim_{t\to+\infty} z_2(t)/z_1(t) = n_2/n_1$ and $\lim_{t\to+\infty} z_3(t)/z_1(t) = n_3/n_1$. Hence the following limit exists:

$$\lim_{t\to+\infty}\left[\frac{1}{(z_2(t)/z_1(t))^2 + (z_3(t)/z_1(t))^2 + 1}\right]^{1/2} = \lim_{t\to+\infty} [z_1(t)\|z(t)\|^{-1}] = n_1.$$

Suppose $n_1 = 1$. Then $\lim_{t\to+\infty} [(z_2(t)/z_1(t))^2 + (z_3(t)/z_1(t))^2] = 0$ and this implies that $n_2 = n_3 = 0$.

Suppose $n_1 \neq 1$. In this case either $\lim_{t\to+\infty} (z_2(t)/z_1(t))^2 \neq 0$ or

$$\lim_{t\to+\infty} (z_3(t)/z_1(t))^2 \neq 0.$$

For definiteness suppose $\lim_{t\to+\infty} (z_2(t)/z_1(t)) \neq 0$. Then

$$\lim_{t\to+\infty} \frac{z_1(t)}{z_2(t)} \quad \text{and} \quad \lim_{t\to+\infty} \frac{z_3(t)}{z_2(t)} = \lim_{t\to+\infty} \frac{z_1(t)}{z_2(t)} \cdot \frac{z_3(t)}{z_1(t)}$$

exist. Therefore the following limit exists:

$$\lim_{t\to+\infty} \left[\frac{1}{(z_1(t)/z_2(t))^2 + (z_3(t)/z_2(t))^2 + 1}\right]^{1/2} = \lim_{t\to+\infty} z_2(t)\|z(t)\|^{-1} = n_2.$$

If $\lim_{t\to+\infty} (z_3(t)/z_1(t))^2 = 0$, then $\lim_{t\to+\infty} z_3(t)\|z(t)\|^{-1}$ exists and $n_3 = 0$. If on the other hand, $\lim_{t\to+\infty} (z_3(t)/z_1(t))^2 \neq 0$, then we find that $\lim_{t\to+\infty} z_3(t)\|z(t)\|^{-1} = n_3$. This proves the sufficiency part of the theorem.

Now suppose that $z_p^+(t)$ enters the origin with direction cosines n_1, n_2, n_3. Then for some index l, $n_l \neq 0$ and therefore $a_k \cdot n \neq 0$ for some k. Then the quotient

$$\frac{\dot z_j(t)}{\dot z_k(t)} = \frac{a_j \cdot z(t) + R_j(t)}{a_k \cdot z(t) + R_k(t)},$$

where R_j denotes the jth component of R, has a well-defined limit as $t \to +\infty$ for all $j \neq k$. Dividing numerator and denominator by $\|z\|$, we see that in the limit we obtain

$$n_j/n_k = (a_j \cdot n)/(a_k \cdot n)$$

for all $j \neq k$, as was to be proved.

3. Qualitative behavior of $\dddot x = f(x, \dot x, \ddot x)$. We have carried out a study of the behavior of the integral curves of the general equation $\dddot x = f(x, \dot x, \ddot x)$, or equivalently, of the third order system of differential equations

$$\dot z_1 = z_2, \quad \dot z_2 = z_3, \quad \dot z_3 = f(z_1, z_2, z_3). \tag{3.1}$$

We assume that $f(0, 0, 0) = 0$ and that $\partial f/\partial z_1$, $\partial f/\partial z_2$ and $\partial f/\partial z_3$ all exist and are continuous in some domain D containing the origin. Defining $\partial f/\partial z_1|_{(0,0,0)} = c$, $\partial f/\partial z_2|_{(0,0,0)} = b$ and $\partial f/\partial z_3|_{(0,0,0)} = a$, we write system (3.1) as

$$\dot z = \begin{pmatrix} \dot z_1 \\ \dot z_2 \\ \dot z_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ c & b & a \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ r(z_1, z_2, z_3) \end{pmatrix} = Az + R(z), \tag{3.2}$$

where $\lim_{\|z\|\to 0} r(z_1, z_2, z_3)\|z\|^{-1} = 0$. We further assume that $r(z_1, z_2, z_3)$ is continuous in D.

First we consider how stability depends upon the signs of the real constants a, b and c appearing in (3.2). Rather surprisingly, it turns out that most cases are unstable. The only exceptions are the following:

Case I. $a < 0; b < 0; c < 0; ab + c > 0;$
Case II. $a < 0; b < 0; c < 0; ab + c = 0;$
Case III. $a = b = c = 0;$
Case IV. $a = c = 0; b < 0;$
Case V. $b = c = 0; a < 0;$
Case VI. $c = 0; a < 0; b < 0.$

All other cases are immediately seen to be unstable because at least one eigenvalue of the matrix A has positive real part. (For definitions of the terms "stability," "instability" and "asymptotic stability" as used here, see for example [6, pp. 31–2].)

Since Bihari omits singular cases ($\det A = 0$) and since we omit unstable cases, the area of common analysis is Cases I and II. Because of the indeterminacy of Case II, Bihari does very little with it, so it is only in Case I that our results are comparable with his. (Cases II–VI are "critical" or "indeterminate" in the sense that stability or instability depends upon the nonlinearity $r(z_1, z_2, z_3)$. This occurs because at least one eigenvalue has zero real part in each of these cases.)

It is to be emphasized that our primary interest in this paper is in whether or not, in stable cases, trajectories actually enter the critical point. Our comments on stability in what follows are therefore only in the nature of providing a guide to some cases in which the question of entering arises, and are not intended to be comprehensive. An extensive literature exists on the determination of whether or not a critical point of a nonlinear nth order system (particularly in nonsingular cases) is stable, and many of these general theorems and techniques could, of course, be profitably applied to our third order systems. (For example, in cases where the nonlinearity is analytic, see the books of Liapunov [9], [10] and in the general case see the books of Hartman [3, Chap. 10], Lefschetz [7] and Malkin [11] (although Malkin also assumes analyticity in his analysis of the difficult critical cases).)

Case I. All the characteristic roots of A have negative real parts. Hence all solutions $z(t)$ of system (3.2) with $\|z(0)\|$ sufficiently small exist for all $t \geqq 0$ and $\lim_{t\to+\infty} z(t) = 0$. There are four subcases to be considered. They correspond respectively to characteristic roots λ_1, λ_2 and λ_3 of A which are (i) all real and equal, (ii) all real with $\lambda_2 = \lambda_3$, (iii) real and distinct, or (iv) λ_1 real and λ_2 and λ_3 complex conjugates. We shall consider case (i) in some detail and then indicate briefly how the other cases differ.

(i) Let $\lambda_1 = \lambda_2 = \lambda_3 = a/3 < 0$. System (3.2) can be transformed into the canonical form

$$\begin{aligned}\dot{u} &= \begin{pmatrix} a/3 & 1 & 0 \\ 0 & a/3 & 1 \\ 0 & 0 & a/3 \end{pmatrix} u + \begin{pmatrix} 0 \\ 0 \\ r\left(u_1, \frac{a}{3}u_1 + u_2, \frac{a^2}{9}u_1 + \frac{2a}{3}u_2 + u_3\right) \end{pmatrix} \\ &= J_1 u + \begin{pmatrix} 0 \\ 0 \\ s(u_1, u_2, u_3) \end{pmatrix}.\end{aligned} \tag{3.3}$$

Applying Theorem 2, we find that a trajectory (integral curve) of system (3.3) which enters the origin must have direction cosines $(1, 0, 0)$ or $(-1, 0, 0)$ at the origin. That is, any integral curve of system (3.3) which enters the origin must do so tangent to the u_1-axis at the origin. The following theorem gives conditions under which *all* trajectories beginning sufficiently close to the origin enter the origin tangent to the u_1-axis.

THEOREM 3. *Let system* (3.3) *satisfy the additional condition that for some* $\varepsilon > 0$ *and all* u *with* $\|u\| < \rho_0$ $(\rho_0 > 0)$, $s(u) = O(\|u\|^{1+\varepsilon})$. *Then for every set of initial values* $k = (k_1, k_2, k_3)$ *with* $\|k\|$ *sufficiently small there exists a unique solution* $\bar{u}(t)$ *of the system such that* $\bar{u}(0) = k$ *and* $\bar{u}(t)$ *enters the origin tangent to the* u_1*-axis.*

Proof. By our general smoothness assumptions on (3.2), for every k in the domain of definition of (3.3) there exists a unique solution $u(t)$ such that $u(0) = k$. Since $\lambda_1 = \lambda_2 = \lambda_3 = a/3 < 0$, any solution $u(t)$ of system (3.3) with $\|u(0)\| = \|k\| < \rho_1$, where $\rho_1 > 0$ is sufficiently small, exists for all $t \geqq 0$, and

$$\lim_{t \to +\infty} \ln \|u(t)\|/t = a/3$$

for any nontrivial solution $u(t)$ of system (3.3) for which $\|u(0)\| < \rho_1$. (See Hartman and Wintner [4, pp. 694–696].)

Throughout the remainder of this proof let $\bar{u}(t)$ designate a nontrivial solution of (3.3) with $\|\bar{u}(0)\| < \rho\,(\rho_0, \rho_1)$, where $\rho > 0$ is less than ρ_1 and small enough so that $\|\bar{u}(t)\| < \rho_0$ for all $t \geqq 0$. Then there exists a $\delta > 0$ such that

$$S(\bar{u}(t)) \equiv \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} s(\bar{u}(t)) = O(e^{(a/3-\delta)t}) \quad \text{for } t \geqq 0. \tag{3.4}$$

Since $\bar{u}(t)$ is a solution of (3.3), $\bar{u}(t)$ satisfies the integral equation

$$\bar{u}(t) = e^{J_1 t}k + \int_0^t e^{J_1(t-v)}S(\bar{u}(v))\,dv.$$

Furthermore, because the characteristic roots of J_1 are real, equal and negative, there exist positive constants K and σ such that $\sigma < \delta$ and

$$\|e^{J_1 t}\| \leqq Ke^{(a/3-\sigma)t} \quad \text{for } t \leqq 0. \tag{3.5}$$

Then for any solution $\bar{u}(t)$ of system (3.3), $\int_0^\infty e^{-J_1 v}S(\bar{u}(v))\,dv < \infty$; therefore the above integral equation may be written as

$$\bar{u}(t) = e^{J_1 t}k^* - \int_t^\infty e^{J_1(t-v)}S(\bar{u}(v))\,dv \quad \text{for some } k^*. \tag{3.6}$$

From (3.4) and (3.5) it follows that

$$\bar{u}(t) = e^{J_1 t}k^* + O(e^{(a/3-\delta)t}) \quad \text{for } t \geqq 0. \tag{3.7}$$

If we assume that $\bar{u}(t)$ is not the trivial solution (so that $k^* \neq 0$), it follows from (3.7) that

$$\lim_{t\to+\infty} \frac{\dot{\bar{u}}_2(t)}{\dot{\bar{u}}_1(t)} = \lim_{t\to+\infty} \frac{a\bar{u}_2/3 + \bar{u}_3}{a\bar{u}_1/3 + \bar{u}_2}$$

$$= \lim_{t\to+\infty} \frac{(ak_2^*/3 + (at/3 + 1)k_3^*)e^{at/3} + O(e^{(a/3-\delta)t})}{\left[\frac{ak_1^*}{3} + \left(\frac{at}{3} + 1\right)k_2^* + \left(\frac{at}{6} + 1\right)tk_3^*\right]e^{at/3} + O(e^{(a/3-\delta)t})} = 0,$$

and similarly,

$$\lim_{t\to+\infty} \dot{\bar{u}}_3(t)/\dot{\bar{u}}_1(t) = 0.$$

The theorem now follows from Theorem 2.

The theorem above is somewhat similar to Bihari's Theorem 4.1 [1, p. 282]. However, Bihari's result, being stated in terms of (3.3) as transformed into spherical coordinates, is less readily applicable, and his proof is by purely geometrical arguments.

We conclude our discussion of case (i) by presenting an example which illustrates that requiring $s(u) = o(\|u\|)$ as $\|u\| \to 0$ is not sufficient to insure that the behavior of the integral curves of the linear system $\dot{u} = J_1 u$ and that of the integral curves of a nonlinear system having the form of system (3.3) are essentially the same in a small neighborhood of the origin. Notice that in the linear case, the origin is a "node"—i.e., all trajectories beginning sufficiently close to the origin enter it. (For detailed behavior of trajectories in this and other linear cases, the reader is referred to the paper by Reyn [15].) Our example is of a perturbed system which has at least one solution which approaches the origin but does not enter it.

Example 1. Consider the differential equation

$$\dddot{x} = -3\ddot{x} - 3\dot{x} - x + r(z_1, z_2, z_3),$$

where

$$r(z_1, z_2, z_3) = \frac{-1}{|\ln Z|}\left[z_1 + z_2 + \frac{1}{|\ln Z|}\left(-z_1 + \frac{z_1^2(z_1 + z_2)}{Z^2}\left(1 + \frac{1}{|\ln Z|}\right)\right)\right],$$

$$r(0,0,0) = 0, \quad z_1 = x, \quad z_2 = \dot{x}, \quad z_3 = \ddot{x}, \quad \text{and} \quad Z^2 = z_1^2 + (z_1 + z_2)^2.$$

Transformation to canonical form yields

$$\dot{u} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix} u + \begin{pmatrix} 0 \\ 0 \\ s(u_1, u_2, u_3) \end{pmatrix}, \tag{3.8}$$

where

$$s(u_1, u_2, u_3) = \frac{-1}{|\ln R|}\left[u_2 + \frac{1}{|\ln R|}\left(-u_1 + \frac{u_1^2 u_2}{R^2}\left(1 + \frac{1}{|\ln R|}\right)\right)\right]$$

with $R^2 = u_1^2 + u_2^2$ and $s(0,0,0) = 0$. In cylindrical coordinates ($R^2 = u_1^2 + u_2^2$, $\theta = \tan^{-1}(u_2/u_1)$ and $u_3 = u_3$) we have

$$\begin{aligned} \dot{R} &= -R + R\sin\theta\cos\theta + u_3\sin\theta, \\ R\dot{\theta} &= u_3\cos\theta - R\sin^2\theta, \\ \dot{u}_3 &= -u_3 + w(R,\theta,u_3), \end{aligned} \tag{3.9}$$

where

$$w(R,\theta,u_3) = \frac{-R}{|\ln R|}\left[\sin\theta + \frac{\cos\theta}{|\ln R|}\left(-1 + \cos\theta\sin\theta\left(1 + \frac{1}{|\ln R|}\right)\right)\right].$$

Notice that $s(u) = o(\|u\|)$ as $\|u\| \to 0$; but there does not exist any $\varepsilon > 0$ such that $s(u) = O(\|u\|^{1+\varepsilon})$ as $\|u\| \to 0$.

Now $u_3(t) = -R\cos\theta/|\ln R|$ is a particular solution of $\dot{u}_3 = -u_3 + w(R,\theta,u_3)$. For this particular solution, for $0 < R < e^{-1}$, and for initial value $R_0 \neq 0$ sufficiently small so that the corresponding particular solution of system (3.9) is defined for all $t \geqq 0$ and approaches the origin as $t \to +\infty$, straightforward analysis shows that as $t \to +\infty$, $|\theta(t)| \to +\infty$. Hence $\lim_{t\to+\infty} u_2(t)/u_1(t) = \lim_{t\to+\infty} \tan\theta(t)$ does not exist and therefore this solution cannot enter the origin.

(ii) In this case $\lambda_1, \lambda_2, \lambda_3 < 0$ and $\lambda_1 \neq \lambda_2 = \lambda_3$. Application of Theorem 2 (after transformation of the system into canonical form analogous to (3.3)) shows that any integral curve which enters the origin must do so tangent to either the u_1-axis or the u_2-axis. Theorem 3 carries over with only the small change that the entering may be along either the u_1- or u_2-axis, and again an example can be constructed to show that if the hypothesis on $s(u)$ is weakened to $s = o(\|u\|)$, integral curves approaching the origin need not enter it.

(iii) Now consider the case where the eigenvalues λ_i of A are negative and distinct. In the linear case, the origin is a nondegenerate node—all trajectories enter the origin tangent to one or another of the three characteristic directions. Unlike the situation for the degenerate cases treated above, one can prove—under the general assumptions made at the beginning of § 3—that it is also true for the perturbed system that all nontrivial solutions $u(t)$ with $\|u(0)\|$ sufficiently small exist for all $t \geqq 0$ and enter the origin.

(iv) Here we have $\lambda_1 < 0$, $\lambda_2 = \alpha + i\beta$ and $\lambda_3 = \alpha - i\beta$, where $\alpha < 0$ and $\beta > 0$.

Because of the spiraling behavior, even in the linear case all trajectories need not enter the origin. Investigation of the perturbed system shows that (again under our general assumptions) there is always at least one trajectory that does enter. Any trajectories that enter the origin must, of course, do so along the positive or negative u_1-axis. (This is a simple consequence of Theorem 2.)

Case II. $a < 0, b < 0, c < 0$; $ab + c = 0$. The eigenvalues of A are then a and $\pm i\beta$, where $\beta = \sqrt{-b}$. System (3.2) may be put into the canonical form

$$\dot{u} = \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & \beta \\ 0 & -\beta & 0 \end{pmatrix} u + \begin{pmatrix} \beta \\ -\beta \\ -a \end{pmatrix} s(u). \tag{3.10}$$

The fact that the nonlinear terms in the equations are all (except for a constant factor) identical might be expected to limit somewhat the possible sorts of behavior. In fact, however, the critical point at the origin may be stable, asymptotically stable or unstable, depending on the function $s(u)$. (A similar remark will be seen to hold in most of the following cases.)

In the linear case, $s(u) \equiv 0$, the critical point at the origin is clearly stable and a general trajectory may be described as a "converging circular cylindrical spiral." Introduction of the perturbation $s(u)$ may or may not cause instability. In any case Theorem 2 gives us the immediate result that any trajectories entering the origin may only do so along the u_1-axis. In case s is a function of u_2 and u_3 only, the last two equations of (3.10) are independent of $u_1(t)$ and we can get results such as the following.

THEOREM 4. *In system* (3.10) *let* $s(u) = h(u_2, u_3)$. *Assume that the critical point* $(0, 0)$ *of*

$$\begin{pmatrix} \dot{u}_2 \\ \dot{u}_3 \end{pmatrix} = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix} \begin{pmatrix} u_2 \\ u_3 \end{pmatrix} + \begin{pmatrix} -\beta \\ -a \end{pmatrix} h(u_2, u_3) \tag{3.11}$$

is asymptotically stable and that there exists a positive integer n such that, for any solution $(u_2(t), u_3(t))$ *of* (3.11),

$$t^{-n-1} \leqq (u_2^2(t) + u_3^2(t))^{1/2} \quad \text{for } t \geqq t_0 > 1$$

and

$$\lim_{t \to +\infty} t^{n+1} \int_{t_0}^{t} e^{a(t-v)} |h(u_2(v), u_3(v))| \, dv < \infty \quad \textit{for } t_0 > 1.$$

Then only the integral curve coinciding with the positive u_1-axis and the integral curve coinciding with the negative u_1-axis enter the origin.

Proof. Let $u(t)$ be any solution of the system (3.10) such that $u_2^2(t_0) + u_3^2(t_0) \neq 0\, (t_0 > 1)$. Then it follows that

$$\lim_{t \to \infty} \frac{|u_1(t)|}{(u_2^2(t) + u_3^2(t))^{1/2}}$$

$$\leqq \lim_{t \to +\infty} \left[e^{a(t-t_0)} |u_1(t_0)| + \beta \int_{t_0}^{t} e^{a(t-v)} |h(u_2(v), u_3(v))| \, dv \right] / t^{-n-1} < \infty.$$

The result follows since the limit as $t \to +\infty$ of the quotient $(u_2^2(t) + u_3^2(t))/u_1^2(t)$, if it exists, cannot be zero.

Remark. We wish to justify the reasonableness of the hypothesis that there exists a positive integer n such that $t^{-n-1} \leqq (u_2^2(t) + u_3^2(t))^{1/2}$ for $t \geqq t_0 > 1$. By asymptotic stability, $(u_2^2(t) + u_3^2(t))^{1/2}$ will in general approach zero "like" some power of t; this power, however, cannot be arbitrarily large, since if $(u_2^2(t) + u_3^2(t))^{1/2}$ went to zero "like" e^{-t} that would imply that the real parts of the characteristic roots corresponding to u_2 and u_3 are negative.

It is clear that under the hypotheses of Theorem 4 the system (3.10) is asymptotically stable. In fact, since (3.11) is independent of u_1, stability or instability of

(3.10) depends upon stability or instability of (3.11). Pliss [13] has shown that under very general conditions, study of the stability of a system with n eigenvalues having zero real parts (and the rest negative real parts) may be similarly reduced to the study of the stability of an n-dimensional system. The method of reduction involves determination of an invariant surface for the full system and, for any given system, is likely to be difficult to carry out in practice.

Case III. $a = b = c = 0$. Then system (3.2) is already in canonical form. In the linear case ($r(z) \equiv 0$), all points on the z_1-axis are critical points. This axis forms a so-called "shear line." No critical point is stable.

Adding the nonlinearity $r(z)$, we introduce the possibility that the critical point at the origin may be isolated. In fact this occurs if and only if there is a deleted neighborhood of $z_1 = 0$ within which $r(z_1, 0, 0) \neq 0$. Assuming that this is the case, we may then inquire as to whether this isolated critical point at the origin is stable. It is easy to see that in most cases (for example, whenever the origin is an isolated zero of $r(z_1, z_2, z_3)$) the instability of the linear case carries over into the nonlinear one. It is possible, however, that the introduction of a nonlinear perturbation may convert the origin into a stable critical point, as the following example shows.

Example 2. Consider the differential equation $\dddot{x} = -(\dot{x})^3$. Letting $z_1 = x$, $z_2 = \dot{x}$ and $z_3 = \ddot{x}$, we obtain a third order system which can be written

$$\frac{dz_1}{z_2} = \frac{dz_2}{z_3} = \frac{dz_3}{-z_2^3}. \tag{3.12}$$

Solving the right-hand equality, we obtain as a first integral

$$z_3^2 + z_2^4/2 = c_1^2. \tag{3.13}$$

Solving (3.13) for z_2 and substituting into (3.12), we have

$$dz_1 = dz_3/-\sqrt{2}(c_1^2 - z_3^2)^{1/2}.$$

Therefore, for $c_1 \neq 0$,

$$z_1 + (1/\sqrt{2}) \sin^{-1} (z_3/|c_1|) = c_2. \tag{3.14}$$

The solutions of (3.12) for fixed $c_1 (\neq 0)$ and c_2 are curves which are the intersections of the cylinders defined by (3.13) and (3.14). For $c_1 = 0$ the solutions are all points $(k, 0, 0,)$—i.e., points on the z_1-axis. Typical trajectories are sketched in Fig. 2. Clearly each point of the z_1-axis is a stable critical point.

We leave open the question of whether for some perturbation $r(z)$ the origin may become an *asymptotically* stable critical point. It seems to us highly unlikely.

Of course, even in unstable cases *some* trajectories may approach the origin, and the question as to whether or not they enter the origin becomes pertinent. Theorem 2 is unfortunately not applicable here (nor in any of the singular cases to follow), but considering Case III as a limiting form of Case I (i) we conjecture that integral curves entering the origin must do so tangent to the u_1-axis.

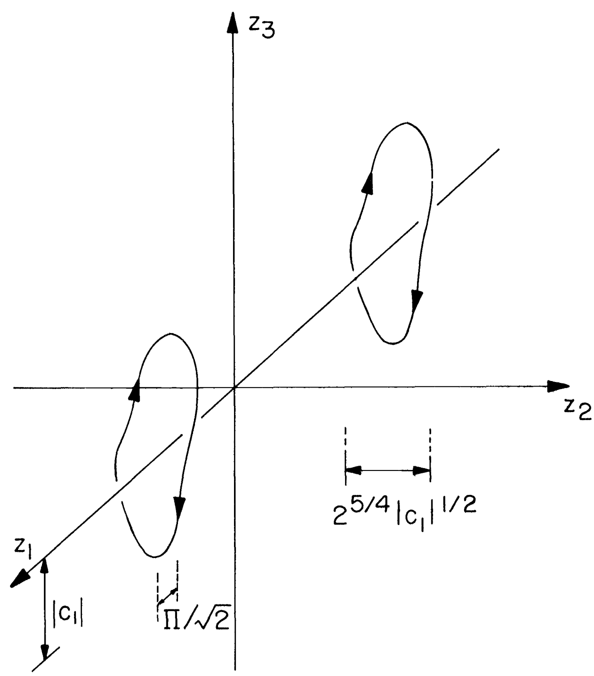

FIG. 2

Case IV. $a = c = 0$; $b < 0$. The eigenvalues of A are, in this case, 0 and $\pm i\beta$, where $\beta = \sqrt{-b}$; and the canonical form of the system is

$$\dot{u} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \beta \\ 0 & -\beta & 0 \end{pmatrix} u + \begin{pmatrix} -1/b \\ 1/b \\ 0 \end{pmatrix} s(u). \tag{3.15}$$

The linear case ($s(u) \equiv 0$) is neutrally stable. (The u_1-axis forms a "line of centers.") One therefore expects that the introduction of the nonlinearity $s(u)$ could induce either asymptotically stable or unstable behavior. This is in fact the case.

Cetaev's theorem [6, p. 39] may be used to readily derive sufficient conditions for instability. For example, we find that if the origin is an isolated zero of $s(u)$, then the origin is an isolated, unstable, critical point of (3.15).

The following example shows that asymptotic stability is also a possibility.

Example 3. In system (3.15) let $s(u) = b(u_1 - u_2)^3$. Let $\rho^2 = u_1^2 + u_2^2 + u_3^2$. Then $\rho\dot{\rho} = -(u_1 - u_2)^4$. Hence for $\rho \neq 0$ and $u_1 \neq u_2$, $\dot{\rho} < 0$. Let $\theta = \tan^{-1}(u_2/u_1)$. Then for $\rho \neq 0$, $u_3 \neq 0$ and $u_1 = u_2$, $\dot{\theta} = \beta u_3/2u_1 \neq 0$. Finally, for $\rho \neq 0$, $u_3 = 0$ and $u_1 = u_2$, $\dot{u}_3 = \beta u_2 \neq 0$. Thus for this example system (3.15) has an isolated critical point at the origin which is asymptotically stable.

For this case no results concerning entering have been established. However, by considering this case to be a limiting case of Case I (iv), Case II or Case VI (iii) one is inclined to predict that there are instances in which integral curves enter the origin tangent to the line $u_2 = u_3 = 0$. We have been unable to find an example of such an instance.

Case V. $b = c = 0$; $a < 0$. The eigenvalues of A are $0, 0, a$, and the canonical form of the system is

$$\dot{u} = \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} u + \begin{pmatrix} 1/a^2 \\ -1/a^2 \\ -1/a \end{pmatrix} s(u). \tag{3.16}$$

In the linear case ($s(u) \equiv 0$) all points on the u_2-axis are unstable critical points. Instability is therefore the normal situation, and as in Cases III and IV, it is easy to show that if the origin is an isolated zero of $s(u)$, then the origin is an isolated unstable critical point of (3.16).

Turning to the question of entering, we find (as an immediate application of Theorem (*) of Hartman and Wintner [4, p. 695]) that there always exists a solution $u(t)$ of (3.16) which enters the origin tangent to the u_1-axis. We also may obtain a theorem quite analogous to Theorem 4 for Case II—i.e., stating that under the same hypotheses as in Theorem 4, only the integral curves coinciding with the positive and negative u_1-axes enter the origin.

Case VI. $c = 0$; $a < 0$, $b < 0$. The eigenvalues of A are 0 and $(1/2) \cdot (a \pm (a^2 + 4b)^{1/2})$. The canonical form of the system will depend upon the sign of $(a^2 + 4b)$, but these subcases have certain common features which we first discuss. The linear case is essentially stable; all points along the z_1-axis (the u_1-axis in the canonical system) are stable critical points. The stability of any one critical point is of the "neutral" type, however, so that an arbitrary perturbation is likely to cause instability. In fact, as in Cases III–V, if the origin is an isolated zero of $r(z)$, then the origin is an unstable isolated critical point of (3.2). It is possible, however, that an isolated critical point at the origin may be stable or even asymptotically stable. This fact will become clear as we look closer at the three subcases.

Before doing this, we note that a natural way to study stability in this case is through the "product-space" approach (see, for example, Lefschetz [7]) and a theorem which we will find useful is the following.

THEOREM 5. *Consider the system of $q + 1$ equations*

$$\dot{y} = g(y) + h(y, z), \tag{3.17α}$$

$$\dot{z} = Qz + k(y, z), \tag{3.17β}$$

where z and k are q-dimensional ($q \geqq 1$) and Q is a constant, stable $q \times q$ matrix. Assume that the following conditions hold:

1. *$g(y)$, $h(y, z)$ and $k(y, z)$ are each continuously differentiable in some domain containing the origin of the appropriate space.*
2. $g(0) = h(0, 0) = 0$; $k(0, 0) = 0$.
3. $g(y) = o(|y|)$ as $|y| \to 0$.
4. *There exist positive constants ρ_1 and ρ_2 such that*

$$\operatorname{sgn} g(y) = \operatorname{sgn} [g(y) + h(y, z)]$$

whenever $|y| < \rho_1$ and $\|z\| < \rho_2$.

5. *There exist positive constants* ρ_3 *and* K, *and nonnegative continuous functions* $l(y)$ *and* $m(z)$, *with* $l(y) = o(|y|)$ *as* $|y| \to 0$, *such that*

$$\|k(y, z)\| \leqq Kl(y)m(z) \quad \textit{whenever } \|(y, z)\| < \rho_3 .$$

Then the critical point at the origin of (3.17) *is stable* (*asymptotically stable*) *if the critical point at* $x = 0$ *of the equation*

$$\dot{x} = g(x) \tag{3.18}$$

is stable (*asymptotically stable*).

A proof of this theorem may be found in [16]. Essentially, the technique is to invoke a converse Lyapunov theorem for (3.18) and, using the Lyapunov function thus known to exist, to obtain asymptotic information about y and ultimately z. We omit the proof here since the theorem is very similar to one given by Munir [12] in a somewhat more general context.

In looking at the three subcases, we shall, as in Case I, present in some detail results for the first case considered and then only indicate briefly how the other cases may differ.

(i) $a^2 + 4b = 0$. The canonical form of the system is then

$$\dot{u} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a/2 & 1 \\ 0 & 0 & a/2 \end{pmatrix} u + \begin{pmatrix} 4/a^2 \\ -4/a^2 \\ 2/a \end{pmatrix} s(u). \tag{3.19}$$

In the special case that s is a function of u_1 only, we may readily apply Theorem 5 with $q = 2$. Furthermore, although Theorem 2 is not applicable (because of the singularity of A) we can get some information on entering under hypotheses similar to those of Theorem 4. To be precise, we obtain the following result.

THEOREM 6. *In system* (3.19) *let* $s(u) = g(u_1)$.

(a) *The critical point at the origin is stable if and only if* $u_1 = 0$ *is a stable critical point of*

$$\dot{u}_1 = 4g(u_1)/a^2. \tag{3.20}$$

Furthermore, if the critical point $u_1 = 0$ *of system* (3.20) *is asymptotically stable, then the critical point at the origin of system* (3.19) *is asymptotically stable.*

(b) *Let the critical point* $u_1 = 0$ *of system* (3.20) *be asymptotically stable and assume there exists a positive integer* n *such that for any solution* $u_1(t)$ *of* (3.20), $t^{-n-1} \leqq |u_1(t)|$ *for* $t \geqq t_0 > 1$ *and*

$$\lim_{t \to +\infty} t^{n+2} \int_{t_0}^{t} e^{a(t-v)/2} |g(u_1(v))| \, dv = 0 \quad \textit{for } t_0 > 1.$$

Then all integral curves lying in the u_2, u_3*-plane enter the origin tangent to the* u_2*-axis while all integral curves not lying in the* u_2, u_3*-plane enter the origin tangent to the* u_1*-axis.*

Proof. Statement (a) follows immediately from Theorem 5. It remains only to prove (b).

Suppose that $u_1(t) = 0$ for some t. Then from system (3.19) with $s(u) = g(u_1)$ we see that

$$\dot{u}_1(t) = 4g(u_1(t))/a^2 = 0,$$
$$\dot{u}_2(t) = au_2(t)/2 + u_3(t),$$
$$\dot{u}_3(t) = au_3(t)/2,$$

since $g(0) = 0$. Hence any solution which begins (when $t = 0$) on the u_2, u_3-plane remains on the u_2, u_3-plane and satisfies

$$u_2(t) = (u_2(0) + u_3(0)t)e^{at/2} \quad \text{and} \quad u_3(t) = u_3(0)e^{at/2}.$$

If $u_3(0) = 0$, then the integral curve coincides with a u_2-semiaxis; if $u_3(0) \neq 0$, then $\lim_{t \to +\infty} u_3(t)/u_2(t) = 0$. Therefore all integral curves lying in the u_2, u_3-plane enter the origin tangent to the u_2-axis.

Now consider an integral curve which does not lie in the u_2, u_3-plane; i.e., suppose that $u_1(t_0) \neq 0$ for $t_0 > 1$. Then

$$u_3(t) = u_3(t_0)e^{a(t-t_0)/2} + 2\int_{t_0}^{t} e^{a(t-v)/2} g(u_1(v))\, dv/a, \qquad t \geqq t_0 > 1.$$

Hence for $t \geqq t_0 > 1$,

$$|u_3(t)| \leqq |u_3(t_0)|e^{a(t-t_0)/2} - \frac{2}{a}\int_{t_0}^{t} e^{a(t-v)/2}|g(u_1(v))|\, dv.$$

Then it follows immediately from the hypotheses that

$$\lim_{t \to +\infty} |u_3(t)|/|u_1(t)| = 0.$$

Now

$$u_2(t) = e^{a(t-t_0)/2}[u_2(t_0) + (t - t_0)u_3(t_0)]$$
$$+ \int_{t_0}^{t} e^{a(t-v)/2} g(u_1(v)) \left[\frac{-4}{a^2} + \frac{2(t-v)}{a}\right] dv.$$

Hence

$$|u_2(t)| \leqq e^{a(t-t_0)/2}[|u_2(t_0)| + (t - t_0)|u_3(t_0)|]$$
$$+ \int_{t_0}^{t} e^{a(t-v)/2}|g(u_1(v))| \; \frac{4}{a^2} - \frac{2(t-v)}{a} \; dv.$$

Finally, if we replace the term $-2(t - v)/a$ in the integrand by the larger quantity $-2t/a$ and then divide by $|u_1(t)|$ and apply the hypotheses, we get

$$\lim_{t \to +\infty} |u_2(t)|/|u_1(t)| = 0.$$

This completes the proof.

Of course, although consideration of the special case $s(u) = s(u_1)$ leads to the most definitive results, this choice does not provide the only possibility of stable behavior. For example, it is clear that the origin is stable if $s(u)$ is a function only of

u_2 and u_3. Furthermore, examples (admittedly rather complicated ones) in which the origin is an asymptotically stable isolated critical point can be constructed.

(ii) $a^2 + 4b > 0$. The canonical form is

$$\dot{u} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} u + \begin{pmatrix} \lambda_3 - \lambda_2 \\ -\lambda_3 \\ \lambda_2 \end{pmatrix} s(u), \tag{3.21}$$

where $\lambda_{2,3} = (1/2)(a \pm (a^2 + 4b)^{1/2})$.

Stability results similar to those for subcase (i) hold here. This is not surprising, since in both instances there is one zero eigenvalue and there are two eigenvalues with negative real parts. In particular, we can prove a theorem completely analogous to Theorem 6; however, in this case the statement on entering must be modified to except the particular solutions coinciding with the positive and negative u_3-axes.

(iii) $a^2 + 4b < 0$. Although the canonical form is now

$$\dot{u} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a/2 & \beta \\ 0 & -\beta & a/2 \end{pmatrix} u + \begin{pmatrix} \beta \\ -\beta \\ a/2 \end{pmatrix} s(u), \tag{3.22}$$

where $\beta = (1/2)(-(a^2 + 4b^2))^{1/2}$, we again have one zero eigenvalue and two with negative real parts, and therefore stability results similar to those for subcase (i) apply. Again in the case that $s(u) = s(u_1)$, we can prove a theorem analogous to Theorem 6, but the conclusion on entering is that *no* integral curve lying in the u_2, u_3-plane enters the origin whereas all integral curves not lying in the u_2, u_3-plane enter the origin tangent to the u_1-axis. One most easily verifies the spiraling behavior of trajectories lying in the u_2, u_3-plane by transforming (3.22) into cylindrical coordinates $u_1 = u_1$, $R^2 = u_2^2 + u_3^2$ and $\theta = \tan^{-1}(u_3/u_2)$. One then readily determines that, under the hypotheses of Theorem 6, in this case all solutions lying in the u_2, u_3-plane satisfy $\lim_{t\to\infty} |\theta(t)| = \infty$, so that $\lim_{t\to\infty} \tan\theta(t)$ does not exist. The asymptotic behavior of curves not lying in the u_2, u_3-plane is then obtained by showing (as in the proof of Theorem 6) that $\lim_{t\to\infty} R(t)/|u_1(t)| = 0$.

4. Summary. We have here presented some results of a study of the behavior of solution curves of third order autonomous differential equations in the neighborhood of a critical point. Without loss of generality, we have taken the critical point to be at the origin. We have chiefly concentrated on one aspect of the problem; namely, whether, in the case that the critical point is stable, all or some integral curves enter the critical point. Results in the "determinate" cases (where the matrix A associated with the unperturbed system has no eigenvalue with zero real part) are fairly complete. In other cases we have attempted to indicate possible sorts of behavior and to point out areas requiring further study.

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SIAM J. MATH. ANAL.
Vol. 2, No. 2, May 1971

GLOBAL REDUCTION OF LINEAR DIFFERENTIAL SYSTEMS INVOLVING A SMALL SINGULAR PARAMETER*

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Abstract. The splitting of linear holomorphic differential systems involving a small singular parameter into lower order systems has been discussed by Sibuya and Wasow.

The systems which are considered are of the form $\varepsilon^m(du/dz) = A(z, \varepsilon)u$, where $A(z, \varepsilon) \sim \sum_0^\infty A_\nu(z)\varepsilon^\nu$ as $\varepsilon \to 0$ and the eigenvalues of the lead matrix $A_0(z)$ are blockwise distinct in a sufficiently small neighborhood of a point z_0. As stated in a reduction theorem of Sibuya's, such a system is locally equivalent to two systems of the same type and of an order less than n. In this paper by using holomorphic block diagonalization and triangularization of holomorphic matrices the methods employed by Sibuya and Wasow are extended. This enables us to prove a global version of the reduction theorem of Sibuya. By applying the global reduction theorem we obtain theorems on asymptotic expansions and factorizations of differential equations involving a small parameter. These theorems are related to results of Langer and Erdélyi.

Introduction. We consider systems of n ordinary first order linear homogeneous differential equations involving a small parameter

$$(*) \qquad \varepsilon^m \frac{du}{dz} = A(z, \varepsilon)u.$$

Here $A(z, \varepsilon)$ is an $n \times n$ matrix, u is an n-vector and m is a positive integer. The matrix $A(z, \varepsilon)$ is holomorphic in z and ε for z in a domain D and ε in a sector $|\arg \varepsilon| \leqq \delta$, $0 < |\varepsilon| \leqq \varepsilon_1$. Furthermore, $A(z, \varepsilon)$ has the asymptotic expansion

$$(**) \qquad A(z, \varepsilon) \sim \sum_0^\infty A_\nu(z)\varepsilon^\nu \quad \text{as } \varepsilon \to 0$$

uniformly for $z \in D$, $|\arg \varepsilon| \leqq \delta$.

The asymptotic behavior of solutions of $(*)$ as $\varepsilon \to 0$, $|\arg \varepsilon| \leqq \delta$, is known in the case $n = 2$ and in the case that the lead matrix $A_0(z)$ in $(**)$ has n different eigenvalues in D (cf. Langer [2]). If some of these eigenvalues coincide at points z in D, this asymptotic behavior is not known except in special cases. In the case that the eigenvalues of the lead matrix $A_0(z)$ are blockwise distinct, Sibuya [4] has shown that the system $(*)$ is locally equivalent to two systems of the same type of an order less than n. In this paper we prove a global version of this reduction theorem.

The splitting of system $(*)$ into systems of lower order can be used in the study of the asymptotic behavior of solutions of $(*)$. Another application could be the study of multipoint boundary value problems for $(*)$.

In § 1 we consider the triangularization of holomorphic matrices. This and a result of Sibuya [5] on block diagonalization of holomorphic matrices are the tools employed in the proof of the global reduction theorem. In § 2 we obtain an inequality involving a fundamental system of solutions of $(*)$ in the case that $A(z, \varepsilon)$ is an upper triangular matrix independent of ε. Using the results of § 1

* Received by the editors February 11, 1970, and in revised form September 10, 1970.
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and § 2, in § 3 we prove a global version of an existence theorem of Sibuya for asymptotic expansions of a nonlinear differential equation. From this theorem we deduce the global reduction theorem. The proof is similar to the proof of Sibuya (see the exposition given by Wasow in [6, Chap. 7]).

The result of Langer mentioned above with slightly more restrictive conditions is obtained in § 5 as a special case of the reduction theorem. Moreover, in § 5 we present an analytic factorization theorem for nth order linear homogeneous differential equations involving a small parameter. This theorem is an extension of the results of Langer [3] and Erdélyi [1] on asymptotic factorization of such equations.

The following terminology is used:

I_n denotes the $n \times n$ identity matrix. The norm of an $n \times m$ matrix A with elements $a_{jh}, j = 1, \cdots, n; h = 1, \cdots, m$, is defined by

$$\|A\| = \max_{j=1,\cdots,n} \sum_{h=1}^{m} |a_{jh}|.$$

A matrix $A(z)$ is said to be holomorphic in a set D of the complex z-plane if all its elements are holomorphic in z in a domain containing D. A matrix $A(z)$ is said to be algebroid in a set D if there exists a domain containing D such that the elements of $A(z)$ are algebroid in this domain.

$S(\varepsilon_0, \delta_0)$ denotes the sector in the complex ε-plane where $0 < |\varepsilon| \leqq \varepsilon_0$ and $|\arg \varepsilon| \leqq \delta_0$.

1. Triangularization of holomorphic matrices. First we prove the following lemma.

LEMMA 1. *Let $C(z)$ be an $m \times n$ matrix which is holomorphic in a compact set D. Suppose* rank $C(z) \leqq n - 1$ *for all points z in D. Then there exists a holomorphic n-vector $v(z)$ in D such that*

$$C(z)v(z) = 0 \tag{1.1}$$

and $v(z) \neq 0$ for all points z in D.

Proof. Let r be the maximum of the rank of $C(z)$ in D and let z_0 be a point in D such that rank $C(z_0) = r$. Then there exists an $r \times r$ submatrix $C^*(z)$ of $C(z)$ such that rank $C^*(z_0) = r \leqq n - 1$. The zeros of det $C^*(z)$ are isolated in D. By deleting these zeros from D we obtain a set D_1 in which $C^*(z)$ is nonsingular and rank $C(z) = r$. Therefore in D_1, (1.1) is equivalent to the equation $C_1(z)v(z) = 0$, where $C_1(z)$ is the $r \times n$ submatrix of $C(z)$ which has $C^*(z)$ as a submatrix.

The r components of $v(z)$, which in this equation are multiplied by elements of $C^*(z)$, may be expressed in the $n - r$ other components of $v(z)$ by means of Cramer's rule. Taking these $n - r$ components equal to zero except for one which is chosen to be det $C^*(z)$, we obtain a solution $\tilde{v}(z)$ of (1.1) in D_1. This vector is holomorphic in D, but it may vanish in the isolated zeros $z_1, \cdots, z_k$ of det $C^*(z)$.

Let β_j be the minimum of the orders of the zero z_j of the components of $\tilde{v}(z)$. Let $v(z) = \prod_{j=1}^{k} (z - z_j)^{-\beta_j}\tilde{v}(z)$. Now $v(z)$ is holomorphic in D and $v(z) \neq 0$ throughout D. Since $v(z)$ satisfies (1.1) in D_1, it also satisfies (1.1) in D.

Using Lemma 1 we can prove the following theorem on triangularization of holomorphic matrices.

THEOREM 1. *Suppose $A(z)$ is a square matrix which is holomorphic in a compact set D. Then in D there exist an upper triangular matrix $B(z)$ and a nonsingular matrix $C(z)$ satisfying*

$$B(z) = C^{-1}(z)A(z)C(z) \tag{1.2}$$

in D. Furthermore $B(z)$ and $C(z)$ are algebroid in D with the same algebraic singularities as the eigenvalues of $A(z)$.

Proof. Let $\lambda_1(z), \cdots, \lambda_n(z)$ be the eigenvalues of the $n \times n$ matrix $A(z)$. They are algebroid functions in D. Consider the Riemann surface of these eigenvalues above D and use a uniformization parameter z^* as a new variable. Let D^* be the domain of definition of z. We shall write $A(z^*)$ and $\lambda_j(z^*)$ instead of $A(z(z^*))$ and $\lambda_j(z(z^*))$.

The rank of the matrix $A(z^*) - \lambda_1(z^*)I_n$ is less than n. So by Lemma 1 there exists a holomorphic vector $v(z^*)$ in D^* which is an eigenvector belonging to the eigenvalue $\lambda_1(z^*)$ of $A(z^*)$ on D^*. Next we construct an orthogonal basis $v_1(z^*), \cdots, v_n(z^*)$ in D^* of holomorphic vectors. The condition that a vector $v(z^*)$ is orthogonal to r holomorphic vectors $\tilde{v}_j(z^*)$, $j = 1, \cdots, r$, with $r < n$ can be written in the form (1.1). Hence according to Lemma 1 there exists a holomorphic vector $v(z^*)$ in D^* satisfying this condition with $v(z^*) \neq 0$ in D^*. So the orthogonal basis mentioned above exists.

Let $V_1(z^*)$ be the matrix with columns $v_1(z^*), \cdots, v_n(z^*)$. Then $V_1(z^*)$ is nonsingular and holomorphic in D^* and

$$V_1^{-1}(z^*)A(z^*)V_1(z^*) = \begin{bmatrix} \lambda_1(z^*) & A_{n-1}^{12}(z^*) \\ 0 & A_{n-1}^{22}(z^*) \end{bmatrix},$$

where $A_{n-1}^{22}(z^*)$ is a holomorphic $(n-1) \times (n-1)$ matrix with eigenvalues $\lambda_2(z^*), \cdots, \lambda_n(z^*)$ in D. The $1 \times (n-1)$ matrix $A_{n-1}^{12}(z^*)$ is also holomorphic in D^*.

We apply the same process to $A_{n-1}^{22}(z^*)$ instead of $A(z^*)$ and thus obtain a matrix $V_2(z^*)$ etc. Let $W_j(z^*)$ denote the matrix $\operatorname{diag}\{I_{j-1}, V_j(z^*)\}$, $j = 1, \cdots$ $n-1$. Choosing $C(z^*) = W_1(z^*) \cdots W_{n-1}(z^*)$, we can easily deduce the assertions of the theorem.

Finally we mention the following theorem on holomorphic block diagonalization of Sibuya (cf. [5]) which will be used in § 3.

THEOREM 2. *Let $A(z)$ be a holomorphic $n \times n$ matrix in a compact simply-connected set D. Let the characteristic polynomial of $A(z)$ be equal to the product of two coprime polynomials $c_1(\lambda, z)$ and $c_2(\lambda, z)$ of degrees n_1 and n_2 in λ with holomorphic coefficients in D.*

Then there exist an $n \times n$ matrix $C(z)$, an $n_1 \times n_1$ matrix $B_1(z)$ and an $n_2 \times n_2$ matrix $B_2(z)$, all holomorphic in D, such that $C^{-1}(z)A(z)C(z) = \operatorname{diag}\{B_1(z), B_2(z)\}$, $z \in D$, and the characteristic polynomial of $B_j(z)$ is $c_j(\lambda, z)$, $j = 1, 2$.

2. A basic inequality. In this section we prove an inequality for integrals related to the differential equation (2.2) where $T(z)$ is an upper triangular matrix. The role of this inequality (see Lemma 2) in the proof of the global reduction theorem is similar to that of Lemma 27.1 of Wasow [6] in the proof of Sibuya's theorem.

First we formulate a condition $\mathrm{H}(\delta)$ which is used in Lemma 2. This condition is similar to that on "associated regions" used by Langer [2] in obtaining the asymptotic expansions for solutions of (*) in the case of n different eigenvalues of $A(z, 0)$. At the end of this section we present a condition $\mathrm{H}^*(\delta)$ which implies $\mathrm{H}(\delta)$.

Let D be a simply-connected domain with boundary C which is piecewise smooth. Let δ be a positive number.

DEFINITION 1. The domain D and the N algebroid functions $t_1(z), \cdots, t_N(z)$ in $D \cup C$ are said to satisfy *condition* $\mathrm{H}(\delta)$ if these functions do not have zeros in D and if there exist branchcuts $L_1, \cdots, L_q$ in D such that $\tilde{D} = D \setminus \bigcup_{j=1}^{q} L_j$ is simply-connected and such that there exist branches of $t_1(z), \cdots, t_N(z)$ in $\tilde{D}$ with the following property:

There exists a point z^* on C such that any point z in D can be connected with z^* by a smooth curve in D which does not intersect $L_1, \cdots, L_q$ and satisfies the inequality

$$|\arg t_j(\zeta)\, d\zeta| \leqq \tfrac{1}{2}\pi - \delta, \qquad j = 1, \cdots, N, \tag{2.1}$$

if ζ runs from z to z^* along this curve. If z is a point of a branchcut different from a branchpoint there have to be two such curves, one on each side of the cut.

Let $T(z)$ be an algebroid upper triangular matrix in D with diagonal elements $t_1(z), \cdots, t_N(z)$ satisfying condition $\mathrm{H}(\delta)$. Let the branchpoints of $T(z)$ be the same as those of $t_h(z)$, $h = 1, \cdots, N$. We denote these points by $\alpha_1, \cdots, \alpha_q$. In the sequel we choose the branch of $T(z)$ which corresponds to the branches of $t_h(z)$ mentioned in condition $\mathrm{H}(\delta)$.

Consider the matrix differential equation

$$\varepsilon^m \frac{dV}{dz} = T(z)V. \tag{2.2}$$

Let $V(z, \zeta, \varepsilon)$ be the fundamental matrix solution of (2.2) in D^* with the property

$$V(\zeta, \zeta, \varepsilon) = I_N \tag{2.3}$$

for any point ζ in $\tilde{D}$.

The following lemma contains an inequality on which the discussion in § 3 is based.

LEMMA 2. *Let η, ε_1 and δ_1 be positive constants $0 < \eta \leqq 1$, $m\delta_1 < \delta$. Let $\chi(z, \varepsilon)$ be holomorphic for $z \in \tilde{D}$, $\varepsilon \in S(\varepsilon_1, \delta_1)$ such that*

$$\|\chi(z, \varepsilon)\| \leqq R\left[1 + \sum_{k=1}^{q} |\varepsilon^{-m}(z - \alpha_k)|^{-1+\eta}\right] \tag{2.4}$$

for $z \in \tilde{D}$, $\varepsilon \in S(\varepsilon_1, \delta_1)$. Here R is a constant independent of z.

Then there exists a positive constant K independent of χ, z and ε such that

$$\left\| \int_{z^*}^{z} V(z, \zeta, \varepsilon)\chi(\zeta, \varepsilon)\, d\zeta \right\| \leqq KR|\varepsilon^m| \tag{2.5}$$

for $z \in \tilde{D}$ and $\varepsilon \in S(\varepsilon_1, \delta_1)$. In (2.5) the path of integration lies in $\tilde{D}$.

Proof. Let $v_{hj}(z, \zeta, \varepsilon)$ be the element at the place (h, j) of $V(z, \zeta, \varepsilon)$. Then we derive from (2.2) and (2.3),

$$\begin{aligned} &v_{Nj}(z, \zeta, \varepsilon) = 0, \cdots, v_{j+1,j}(z, \zeta, \varepsilon) = 0, \\ &v_{jj}(z, \zeta, \varepsilon) = \exp \varepsilon^{-m} \int_{\zeta}^{z} t_j(\zeta_1)\, d\zeta_1, \qquad j = 1, \cdots, N, \end{aligned} \tag{2.6}$$

and

$$v_{hj}(z, \zeta, \varepsilon) = \varepsilon^{-m} \sum_{l=h+1}^{j} \int_{\zeta}^{z} t_{hl}(\zeta_2) v_{lj}(\zeta_2, \zeta, \varepsilon) \exp\left\{\varepsilon^{-m} \int_{\zeta_2}^{z} t_h(\zeta_1)\, d\zeta_1\right\} d\zeta_2 \tag{2.7}$$

if $h < j$.

Consider a point z in $\tilde{D}$ and a curve L from z to z^* such that (2.1) holds on L. We may transform L into a curve L' from z to z^* such that the arc length $\sigma(\zeta)$ along L' satisfies $|\sigma(\zeta_2) - \sigma(\zeta_1)| \leqq 2|\zeta_2 - \zeta_1|$ for ζ_1 and ζ_2 on L' in sufficiently small neighborhoods of $\zeta = \alpha_j$, $j = 1, \cdots, q$. The curve L' may be constructed in such a way that (2.1) remains valid if we replace δ by a constant δ_2, where $m\delta_1 < \delta_2 < \delta$.

Since the functions $t_j(z)$, $j = 1, \cdots, N$, do not have zeros in D, there exists a constant k_1 such that $|t_j(\zeta)| \geqq k_1 > 0$, $\zeta \in D$, $j = 1, \cdots, N$. Now it is easily verified that there exists a positive constant k_2 such that

$$\operatorname{Re} \varepsilon^{-m} \int_{\zeta_1}^{\zeta_2} t_j(\zeta_3)\, d\zeta_3 \geqq k_2 |\varepsilon|^{-m} \{\sigma(\zeta_2) - \sigma(\zeta_1)\} \tag{2.8}$$

if ζ_1 and ζ_2 lie on L', ζ_1 lies between z and ζ_2 on L' and $|\arg \varepsilon| \leqq \delta_1$.

From (2.6) we obtain for these points ζ_1 and ζ_2 the following inequality:

$$|v_{jj}(\zeta_1, \zeta_2, \varepsilon)| \leqq \exp - k_2 |\varepsilon|^{-m} \{\sigma(\zeta_2) - \sigma(\zeta_1)\}. \tag{2.9}$$

From (2.7), (2.8) and (2.9) it can be proved by induction that there exist constants K_{hj}, $h = j - 1, j - 2, \cdots, 1$, such that

$$|v_{hj}(\zeta_1, \zeta_2, \varepsilon)| \leqq K_{hj} \sum_{l=1}^{j-h} |\varepsilon|^{-ml} \{\sigma(\zeta_2) - \sigma(\zeta_1)\}^l \exp -k_2 |\varepsilon|^{-m} \{\sigma(\zeta_2) - \sigma(\zeta_1)\} \tag{2.10}$$

if $h < j$, ζ_1 and ζ_2 lie on L' and ζ_1 lies between z and ζ_2 on L', $|\arg \varepsilon| \leqq \delta_1$.

Now we choose L' as the path of integration in the left-hand side of (2.5) and we take $\sigma(\zeta)$ as a new variable of integration. Using (2.4), (2.9) and (2.10), we obtain the assertion of the lemma.

The condition $\mathrm{H}(\delta)$ used above may be replaced by the following *condition* $\mathrm{H}^*(\delta)$.

DEFINITION 2. The domain D and the functions $t_1(z), \cdots, t_N(z)$ in $\bar{D}$ satisfy *condition* $\mathrm{H}^*(\delta)$ if:

(i) the functions $t_1(z), \cdots, t_N(z)$ have no zeros in D;

(ii) they are the roots of an equation

$$t^N + \sum_{j=0}^{N-1} \beta_j(z) t^j = 0 \tag{2.11}$$

with holomorphic coefficients $\beta_j(z)$, $j = 0, \cdots, N-1$, in D;

(iii) any pair of roots $t(z)$ and $t^*(z)$ of (2.11) satisfies the inequality

$$|\arg t(z)/t^*(z)| \leqq \pi - 2\delta$$

in all points z in D;

(iv) there exists a point z^* on C such that the sectors

$$|\arg(\zeta - z^*) + \arg - t_j(z^*)| \leqq \tfrac{1}{2}\pi - \delta, \qquad j = 1, \cdots, N,$$

contain the restriction to D of a neighborhood of $\zeta = z^*$;

(v) if $z \in C$, $z \neq z^*$, there exists a line segment with endpoint z in D such that on this segment

$$|\arg t_j(z) + \arg(\zeta - z)| < \tfrac{1}{2}\pi - \delta, \qquad j = 1, \cdots, N.$$

We prove the following property of this condition: *Condition* H*(δ) *implies condition* H(δ).

Proof. Let $\alpha_1, \cdots, \alpha_q$ be the branchpoints of $t_1(z), \cdots, t_N(z)$ in D. Consider a point z in D and suppose that the principal value of $|\arg t_j(z)/t_h(z)|$ is maximal for $j = j_0$, $h = h_0$.

Then we define

$$\varphi(z) = -\tfrac{1}{2}\arg t_{j_0}(z)t_{h_0}(z). \tag{2.12}$$

On account of (iii) we have

$$|\varphi(z) + \arg t(z)| \leqq \tfrac{1}{2}\pi - \delta \tag{2.13}$$

for any root $t(z)$ of (2.11).

Let the branchcut L_k, $k = 1, \cdots, q$, be the curve with starting point $z = \alpha_k$ such that its tangent vector at any point z coincides with the half-line $\arg(\zeta - z) = \varphi(z) + \pi$. Let $t(\zeta)$ be one of the roots of (2.11) and consider $\int_{\alpha_k}^{z} t(\zeta)\, d\zeta$ along a curve on the Riemann surface of $t(\zeta)$ whose projection is L_k. The real part of this integral decreases if z runs along that curve from α_k onwards because $|t(\zeta)|$ has a positive lower bound in D (cf. (i)) and because of (2.13). Hence L_k is not closed and L_k is of finite length in D. So L_k connects α_k with a point on C.

If z_0 is an arbitrary point in D, $z_0 \notin C$, we construct a curve L in D with starting point z_0 such that its tangent vector at any point z coincides with the half-line $\arg(\zeta - z) = \varphi(z)$.

Next suppose $z_0 \in C$, $z_0 \notin L_k$, $z_0 \neq z^*$. Then we choose the curve L such that the first part of it coincides with the line segment mentioned in condition (v) and does not intersect a branchcut. The second part of L is chosen in the same way as above. If z_0 is the endpoint of a curve L_k, we choose L in the same way as if $z \notin C$. Now (2.12), (2.13) and (v) imply that (2.1) holds if ζ runs from z to z^* along L.

From the constructions of L and L_k it follows that L does not intersect a branchcut. If z_0 lies on L_k, the curve L partly coincides with L_k. Such curves L are counted twice. From conditions (iv) and (v) it follows that the endpoints of the curves L coincide with z^*. Hence condition H(δ) is satisfied.

3. An existence theorem for a nonlinear differential equation. In this section we prove an extension of the fundamental lemma of Sibuya in [4] (cf. Theorem 26.1 in [6]).

THEOREM 3. *Let m and N be positive integers, w_0, ε_0 and δ positive constants and D a simply-connected domain in the complex z-plane with a piecewise smooth boundary C. Let w be an N-vector and $f(z, w, \varepsilon)$ be an N-vector holomorphic in z, w and ε for $z \in D$, $\|w\| \leqq w_0$ and $\varepsilon \in S(\varepsilon_0, \delta)$. Suppose*

$$f(z, w, \varepsilon) \sim \sum_{j=0}^{\infty} a_j(z, w)\varepsilon^j$$

as $\varepsilon \to 0$ uniformly for $\varepsilon \in S(\varepsilon_0, \delta)$, $z \in D$, $\|w\| \leqq w_0$.

Let $f_j(z, w, \varepsilon)$, $j = 1, \cdots, N$, denote the components of $f(z, w, \varepsilon)$ and let $A(z)$ be the Jacobian matrix

$$\left\{ \lim_{\substack{\varepsilon \to 0 \\ |\arg \varepsilon| \leqq \delta}} \frac{\partial f_j}{\partial w_k}(z, w, \varepsilon)|_{w=0} \right\}.$$

Denote the characteristic polynomial of $A(z)$ by $p(\lambda, z)$. Suppose $p(0, z) \neq 0$ for $z \in D$ and

$$p(\lambda, z) = \prod_{h=1}^{r} p_h(\lambda, z), \tag{3.1}$$

where $p_1(\lambda, z), \cdots, p_r(\lambda, z)$ are r mutually coprime polynomials in λ, the coefficients of which are holomorphic in D. Let $p_h(\lambda, z)$ be of degree N_h in λ and let the zeros of $p_h(\lambda, z)$ be denoted by $\lambda_{hj}(z)$, $j = 1, \cdots, N_h$. Assume that the functions $\lambda_{hj}(z)$, $j = 1, \cdots, N_h$, satisfy condition $\mathrm{H}(\delta_0)$ *or* $\mathrm{H}^*(\delta_0)$ (cf. Definitions 1 and 2 in § 2) *if $h = 1, \cdots, r$. Here δ_0 is a constant with $\delta_0 > m\delta$.*

Suppose that the differential equation

$$\varepsilon^m \frac{dw}{dz} = f(z, w, \varepsilon) \tag{3.2}$$

is formally satisfied in D by a formal series

$$\sum_{\nu=1}^{\infty} w_\nu(z)\varepsilon^\nu, \tag{3.3}$$

where $w_\nu(z)$, $\nu = 1, 2, \cdots$, are holomorphic functions in D and the norms of the partial sums in (3.3) *have an upper bound less than w_0 for $z \in D$, $\varepsilon \in S(\varepsilon_0, \delta)$.*

Then there exists a solution $w = \varphi(z, \varepsilon)$ of (3.2) *for $z \in D$, $\varepsilon \in S(\varepsilon_0^*, \delta)$, $0 < \varepsilon_0^* \leqq \varepsilon_0$, such that $\varphi(z, \varepsilon)$ possesses the asymptotic expansion* (3.3) *as $\varepsilon \to 0$ uniformly for $z \in D$, $|\arg \varepsilon| \leqq \delta$.*

Proof. We assume that condition $\mathrm{H}(\delta_0)$ holds, since condition $\mathrm{H}^*(\delta_0)$ implies $\mathrm{H}(\delta_0)$. There exists a function $\varphi^*(z, \varepsilon)$ which is holomorphic in z and ε for $z \in D$, $\varepsilon \in S(\varepsilon_0, \delta)$, which has the asymptotic expansion (3.3) as $\varepsilon \to 0$ uniformly for $z \in D$, $|\arg \varepsilon| \leqq \delta$, whereas $\|\varphi^*(z, \varepsilon)\| \leqq w_1 < w_0$ for $z \in D$, $\varepsilon \in S(\varepsilon_1, \delta)$. Here ε_1 is a constant such that $0 < \varepsilon_1 \leqq \varepsilon_0$. Substituting

$$u = w - \varphi^*(z, \varepsilon) \tag{3.4}$$

in (3.2) we obtain the following equation in u (cf. [6, § 27.1]):

$$\varepsilon^m \frac{du}{dz} = b(z, \varepsilon) + B(z, \varepsilon)u + g(z, u, \varepsilon), \tag{3.5}$$

where $B(z, \varepsilon)$ is an $N \times N$ matrix and $b(z, \varepsilon)$ and $g(z, u, \varepsilon)$ are N-vectors holomorphic in z, ε and u for $z \in D$, $\varepsilon \in S(\varepsilon_1, \delta)$, $\|u\| \leqq u_0$, u_0 being some positive constant. Further,

$$B(z, \varepsilon) = A(z)(1 + o(1)), \quad b(z, \varepsilon) = O(\varepsilon^\mu) \quad \text{as } \varepsilon \to 0 \tag{3.6}$$

uniformly for $z \in D$, $|\arg \varepsilon| \leqq \delta$. Here μ is any positive integer.

Finally,

$$g(z, 0, \varepsilon) = 0, \quad \frac{\partial}{\partial u_j} g(z, u, \varepsilon)|_{u=0} = 0, \qquad j = 1, \cdots, N, \tag{3.7}$$

for $z \in D$, $\varepsilon \in S(\varepsilon_1, \delta)$.

We only have to show that (3.5) admits of a solution $u(z, \varepsilon)$ with the property

$$u(z, \varepsilon) = O(\varepsilon^\mu) \quad \text{as } \varepsilon \to 0 \tag{3.8}$$

uniformly for $z \in D$, $|\arg \varepsilon| \leqq \delta$, for any positive integer μ.

By Theorem 2 there exists a nonsingular holomorphic matrix $R(z)$ in D such that

$$R^{-1}(z)A(z)R(z) = \tilde{A}(z) = \operatorname{diag}\{A_1(z), \cdots, A_r(z)\},$$

where $A_h(z)$ is a holomorphic $N_h \times N_h$ matrix with characteristic polynomial $p_h(\lambda, z)$, $h = 1, \cdots, r$. The substitution $u(z) = R(z)\tilde{u}(z)$ transforms (3.5) into an equation of the same type for $\tilde{u}$ with the same properties for the functions $\tilde{b}$, $\tilde{B}$ and $\tilde{g}$ corresponding to b, B and g. In (3.6) we can replace B, A and b by $\tilde{B}$, $\tilde{A}$ and $\tilde{b}$. It follows that it is sufficient to consider (3.5) in the case that $A(z) = \operatorname{diag}\{A_1(z), \cdots, A_r(z)\}$ in (3.6).

Applying Theorem 1 we infer that there exist algebroid $N_h \times N_h$ matrices $S_h(z)$ and $T_h(z)$ such that $S_h(z)$ is nonsingular in D and

$$S_h^{-1}(z)A_h(z)S_h(z) = T_h(z), \tag{3.9}$$

whereas $T_h(z)$ is upper triangular with $\lambda_{hj}(z), j = 1, \cdots, N_h$, on the diagonal. Here $h = 1, \cdots, r$. The branchpoints of $S_h(z)$ and $T_h(z)$ are those of the functions $\lambda_{hj}(z)$, $j = 1, \cdots, N_h$. Denote these points by α_{hk}, $k = 1, \cdots, q_h$. Let L_{hk}, $k = 1, \cdots, q_h$ be the branchcuts of condition $\mathrm{H}(\delta_0)$ for $\lambda_{hj}(z), j = 1, \cdots, N_h$. Let D_h be the domain obtained by deleting from D these cuts and let D^* be the intersection of $D_1, \cdots, D_r$. In D_h we choose the branches of the matrices $S_h(z)$ and $T_h(z)$ which correspond to the choice for the branches of the functions $\lambda_{hj}(z)$, $j = 1, \cdots, N_h$, in condition $\mathrm{H}(\delta_0)$. Finally we define

$$S(z) = \operatorname{diag}\{S_1(z), \cdots, S_r(z)\}, \quad T(z) = \operatorname{diag}\{T_1(z), \cdots, T_r(z)\}.$$

Substituting

$$u(z) = S(z)v(z) \tag{3.10}$$

in (3.5) we obtain the differential equation

$$\varepsilon^m \frac{dv}{dz} = c(z, \varepsilon) + T(z, \varepsilon)v + h(z, v, \varepsilon), \tag{3.11}$$

where

$$T(z, \varepsilon) = S^{-1}(z)B(z, \varepsilon)S(z) - \varepsilon^m S^{-1}(z)\frac{dS}{dz}. \tag{3.12}$$

The matrix $T(z, \varepsilon)$ and the N-vectors $c(z, \varepsilon)$ and $h(z, v, \varepsilon)$ are algebroid in z with branchpoints in $z = \alpha_{hj}$, $h = 1, \cdots, r$, $j = 1, \cdots, q_h$, and holomorphic in ε and v for $\varepsilon \in S(\varepsilon_1, \delta)$ and $\|v\| < v_0$, where v_0 is a positive constant. Further we have

$$c(z, \varepsilon) = O(\varepsilon^\mu) \quad \text{as } \varepsilon \to 0 \tag{3.13}$$

uniformly for $z \in D$, $|\arg \varepsilon| \leqq \delta$ and any positive integer μ, and

$$h(z, 0, \varepsilon) = 0, \quad \frac{\partial}{\partial v_j} h(z, v, \varepsilon)|_{v=0} = 0, \qquad j = 1, \cdots, N, \tag{3.14}$$

if $z \in D$, $\varepsilon \in S(\varepsilon_1, \delta)$. From (3.12) and (3.6) we deduce that there exist a positive constant $\eta \leqq 1$ and a function $\varepsilon(\gamma)$ defined for $\gamma > 0$ such that $0 < \varepsilon(\gamma) \leqq \varepsilon_1$ and

$$\|T(z, \varepsilon) - T(z)\| \leqq \gamma\left[1 + \sum_{h=1}^{r} \sum_{k=1}^{q_h} |\varepsilon^{-m}(z - \alpha_{hk})|^{-1+\eta}\right] \tag{3.15}$$

if $z \in D$ and $\varepsilon \in S(\varepsilon(\gamma), \delta)$. Here $T(z) = T(z, 0)$.

Let $V_h(z, \zeta, \varepsilon)$ be the fundamental matrix solution of the matrix differential equation

$$\varepsilon^m \frac{dV}{dz} = T_h(z)V, \tag{3.16}$$

with the property that $V_h(\zeta, \zeta, \varepsilon) = I_{N_h}$. Here z and ζ are points in D_h.

If d is any N-vector, we denote by $d_{(1)}$ the vector consisting of the first N_1 components of d, $d_{(2)}$ the vector consisting of the next N_2 components of d, etc. So $d = \{d_{(1)}, \cdots, d_{(r)}\}$.

If v is any N-vector, we define

$$\rho(\zeta, v, \varepsilon) = c(\zeta, \varepsilon) + \{T(\zeta, \varepsilon) - T(\zeta)\}v + h(\zeta, v, \varepsilon). \tag{3.17}$$

Now consider the following system of integral equations for the N-vector $v(z, \varepsilon)$:

$$v_{(h)}(z, \varepsilon) = \varepsilon^{-m} \int_{z_h^*}^{z} V_h(z, \zeta, \varepsilon)\rho_{(h)}(\zeta, v(\zeta, \varepsilon), \varepsilon)\, d\zeta, \qquad h = 1, \cdots, r. \tag{3.18}$$

Here z_h^* denotes the point z^* mentioned in condition $\mathrm{H}(\delta_0)$ for the functions $\lambda_{hj}(z)$, $j = 1, \cdots, N_h$. Furthermore, the path of integration from z_h^* to z lies entirely in D_h.

Any solution $v(z, \varepsilon)$ of (3.18) which is continuous in D^* satisfies (3.11). In order to construct such a solution we define

$$\begin{aligned} &v(z, \varepsilon, 0) = 0, \\ &v_{(h)}(z, \varepsilon, \nu) = \varepsilon^{-m} \int_{z_h^*}^{z} V_h(z, \zeta, \varepsilon)\rho_{(h)}(\zeta, v(\zeta, \varepsilon, \nu - 1), \varepsilon)\, d\zeta, \\ &\qquad\qquad h = 1, \cdots, r, \quad \nu = 1, 2, \cdots. \end{aligned} \tag{3.19}$$

Then $v_{(h)}(z, \varepsilon, \nu)$ is continuous in D_h with continuous boundary values on the cuts L_{hk}, $k = 1, \cdots, q_h$.

The differential equation (3.16) and its fundamental matrix solution $V_h(z, \zeta, \varepsilon)$ have been considered in § 2 (cf. (2.2)). Let K_h be the constant K as in Lemma 2 corresponding to the choice $T(z) = T_h(z)$ in § 2. Let $\tilde{K}$ be the maximum of $K_1, \cdots, K_r$. From (3.17), (3.14) and (3.15) we deduce that there exist constants ε_3 and v_1 such that $\varepsilon_3 \leqq \varepsilon_2$, $0 < v_1 < v_0$ and

$$\|\rho(z, v, \varepsilon) - \rho(z, \tilde{v}, \varepsilon)\| \leqq \tfrac{1}{2}\tilde{K}^{-1}\|v - \tilde{v}\|\left[1 + \sum_{h=1}^{r}\sum_{k=1}^{q_h} |\varepsilon^{-m}(z - \alpha_{hk})|^{\eta - 1}\right], \tag{3.20}$$

if $z \in D^*$, $\|v\| < v_1$, $\|\tilde{v}\| < v_1$, $\varepsilon \in S(\varepsilon_3, \delta)$. We can infer from (3.17), (3.13) and (3.14) that there exist positive constants C_μ such that

$$\|\rho(z, 0, \varepsilon)\| \leqq C_\mu|\varepsilon^\mu|, \qquad \mu = 1, 2, \cdots, \tag{3.21}$$

if $z \in D^*$, $\varepsilon \in S(\varepsilon_3, \delta)$.

Applying Lemma 2 and using (3.19), (3.21) and (3.20) we deduce that there exist constants ε_μ^*, $\mu = 1, 2, \cdots$, such that

$$\|v(z, \varepsilon, \nu) - v(z, \varepsilon, \nu - 1)\| \leqq 2^{1-\nu}\tilde{K}C_\mu|\varepsilon^\mu|, \qquad \nu = 1, 2, \cdots, \tag{3.22}$$

if $z \in D^*$, $\varepsilon \in S(\varepsilon_\mu^*, \delta)$, where $0 < \varepsilon_\mu^* \leqq \varepsilon_3$ and $2\tilde{K}C_\mu(\varepsilon_\mu^*)^\mu < v_1$. Hence $\lim_{\nu \to \infty} v(z, \varepsilon, \nu) = v(z, \varepsilon)$ exists if $z \in D^*$, $\varepsilon \in S(\varepsilon_1^*, \delta)$. Moreover, since $v_{(h)}(z, \varepsilon, \nu)$ is continuous in D_h with continuous boundary values, the function $v_{(h)}(z, \varepsilon)$ has the same properties. Furthermore, $v(z, \varepsilon)$ is a holomorphic solution of (3.18) and (3.11) for $z \in D^*$, $\varepsilon \in S(\varepsilon_1^*, \delta)$, whereas $v(z, \varepsilon) = O(\varepsilon^\mu)$ as $\varepsilon \to 0$ uniformly for $z \in D^*$, $|\arg \varepsilon| \leqq \delta$ for any positive integer μ.

Using the substitution (3.10) we obtain a holomorphic solution $u(z, \varepsilon)$ of (3.5) with the desired property (3.8) for $z \in D^*$, $\varepsilon \in S(\varepsilon_1^*, \delta)$. The block diagonal form of $S(z)$ and (3.10) imply that $u_{(h)}(z)$ is continuous in D_h with continuous boundary values on the cuts L_{hk}, $k = 1, \cdots, q_h$. However, the possibility that these boundary values are different at both sides of a cut is not excluded a priori.

Since the right-hand side of (3.5) depends holomorphically on z and u, (3.5) with initial condition $u(z_0) = \tilde{u}$ at any point z_0 in D and for any $\tilde{u}$ with $\|\tilde{u}\| < u_0$ has a unique holomorphic solution in a neighborhood of z_0 in D. Choosing $z_0 = \alpha_{hk}$ we see that $u(z, \varepsilon)$ is one-valued on L_{hk} between α_{hk} and the first point of intersection of L_{hk} and another cut $L_{jl}, j \neq h$. Denoting the latter point by β_1 we see that the boundary values on both sides of L_{hk} of $u_{(h)}(z, \varepsilon)$ in the point β_1 coincide. Choosing $z_0 = \beta_1$ we deduce that $u(z, \varepsilon)$ is one-valued on L_{hk} between $z = \beta_1$ and the next point of intersection of L_{hk} with another cut. Hence $u_{(h)}(z, \varepsilon)$ is continuous in D. So $u(z, \varepsilon)$ is a continuous solution of (3.5) for $z \in D$, $\varepsilon \in S(\varepsilon_1^*, \delta)$. Moreover, it is clear that the estimate (3.8) holds for any positive integer μ uniformly for $z \in D$ and $\varepsilon \in S(\varepsilon_1^*, \delta)$. This completes the proof.

4. The global reduction theorem. Using the existence theorem of § 3 we may prove a global analogue of the theorem of Sibuya (cf. [4] or Theorem 26.2 in [6]). Since the proof very much resembles the proof of Wasow of Sibuya's result, it is not presented here.

THEOREM 4. *Let m and n be positive integers, let δ, δ_0 and ε_0 be positive constants with $\delta_0 > m\delta$ and let D be a simply-connected domain in the complex z-plane with boundary C which is piecewise smooth.*

Suppose $A(z, \varepsilon)$ is an $n \times n$ matrix which is holomorphic in z and ε for $z \in D$ and $\varepsilon \in S(\varepsilon_0, \delta)$. Suppose

$$A(z, \varepsilon) \sim \sum_{\nu=0}^{\infty} A_\nu(z)\varepsilon^\nu \quad as\ \varepsilon \to 0 \tag{4.1}$$

uniformly for $z \in D$ and $|\arg z| \leqq \delta$.

Assume that the characteristic polynomial of $A_0(z)$ is the product of two mutually coprime polynomials in λ, viz. $c_1(\lambda, z)$ and $c_2(\lambda, z)$. Let $p(\lambda, z)$ be the resultant of $c_1(\lambda + \mu, z)$ and $c_2(\mu, z)$, the latter polynomials being considered as polynomials in μ.

Suppose that (3.1) *holds, where $p_1(\lambda, z), \cdots, p_r(\lambda, z)$ are mutually coprime polynomials in λ with coefficients which are holomorphic in D. Let the polynomial $p_h(\lambda, z)$ be of degree N_h in λ and let the zeros of the polynomial $p_h(\lambda, z)$ be denoted by $\lambda_{hj}(z)$, $j = 1, \cdots, N_h$. Assume that D and the functions $\lambda_{hj}(z)$, $j = 1, \cdots, N_h$, satisfy condition* $\mathrm{H}(\delta_0)$ *or* $\mathrm{H}^*(\delta_0)$ (cf. Definitions 1 and 2 of § 2) *if $h = 1, \cdots, r$. Assume the same for the functions $-\lambda_{hj}(z)$.*

Then there exist a constant ε_1, $0 < \varepsilon_1 \leqq \varepsilon_0$, and an $n \times n$ matrix $P(z, \varepsilon)$ holomorphic in both variables for $z \in D$, $\varepsilon \in S(\varepsilon_1, \delta)$ with the following properties:

The differential equation

$$\varepsilon^m \frac{du}{dz} = A(z, \varepsilon)u \tag{4.2}$$

is transformed by the substitution

$$u = P(z, \varepsilon)v \tag{4.3}$$

into the differential equation

$$\varepsilon^m \frac{dv}{dz} = B(z, \varepsilon)v, \tag{4.4}$$

where $B(z, \varepsilon) = \operatorname{diag}\{B^{11}(z, \varepsilon), B^{22}(z, \varepsilon)\}$. Here $B^{11}(z, \varepsilon)$ and $B^{22}(z, \varepsilon)$ are square matrices which are holomorphic in z and ε for $z \in D$, $\varepsilon \in S(\varepsilon_1, \delta)$. Moreover,

$$P(z, \varepsilon) \sim \sum_{\nu=0}^{\infty} P_\nu(z)\varepsilon^\nu, \quad B^{jj}(z, \varepsilon) \sim \sum_{\nu=0}^{\infty} B_\nu^{jj}(z)\varepsilon^\nu, \quad j = 1, 2, \quad as\ \varepsilon \to 0 \tag{4.5}$$

uniformly for $z \in D$, $|\arg \varepsilon| \leqq \delta$. The matrix $P_0(z)$ is nonsingular in D, and the characteristic polynomial of $B^{jj}(z)$ is $c_j(\lambda, z)$, $j = 1, 2$.

Equation (4.4) *is equivalent to the differential equations*

$$\varepsilon^m \frac{dw}{dz} = B^{11}(z, \varepsilon)w \quad and \quad \varepsilon^m \frac{d\tilde{w}}{dz} = B^{22}(z, \varepsilon)\tilde{w}, \tag{4.6}$$

where $w = \{v_1, \cdots, v_{n_1}\}^T$ and $\tilde{w} = \{v_{n_1+1}, \cdots, v_n\}^T$.

Remark 1. The conditions concerning $A_0(z)$ may be replaced by the following equivalent conditions.

Let the eigenvalues of $A_0(z)$ be $\lambda_1(z), \cdots, \lambda_n(z)$. There exists an integer n_1, $1 \leqq n_1 < n$, such that $\lambda_j(z) \neq \lambda_h(z)$ if $z \in D$ and $1 \leqq j \leqq n_1 < h \leqq n$. The set of differences $\lambda_j(z) - \lambda_h(z)$, $1 \leqq j \leqq n_1 < h \leqq n$, can be split up into r subsets, each subset satisfying condition $\mathrm{H}(\delta_0)$ or $\mathrm{H}^*(\delta_0)$. In the same way the set of differences $\lambda_h(z) - \lambda_j(z)$, $1 \leqq j \leqq n_1 < h \leqq n$, can be split up into subsets satisfying condition $\mathrm{H}(\delta_0)$ or $\mathrm{H}^*(\delta_0)$.

Remark 2. If $T(z)$ is a holomorphic nonsingular $n \times n$ matrix in D with the property that $T^{-1}(z)A_0(z)T(z) = \operatorname{diag}\{A^{11}(z), A^{22}(z)\}$, where $A^{jj}(z)$ is a square holomorphic matrix in D with characteristic polynomial $c_j(\lambda, z)$, $j = 1, 2$, then the matrix $P(z, \varepsilon)$ may be chosen such that

$$\lim_{\varepsilon \to 0} P(z, \varepsilon) = T(z) \tag{4.7}$$

uniformly for $z \in D$, $\varepsilon \in S(\varepsilon_1, \delta)$.

5. Applications. In the case that all eigenvalues of the matrix $A_0(z)$ in Theorem 4 are different we obtain the following result.

THEOREM 5. *Let m and n be positive integers, let δ, δ_0 and ε_0 be positive constants, $\delta_0 > m\delta$, and let D be a simply-connected domain in the complex z-plane with boundary C which is piecewise smooth. Let $A(z, \varepsilon)$ be an $n \times n$ matrix which is holomorphic in z and ε for $z \in D$, $\varepsilon \in S(\varepsilon_0, \delta)$ and which satisfies* (4.1) *uniformly for $z \in D$ and $|\arg \varepsilon| \leqq \delta$. Let the eigenvalues of the matrix $A_0(z)$ occurring in* (4.1) *be denoted by $\lambda_1(z), \cdots, \lambda_n(z)$. Assume that $\lambda_j(z) \neq \lambda_h(z)$ if $j \neq h$ and $z \in D$. Suppose that for any choice of j and h with $j \neq h$, the function $\lambda_j(z) - \lambda_h(z)$ satisfies condition* $\mathrm{H}(\delta_0)$ *or* $\mathrm{H}^*(\delta_0)$ (cf. Definitions 1 and 2 in § 2).

Then the matrix differential equation $\varepsilon^m dU/dz = A(z, \varepsilon)U$ possesses a fundamental matrix solution $U(z, \varepsilon)$ of the form

$$U(z, \varepsilon) = \hat{U}(z, \varepsilon) \exp Q(z, \varepsilon) \tag{5.1}$$

for $z \in D$, $\varepsilon \in S(\varepsilon_1, \delta)$ with the following properties:

The $n \times n$ matrices $\hat{U}(z, \varepsilon)$ and $Q(z, \varepsilon)$ are holomorphic in z and ε for $z \in D$, $\varepsilon \in S(\varepsilon_1, \delta)$.

$$Q(z, \varepsilon) = \sum_{\nu=1}^{m} Q_\nu(z)\varepsilon^{-\nu}, \tag{5.2}$$

where $Q_\nu(z)$, $\nu = 1, \cdots, m$, are holomorphic $n \times n$ matrices in D and

$$Q_m(z) = \operatorname{diag}\left\{\int_{z_0}^{z} \lambda_1(\zeta)\, d\zeta, \cdots, \int_{z_0}^{z} \lambda_n(\zeta)\, d\zeta\right\},$$

z_0 being some point in D.

$$\hat{U}(z, \varepsilon) \sim \sum_{\nu=0}^{\infty} \hat{U}_\nu(z)\varepsilon^\nu \quad \text{as } \varepsilon \to 0 \tag{5.3}$$

uniformly for $z \in D$, $|\arg \varepsilon| \leqq \delta$, *where* $\hat{U}_\nu(z)$, $\nu = 0, 1, 2, \cdots$, *are holomorphic* $n \times n$ *matrices and* $\hat{U}_0(z)$ *is nonsingular for* $z \in D$. *The number* ε_1 *is a constant with* $0 < \varepsilon_1 \leqq \varepsilon_0$.

The proof of Theorem 5 is quite similar to the proof of Theorem 26.3 in [6]. In [2] Theorem 5 is proved with slightly different conditions. Instead of condition $H(\delta_0)$, Langer assumed that there exist points z_{jh} on C, $j \neq h$, such that any point $z \in D$ can be connected with z_{jh} by a curve in D with the property that

$$\operatorname{Re} \varepsilon^{-m} \int_z^\zeta \{\lambda_j(\zeta_1) - \lambda_h(\zeta_1)\}\, d\zeta \tag{5.4}$$

increases if ζ runs from z to z_{jh} along this curve for any fixed ε with $|\arg \varepsilon| \leqq \delta$. In this paper we have assumed

$$|\arg \{\lambda_j(\zeta) - \lambda_h(\zeta)\}\, d\zeta| \leqq \tfrac{1}{2}\pi - \delta_0 \tag{5.5}$$

if ζ runs from z to z_{jh} along that curve (cf. (2.1)). Condition (5.5) implies the condition concerning (5.4).

Another special case of Theorem 4 is obtained by considering an nth order differential equation. We state in the following theorem an analytic factorization of such differential equations which is related to the asymptotic factorization theorems of Langer [3] and Erdélyi [1].

THEOREM 6. *Let* m, n, δ_0, δ, ε_0, D *and* C *be as in Theorem* 4. *Let* $a_j(z, \varepsilon)$, $j = 1, \cdots, n$, *be holomorphic functions of* z *and* ε *for* $z \in D$, $\varepsilon \in S(\varepsilon_0, \delta)$ *which have the asymptotic expansions*

$$a_j(z, \varepsilon) \sim \sum_{\nu=0}^{\infty} a_{j\nu}(z)\varepsilon^\nu \quad \text{as } \varepsilon \to 0, \qquad j = 1, \cdots, n, \tag{5.6}$$

uniformly for $z \in D$, $|\arg \varepsilon| \leqq \delta$.

Suppose

$$\lambda^n + \sum_{j=1}^{n} a_{j0}(z)\lambda^{n-j} = c_1(\lambda, z)c_2(\lambda, z), \tag{5.7}$$

where $c_j(\lambda, z)$, $j = 1, 2$, *is a polynomial in* λ *of degree* n_j *with holomorphic coefficients in* D. *Let* $c_1(\lambda, z)$ *and* $c_2(\lambda, z)$ *satisfy the same conditions as in Theorem* 4.

Then the differential equation

$$\varepsilon^{nm}\frac{d^n y}{dz^n} + \sum_{j=1}^{n} \varepsilon^{(n-j)m} a_j(z, \varepsilon)\frac{d^{n-j} y}{dz^{n-j}} = 0 \tag{5.8}$$

is equivalent to the differential equations

$$\varepsilon^{n_1 m}\frac{d^{n_1} y}{dz^{n_1}} + \sum_{j=1}^{n_1} \varepsilon^{(n_1-j)m} \beta_j(z, \varepsilon)\frac{d^{n_1-j} y}{dz^{n_1-j}} = 0 \tag{5.9}$$

and

$$\varepsilon^{n_2 m}\frac{d^{n_2}y}{dz^{n_2}} + \sum_{j=1}^{n_2} \varepsilon^{(n_2-j)m}\gamma_j(z,\varepsilon)\frac{d^{n_2-j}y}{dz^{n_2-j}} = 0 \tag{5.10}$$

in the sense that all linear combinations of solutions of (5.9) *and* (5.10) *are solutions of* (5.8) *and conversely, for* $z \in D$, $\varepsilon \in S(\varepsilon_1, \delta)$, ε_1 *being some constant with* $0 < \varepsilon_1 \leqq \varepsilon$. *The functions* $\beta_j(z, \varepsilon)$, $j = 1, \cdots, n_1$, *and* $\gamma_h(z, \varepsilon)$, $h = 1, \cdots, n_2$, *are holomorphic in* z *and* ε *for* $z \in D$, $\varepsilon \in S(\varepsilon_1, \delta)$, *and*

$$\begin{aligned} \beta_j(z,\varepsilon) &\sim \sum_{\nu=0}^{\infty} \beta_{j\nu}(z)\varepsilon^\nu, \qquad j = 1, \cdots, n_1, \quad \varepsilon \to 0, \\ \gamma_j(z,\varepsilon) &\sim \sum_{\nu=0}^{\infty} \gamma_{j\nu}(z)\varepsilon^\nu, \qquad j = 1, \cdots, n_2, \quad \varepsilon \to 0, \end{aligned} \tag{5.11}$$

uniformly for $z \in D$, $|\arg \varepsilon| \leqq \delta$. *Finally,*

$$\begin{aligned} \lambda^{n_1} + \sum_{j=1}^{n_1} \beta_{j0}(z)\lambda^{n_1-j} &= c_1(\lambda, z), \\ \lambda^{n_2} + \sum_{j=1}^{n_2} \gamma_{j0}(z)\lambda^{n_2-j} &= c_2(\lambda, z). \end{aligned} \tag{5.12}$$

Proof. The differential equation (5.8) is equivalent to the system

$$\begin{aligned} \varepsilon^m \frac{du_j}{dz} &= u_{j+1}, \qquad j = 1, \cdots, n-1, \\ \varepsilon^m \frac{du_n}{dz} &= -\sum_{j=1}^{n} a_{n-j+1}(z,\varepsilon)u_j. \end{aligned} \tag{5.13}$$

This system is a special case of (4.2), the matrix $A(z, \varepsilon)$ being the companion matrix of the polynomial

$$\lambda^n + \sum_{j=1}^{n} a_j(z,\varepsilon)\lambda^{n-j}.$$

This matrix satisfies (4.1), where now $A_0(z)$ is the companion matrix of the polynomial in the left-hand side of (5.7). All assumptions of Theorem 4 are fulfilled. So there exist matrices $P(z, \varepsilon)$ and $B(z, \varepsilon)$ with the properties mentioned in Theorem 4. In particular, (5.13) is transformed into (4.4) by means of the substitution (4.3), where $B(z, \varepsilon)$ has block diagonal form.

Using Remark 2 of § 4 we show that we may choose the matrix $P(z, \varepsilon)$ such that $\lim_{\varepsilon \to 0} P(z, \varepsilon)$ has a special property. Let $C^{11}(z)$ and $C^{22}(z)$ be the companion matrices of the polynomials $c_1(\lambda, z)$ and $c_2(\lambda, z)$. Then

$$R(z)A_0(z) = [\mathrm{diag}\,\{C^{11}(z), C^{22}(z)\}]R(z),$$

where $R(z)$ is the $n \times n$ matrix defined by

$$
(5.14)\quad R(z) = \begin{bmatrix}
\gamma_{n_2} & \gamma_{n_2-1} & \cdots & \cdots & \gamma_1 & 1 & 0 & \cdots & 0 \\
0 & \gamma_{n_2} & \ddots & & & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & & & \ddots & \ddots & 0 \\
0 & \cdots & 0 & \gamma_{n_2} & \gamma_{n_2-1} & \cdots & \cdots & \gamma_1 & 1 \\
\beta_{n_1} & \beta_{n_1-1} & \cdots & \beta_1 & 1 & 0 & \cdots & \cdots & 0 \\
0 & \ddots & \ddots & & \ddots & \ddots & \ddots & & \vdots \\
\vdots & \ddots & \ddots & \ddots & & \ddots & \ddots & \ddots & \vdots \\
\vdots & & \ddots & \ddots & \ddots & & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & 0 & \beta_{n_1} & \beta_{n_1-1} & \cdots & \beta_1 & 1
\end{bmatrix}.
$$

Here the quantities γ_j and β_h are the functions $\gamma_{j0}(z)$ and $\beta_{h0}(z)$, $j = 1, \cdots, n_2$, $h = 1, \cdots, n_1$, defined by means of (5.12). The matrix $R(z)$ is nonsingular since its determinant is the resultant of the coprime polynomials $c_1(\lambda, z)$ and $c_2(\lambda, z)$. So the matrix $T(z) = R^{-1}(z)$ is nonsingular and holomorphic in D and satisfies

$$
(5.15)\qquad T^{-1}(z)A_0(z)T(z) = \operatorname{diag}\{C^{11}(z), C^{22}(z)\}.
$$

In view of Remark 2 following Theorem 4 we may choose $P(z, \varepsilon)$ such that (4.7) holds.

Let $W(z, \varepsilon)$ and $\tilde{W}(z, \varepsilon)$ be fundamental matrix solutions in D of the matrix differential equations

$$
\varepsilon^m \frac{dW}{dz} = B^{11}(z, \varepsilon)W, \qquad \varepsilon^m \frac{d\tilde{W}}{dz} = B^{22}(z, \varepsilon)\tilde{W},
$$

which correspond to equations (4.6). Then the columns of the matrix $U(z, \varepsilon)$ defined by

$$
(5.16)\qquad U(z, \varepsilon) = P(z, \varepsilon) \operatorname{diag}\{W(z, \varepsilon), \tilde{W}(z, \varepsilon)\}
$$

constitute a fundamental system of solutions of (5.13). Putting $u_1 = y$ in (5.13) we see that (5.8) possesses a fundamental system of solutions $y_1, \cdots, y_n$ in D with Wronskian matrix $U(z, \varepsilon)$.

Let $U^*(z, \varepsilon)$ and $P^*(z, \varepsilon)$ be the matrices consisting of the first n_1 columns and $n_1 + 1$ rows of the matrices $U(z, \varepsilon)$ and $P(z, \varepsilon)$. Then

$$U^*(z, \varepsilon) = P^*(z, \varepsilon)W(z, \varepsilon) \tag{5.17}$$

on account of (5.16). Let $U_1(z, \varepsilon)$ denote the matrix which arises by enlarging $U^*(z, \varepsilon)$ with an $(n_1 + 1)$st column consisting of $y, \varepsilon^m y', \cdots, \varepsilon^{n_1 m} y^{(n_1)}$. Then the equation

$$\det U_1(z, \varepsilon) = 0 \tag{5.18}$$

is an n_1th order linear homogeneous differential equation for the function y, which possesses the fundamental system of solutions $y_1, \cdots, y_{n_1}$ in D.

It follows from the construction of $U_1(z, \varepsilon)$ and (5.17) that (5.18) is equivalent to the equation

$$\sum_{j=0}^{n_1} \alpha_j(z, \varepsilon)\varepsilon^{mj} y^{(j)} = 0, \tag{5.19}$$

where $(-1)^{n_1 + j}\alpha_j(z, \varepsilon)$ is the determinant of the matrix which is obtained by omitting the $(j + 1)$st row in $P^*(z, \varepsilon)$. So the functions $\alpha_j(z, \varepsilon)$ have asymptotic expansions of the form (5.11) as $\varepsilon \to 0$ uniformly for $z \in D$, $|\arg \varepsilon| \leqq \delta$.

Let the matrices $P^{jh}(z, \varepsilon)$, $j, h = 1, 2$, denote the matrices into which $P(z, \varepsilon)$ is divided by cuts after the $(n_1 + 1)$st row and column. Similarly the matrices $T^{jh}(z)$ and $R^{jh}(z)$ are defined. Then the function $\alpha_j(z, \varepsilon)(-1)^{n_1 + j}$ is the minor of the element in the $(j + 1)$st row and last column of $P^{11}(z, \varepsilon)$. Equation (4.7) implies that $P^{11}(z, \varepsilon) \to T^{11}(z)$ as $\varepsilon \to 0$. Hence,

$$\lim_{\varepsilon \to 0} a_j(z, \varepsilon) = (-1)^{n_1 + j} \quad \text{(minor of the element at the place } (j + 1, n_1 + 1) \text{ of } T^{11}(z)). \tag{5.20}$$

Since $T = R^{-1}$ we have

$$R^{21}T^{11} + R^{22}T^{21} = 0, \quad R^{11}T^{11} + R^{12}T^{21} = I_{n_1 + 1}.$$

As $\det R^{22} = 1$ (cf. (5.14)) we can deduce from these equations that

$$\{R^{11} - R^{12}(R^{22})^{-1}R^{21}\}T^{11} = I_{n_1 + 1}. \tag{5.21}$$

Therefore T^{11} is nonsingular and the right-hand side of (5.20) is equal to the element at the place $(n_1 + 1, j + 1)$ of $\{T^{11}(z)\}^{-1}$ times $\det T^{11}(z)$. From (5.14) and (5.21) we deduce that this element is equal to $\beta_{n_1 - j, 0}(z)$ if $j = 0, \cdots, n_1 - 1$ and equal to 1 if $j = n_1$. Hence, (5.19) may be written in the form (5.9) where the functions $\beta_j(z, \varepsilon)$ satisfy the assertions of the theorem. Moreover, since (5.18) has the fundamental system of solutions $y_1, \cdots, y_{n_1}$, so has (5.9).

Let $I(n_1, n_2)$ denote the matrix $\begin{bmatrix} 0 & I_{n_1} \\ I_{n_2} & 0 \end{bmatrix}$. By replacing $P(z, \varepsilon)$ by $P(z, \varepsilon)I(n_1, n_2)$ in (4.3) we obtain (4.4) with B^{11} and B^{22} interchanged. The reasoning given above remains valid with some modifications. Among others we have to replace $T(z)$ and $R(z)$ by $T(z)I(n_1, n_2)$ and $I(n_1, n_2)R(z)$ and we have to interchange $W(z)$ and $\tilde{W}(z)$, $\beta_1, \cdots, \beta_{n_1}$ and $\gamma_1, \cdots, \gamma_{n_2}$ and $y_1, \cdots, y_{n_1}$ and $y_{n_1+1}, \cdots, y_n$. Then we obtain (5.10) with coefficients satisfying the assertions of the theorem and with $y_{n_1+1}, \cdots, y_n$ as the fundamental system of solutions.

Remark. The coefficients in the asymptotic expansions (5.11) may be calculated by means of a method of Erdélyi's [1].

Acknowledgment. The author wishes to express his gratitude to Professor A. Erdélyi who has introduced him to this subject and who has given valuable advice.

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SIAM J. MATH. ANAL.
Vol. 2, No. 2, May 1971

A NOTE ON THE CONSTRUCTION OF GENERALIZED WALSH FUNCTIONS*

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Abstract. A straightforward method for constructing the generalized Walsh functions is presented. This method defines the function in parts with each new part being a scaled linear translate of the previously defined portion.

In a recent article Byrnes and Swick [2] presented a method for constructing the Walsh functions. The purposes of this note are twofold. The first is to indicate that the set of functions resulting from that method while being identical with the set defined by Walsh [9] are different from the set used by most authors [1], [3]–[8].[1] The second purpose is to give a straightforward method for constructing the generalized Walsh functions which include the Walsh functions as defined by the majority of authors. These generalized functions are used in Fine integral transforms [5], [8].

The definition of the generalized Walsh function $(0, +\infty)$ is given here for reference:

$$\psi_y(t) = (-1)^S,$$
$$S = \sum_{i+j=1} y_i t_j;$$

$y = \sum_{i=-k}^{+\infty} y_i 2^{-i}$ is the dyadic expansion for y and similarly for t. It is assumed that $y_{-k} = 1$ and $y_i = 0$ for all $i < -k$. The construction procedure is outlined. Clearly $\psi_y(t) = 1$ for $t \in (0, 2^{-k-1})$. However for $x \in (2^{-k-1}, 2^{-k})$, $\psi_y(x) = \psi_y(t + 2^{-k-1}) = -1$ where $t \in (0, 2^{-k-1})$. Continuing in this manner, the construction of $\psi_y(x)$ for $x \in (2^{p-1}, 2^p)$ can be given in terms of the previously defined part of $\psi_y(t)$:

$$\psi_y(x) = \psi_y(t + 2^{p-1}) = (-1)^{y_p}\psi_y(t), \quad t \in (0, 2^{p-1}),$$
$$p = -k, -k+1, \cdots, 0, +1, \cdots.$$

To complete the construction take $\psi_y(0) = +1$ and $\psi_y(t) = 0$ at any jump.

A generalized Walsh function for any y with dyadic rational value is periodic. If $y = C + D/2^q$ where $(D, 2) = 1$, then the period of $\psi_y(t)$ is 2^q.

Finally it is clear how to construct the generalized functions over the integers module α for $\alpha \geqq 2$ (see [3], [8]).

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* Received by the editors May 5, 1970, and in revised form September 24, 1970.

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[1] There are a number of counterexamples to this. Considering $\psi_{230}(t)$ as constructed in [2], [9] one finds that $\psi_{230}(9/1024) = \psi_{230}(55/1024)$, whereas other definitions give $\psi_{230}(9/1024) = -1$ and $\psi_{230}(55/1024) = +1$.

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SIAM J. MATH. ANAL.
Vol. 2, No. 2, May 1971

A BOUNDARY VALUE PROBLEM WHOSE SOLUTION INVOLVES EQUATIONS NONLINEAR IN AN EIGENVALUE PARAMETER*

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Abstract. The nonseparable partial differential equation for one-dimensional damped wave motion with a variable damping coefficient $\varphi_{xx} - \varphi_{tt} = p(x)\varphi_t$ is solved in series form for the usual mixed boundary value problem on a semi-infinite strip. The solution is assumed to have the form: $\varphi = \sum_n c_n \psi_n(x)e^{\lambda_n t}$ from which the differential equation $\psi_n'' - (\lambda_n^2 + \lambda_n p(x))\psi_n = 0$ with $\psi_n(\pm 1) = 0$ is found. Since the preceding equation is not of the classical Sturm–Liouville type, the existence of eigenvalues, eigenfunctions and properties of the relevant eigenfunction expansion are proved using asymptotic expansions and complex variable techniques. The coefficients in the solution series are found by converting the equation for ψ_n to a first order system of differential equations in which the eigenvalue parameter λ_n occurs linearly. The eigenvectors of this system are found to be orthogonal with respect to an indefinite bilinear form so that the coefficients c_n in the solution may be found by exact formula even though the eigenfunctions ψ_n are not orthogonal.

1. Introduction. This paper is concerned with the solution, in series, of equations of the form

$$\varphi_{xx} - \varphi_{tt} = p(x)\varphi_t, \tag{1}$$

where p is a continuously differentiable, nonnegative function of x on the interval $-1 \leqq x \leqq 1$ (denoted by I). The solution of (1) is sought in the semi-infinite strip defined by the inequalities

$$-1 \leqq x \leqq 1, \qquad 0 \leqq t < \infty \tag{2}$$

with the mixed boundary conditions prescribed as follows:

$$\varphi(-1, t) = 0, \tag{3a}$$

$$\varphi(1, t) = 0, \tag{3b}$$

$$\varphi(x, 0) = f(x), \tag{3c}$$

$$\varphi_t(x, 0) = g(x), \tag{3d}$$

where f and g are arbitrary except for certain mild differentiability conditions which will emerge later. In order to simplify matters somewhat, we shall also assume that $f(\pm 1) = p(\pm 1) = 0$.

Equation (1) is not separable unless p reduces to a constant on the interval I; nevertheless, the solution of the problem will be obtained in the series form

$$\varphi(x, t) = \sum_{\lambda_n} \alpha_n \psi_n(x) e^{\lambda_n t}, \tag{4}$$

where the α_n and the λ_n are denumerable sets of complex constants.

Equation (1) is the one-dimensional damped wave equation with a variable damping term p. Problems of this form arise frequently in various disciplines and, hence, no specific physical interpretation will be discussed as the reader may supply his own.

* Received by the editors September 8, 1969, and in final revised form July 24, 1970.

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Substituting (4) into (1), the equation which ψ_n must satisfy is found to be

$$\psi_n'' - (\lambda_n^2 + \lambda_n p)\psi_n = 0 \tag{5}$$

with the boundary conditions

$$\psi_n(\pm 1) = 0 \tag{6}$$

in order that the series (4) satisfy the boundary conditions (3a) and (3b). Equations (5) and (6) thus constitute an "eigenvalue" problem which is not of the classical Sturm–Liouville variety since the eigenvalue parameter λ_n occurs nonlinearly in (5). The principal object of this paper is to establish the existence of a denumerable set of complex eigenvalues and corresponding eigenfunctions for the above problem and to establish the relevant expansion properties necessary for the solution of (1) subject to the desired boundary value problem. A formula for calculating the coefficients α_n of (4) in terms of the functions g and f will also be obtained.

In the sequel, vectors are printed in boldface. Vectors are indexed using subscripts with their components indicated by superscripts, thus

$$\mathbf{z}_n = (z_n^{(1)}, z_n^{(2)}).$$

A dot $\cdot$ between two vectors indicates the real Euclidean inner product even if the vectors are complex-valued; therefore, if $\mathbf{x} = (x^{(1)}, x^{(2)})$ and $\mathbf{y} = (y^{(1)}, y^{(2)})$, then $\mathbf{x} \cdot \mathbf{y} = x^{(1)}y^{(1)} + x^{(2)}y^{(2)}$. The symbol $|\cdot|$ denotes the absolute value of a complex number and the symbol $\|\cdot\|$ denotes the corresponding matrix or vector norm, i.e.,

$$\|\mathbf{z}_n\|^2 = |z_n^{(1)}|^2 + |z_n^{(2)}|^2; \qquad \left\|\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right\|^2 = |a_{11}|^2 + |a_{12}|^2 + |a_{21}|^2 + |a_{22}|^2.$$

The inequality

$$\left\|\int_a^b A\mathbf{x}\,dt\right\|^2 \leqq |b - a| \int_a^b \|A\|^2 \|\mathbf{x}\|^2\,dt,$$

which follows directly from the two easily proved inequalities

$$\left\|\int_a^b \mathbf{x}\,dt\right\|^2 \leqq |b - a| \int_a^b \|x\|^2\,dt, \qquad \|A\mathbf{x}\| \leqq \|A\|\,\|\mathbf{x}\|,$$

has been used several times.

The two remaining boundary conditions, (3c) and (3d), are expressed in series form by the equations

$$f(x) = \sum_{\lambda_n} \alpha_n \psi_n(x), \tag{7a}$$

$$g(x) = \sum_{\lambda_n} \lambda_n \alpha_n \psi_n(x). \tag{7b}$$

Thus, two functions f and g must be simultaneously expanded in terms of the eigenfunctions ψ_n if the desired series solution is to be obtained. The eigenfunctions

ψ_n (provided that they exist) are not orthogonal[1] so that the determination of the coefficients α_n is not straightforward as in the usual case.

Instead of treating (5) directly, we shall convert it to an equivalent system of two first order differential equations which are linear in the eigenvalue parameter λ_n. To this end, let

$$z_n^{(1)} = \psi_n, \qquad z_n^{(2)} = \psi_n'/\lambda_n.$$

Then it is easy to verify that

$$z_n^{(1)\prime} = \lambda_n z_n^{(2)}. \tag{8}$$

Using (5), we derive

$$z_n^{(2)\prime} = (\lambda_n + p) z_n^{(1)}. \tag{9}$$

In vector form, (8) and (9) become

$$\mathbf{z}_n' = \lambda_n A \mathbf{z}_n + B \mathbf{z}_n, \tag{10}$$

where

$$\mathbf{z}_n = \begin{pmatrix} z_n^{(1)} \\ z_n^{(2)} \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ p & 0 \end{pmatrix}.$$

The remaining boundary conditions, represented by (7a) and (7b), may be written in vector form as

$$\begin{pmatrix} g \\ f' \end{pmatrix} = \sum_{\lambda_n} c_n \mathbf{z}_n, \qquad c_n = \lambda_n \alpha_n. \tag{11}$$

Thus, our problem is to expand the vector (g, f') in terms of the eigenvector solutions[2] of (10) so as to determine the coefficients c_n from which the α_n may then be calculated. We shall show in the sequel that such an expansion is possible under certain conditions and that a simple formula for the coefficients c_n can be found.

2. Existence and properties of the eigenvalues and eigenfunctions.

THEOREM 1. *Let* $\mathbf{z}$ *be the solution of the equation*

$$\mathbf{z}' = \lambda A \mathbf{z} + B \mathbf{z} \tag{12}$$

---

[1] Let λ_n be the eigenvalue corresponding to the eigenfunction ψ_n and let λ_m correspond to ψ_m. The usual cross product multiplication process yields, after utilization of (6),

$$(\lambda_m - \lambda_n) \int_{-1}^{+1} (\lambda_m + \lambda_n + p) \psi_n \psi_m \, dx = 0.$$

So, if $\lambda_n \neq \lambda_m$, then

$$\int_{-1}^{+1} (\lambda_m + \lambda_n + p) \psi_n \psi_m \, dx = 0.$$

Thus, the weight function $\lambda_n + \lambda_m + p$ depends on λ_i so that orthogonality in the classical sense fails.

[2] Systems of differential equations having the form (10) have been treated by various authors using methods which are essentially distinct from those which will be used here. See, for instance, [1, pp. 51–128].

subject to the initial condition

$$\mathbf{z}(-1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \lambda \neq 0, \qquad x \in I.$$

Then

$$\mathbf{z} = \begin{pmatrix} \sinh Q(x, \lambda) \\ \dfrac{Q'}{\lambda} \cosh Q(x, \lambda) \end{pmatrix} + O\left(\frac{e^{|\lambda|(x+1)}}{|\lambda|} \right),$$

where

$$Q(x, \lambda) = \lambda(x + 1) + \frac{1}{2} \int_{-1}^{x} p(s)\, ds,$$

and the last term indicates a vector whose norm is of that order.

Proof. Let

$$\zeta^{(1)} = \sinh Q, \qquad \zeta^{(2)} = \zeta^{(1)\prime}/\lambda.$$

Then, by differentiating, it is easy to verify that the vector

$$\boldsymbol{\zeta} = \begin{pmatrix} \zeta^{(1)} \\ \zeta^{(2)} \end{pmatrix}$$

satisfies the differential equation

$$\boldsymbol{\zeta}' = \lambda A \boldsymbol{\zeta} + B \boldsymbol{\zeta} + C \boldsymbol{\zeta} \tag{13}$$

with the initial condition

$$\boldsymbol{\zeta}(-1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

where

$$C = \begin{pmatrix} 0 & 0 \\ \dfrac{p^2}{4\lambda} & \dfrac{p'}{2Q'} \end{pmatrix}.$$

If we set

$$\boldsymbol{\Psi} = \mathbf{z} - \boldsymbol{\zeta},$$

then $\boldsymbol{\Psi}$ satisfies the differential equation

$$\boldsymbol{\Psi}' = \lambda A \boldsymbol{\Psi} + B \boldsymbol{\Psi} - C \boldsymbol{\zeta} \tag{14}$$

with the initial conditions

$$\boldsymbol{\Psi}(-1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

To estimate the order of the vector $\boldsymbol{\Psi}$, we shall construct an integral representation of $\boldsymbol{\Psi}$ utilizing the two vector equations

$$-\mathbf{Y}_i' = \lambda A \mathbf{Y}_i + B \mathbf{Y}_i + D \mathbf{Y}_i, \qquad i = 1, 2, \tag{15}$$

with the initial conditions

$$\mathbf{Y}_1(-1) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \qquad \mathbf{Y}_2(-1) = \begin{pmatrix} -1 \\ 0 \end{pmatrix},$$

where

$$D = \begin{pmatrix} 0 & 0 \\ \dfrac{p^2}{4\lambda} & -\dfrac{p'}{2Q'} \end{pmatrix}.$$

It is easily verified that

$$\mathbf{Y}_1 = \begin{pmatrix} \zeta^{(1)} \\ -\zeta^{(2)} \end{pmatrix} = \begin{pmatrix} \sinh Q \\ -\dfrac{Q'}{\lambda}\cosh Q \end{pmatrix}$$

and

$$\mathbf{Y}_2 = \begin{pmatrix} \cosh Q \\ \dfrac{Q'}{\lambda}\sinh Q \end{pmatrix}.$$

To form the desired representation of $\mathbf{\Psi}$, multiply (14) and (15) by the matrix A ($A^2 = I$) and take the inner product of (14) with $\mathbf{Y}_i$ and (15) with $\mathbf{\Psi}$ thereby deriving the equations

$$\mathbf{Y}_i \cdot A\mathbf{\Psi}' = \lambda \mathbf{Y}_i \cdot \mathbf{\Psi} + \mathbf{Y}_i \cdot AB\mathbf{\Psi} - \mathbf{Y}_i \cdot AC\zeta,$$

$$-\mathbf{\Psi} \cdot A\mathbf{Y}_i' = \lambda \mathbf{\Psi} \cdot \mathbf{Y}_i + \mathbf{\Psi} \cdot AB\mathbf{Y}_i + \mathbf{\Psi} \cdot AD\mathbf{Y}_i.$$

Subtracting the two previous equations and utilizing the boundary conditions, we derive

$$(\mathbf{Y}_i \cdot A\mathbf{\Psi})' = -\mathbf{\Psi} \cdot AD\mathbf{Y}_i - \mathbf{Y}_i \cdot AC\zeta,$$

$$(\mathbf{Y}_i \cdot A\mathbf{\Psi})(x) = -\int_{-1}^{x} \mathbf{\Psi} \cdot AD\mathbf{Y}_i \, dt - \int_{-1}^{x} \mathbf{Y}_i \cdot AC\zeta \, dt.$$

This last equation can be solved for the vector $\mathbf{\Psi}$. The result is

$$\mathbf{\Psi} = \frac{\lambda}{Q'(x)} \begin{pmatrix} Y_2^{(1)}(x) & -Y_1^{(1)}(x) \\ -Y_2^{(2)}(x) & Y_1^{(2)}(x) \end{pmatrix} \left\{ \begin{pmatrix} \displaystyle\int_{-1}^{x} \mathbf{\Psi} \cdot AD\mathbf{Y}_1 \, dt \\ \displaystyle\int_{-1}^{x} \mathbf{\Psi} \cdot AD\mathbf{Y}_2 \, dt \end{pmatrix} + \begin{pmatrix} \displaystyle\int_{-1}^{x} \mathbf{Y}_1 \cdot AC\zeta \, dt \\ \displaystyle\int_{-1}^{x} \mathbf{Y}_2 \cdot AC\zeta \, dt \end{pmatrix} \right\},$$

which can be rewritten, after some algebraic manipulations, in the form

$$\mathbf{\Psi} = \frac{1}{\Delta}\int_{-1}^{x} M_1 \begin{pmatrix} k^{(1)}\Psi^{(1)}(t) \\ k^{(2)}\Psi^{(1)}(t) \end{pmatrix} dt - \frac{1}{\Delta}\int_{-1}^{x} M_2 \begin{pmatrix} k^{(1)}\zeta^{(1)}(t) \\ k^{(2)}\zeta^{(2)}(t) \end{pmatrix} dt, \tag{16}$$

where

$$R(x,t,\lambda) = Q(x) - Q(t) = \lambda(x-t) + \frac{1}{2}\int_t^x p(s)\,ds, \quad k^{(1)} = \frac{p^2}{4\lambda}, \quad k^{(2)} = -\frac{p'}{2Q'},$$

$$\Delta(t) = -\frac{Q'(t)}{\lambda},$$

$$M_1 = \begin{pmatrix} \sinh R & -\Delta(t)\cosh R \\ \Delta(x)\cosh R & -\Delta(x)\Delta(t)\sinh R \end{pmatrix},$$

$$M_2 = \begin{pmatrix} \Delta(t)\cosh R & -\Delta(t)\cosh R \\ -\Delta(t)\Delta(x)\sinh R & \Delta(t)\Delta(x)\sinh R \end{pmatrix}.$$

If we put

$$\boldsymbol{\Psi}(x,\lambda) = e^{|\lambda|(x+1)}\begin{pmatrix} r^{(1)} \\ r^{(2)} \end{pmatrix},$$

then (16) becomes

$$\begin{pmatrix} r^{(1)}(x) \\ r^{(2)}(x) \end{pmatrix} = \frac{1}{\Delta}\int_{-1}^x e^{|\lambda|(t-x)} M_1 \begin{pmatrix} k^{(1)} r^{(1)}(t) \\ k^{(2)} r^{(1)}(t) \end{pmatrix} dt - \frac{e^{-|\lambda|(x+1)}}{\Delta}\int_{-1}^x M_2 \begin{pmatrix} k^{(1)}\zeta^{(1)} \\ k^{(2)}\zeta^{(2)} \end{pmatrix} dt,$$

from which we see that

$$\begin{aligned}
\left\|\begin{pmatrix} r^{(1)} \\ r^{(2)} \end{pmatrix}\right\| &= \left\| \frac{1}{\Delta}\int_{-1}^x e^{|\lambda|(t-x)} r^{(1)} M_1 \begin{pmatrix} k^{(1)} \\ k^{(2)} \end{pmatrix} dt - \frac{e^{-|\lambda|(x+1)}}{\Delta}\int_{-1}^x M_2 \begin{pmatrix} k^{(1)}\zeta^{(1)} \\ k^{(2)}\zeta^{(2)} \end{pmatrix} dt \right\| \\
&\leqq \left\| \frac{1}{\Delta}\int_{-1}^x e^{|\lambda|(t-x)} r^{(1)} M_1 \begin{pmatrix} k^{(1)} \\ k^{(2)} \end{pmatrix} dt \right\| + \left\| \frac{e^{-|\lambda|(x+1)}}{\Delta}\int_{-1}^x M_2 \begin{pmatrix} k^{(1)}\zeta^{(1)} \\ k^{(2)}\zeta^{(2)} \end{pmatrix} dt \right\| \\
&\leqq \sqrt{\frac{2}{|\Delta|^2}\int_{-1}^x e^{2|\lambda|(t-x)} |r^{(1)}|^2 \|M_1\|^2 \left\|\begin{pmatrix} k^{(1)} \\ k^{(2)} \end{pmatrix}\right\|^2 dt} \\
&\quad + \sqrt{\frac{e^{-2|\lambda|(x-1)}}{|\Delta|^2}\int_{-1}^x \|M_2\|^2 \left\|\begin{pmatrix} k^{(1)}\zeta^{(1)} \\ k^{(2)}\zeta^{(2)} \end{pmatrix}\right\|^2 dt}
\end{aligned}$$

or

$$\begin{aligned}
\rho(x) \leqq &\sqrt{\frac{2}{|\Delta|^2}\int_{-1}^x e^{2|\lambda|(t-x)} \rho(t)^2 \|M_1\|^2 \left\|\begin{pmatrix} k^{(1)} \\ k^{(2)} \end{pmatrix}\right\|^2 dt} \\
&+ \sqrt{\frac{e^{2|\lambda|(x+1)}2}{|\Delta|^2}\int_{-1}^x \|M_2\|^2 \left\|\begin{pmatrix} k^{(1)}\zeta^{(1)} \\ k^{(2)}\zeta^{(2)} \end{pmatrix}\right\|^2 dt}
\end{aligned}$$

where we have used the inequality

$$|r^{(1)}| \leqq \sqrt{|r^{(1)}|^2 + |r^{(2)}|^2}.$$

As $|\lambda| \to \infty$, we see that

$$|\Delta|^2 = O(1), \qquad \left\| \begin{pmatrix} k^{(1)} \\ k^{(2)} \end{pmatrix} \right\|^2 = O\left(\frac{1}{|\lambda|^2}\right),$$

$$\|M_i\|^2 = O(e^{2|\lambda|(x-t)}), \qquad i = 1, 2,$$

$$\|\zeta_i\|^2 = O(e^{2|\lambda|(x+1)}), \qquad i = 1, 2,$$

so that, if

$$\rho^* = \max_{x \in I} \rho(x),$$

then

$$\rho^* \leqq \rho^* \sqrt{2 \int_{-1}^{+1} \frac{e^{2|\lambda|(t-x)}}{\Delta(x)^2} \|M_1\|^2 \left\| \begin{pmatrix} k^{(1)} \\ k^{(2)} \end{pmatrix} \right\|^2 dt} + \sqrt{2 \int_{-1}^{+1} \frac{e^{-2|\lambda|(x+1)}}{\Delta(x)^2} \left\| \begin{pmatrix} k^{(1)}\zeta^{(1)} \\ k^{(2)}\zeta^{(2)} \end{pmatrix} \right\|^2 dt}.$$

Both integrals of the previous equation are readily seen to be $O(1/|\lambda|^2)$. Thus,

$$\rho^* \leqq \rho^* O\left(\frac{1}{|\lambda|}\right) + O\left(\frac{1}{|\lambda|}\right),$$

which implies that

$$\rho^* \leqq \frac{O(1/|\lambda|)}{1 - O(1/|\lambda|)} = O\left(\frac{1}{|\lambda|}\right).$$

Hence

$$\mathbf{\Psi} = e^{|\lambda|(x+1)} O\left(\frac{1}{|\lambda|}\right),$$

from which the theorem follows at once.

LEMMA 1.1. *Let ψ be the solution of the equation*

$$\psi'' - (\lambda^2 + \lambda p)\psi = 0 \tag{17}$$

satisfying the initial conditions

$$\psi(-1) = 0, \qquad \psi'(-1) = 1. \tag{18}$$

Then

$$\psi(x, \lambda) = \frac{1}{\lambda} \sinh Q(x, \lambda) \left\{ 1 + O\left(\frac{1}{|\lambda|}\right) \right\}, \qquad \lambda \neq 0. \tag{19}$$

Proof. The first component, $z^{(1)}$, of (12) satisfies the same equation as ψ and initial conditions $z^{(1)}(-1) = 0$, $z^{(1)\prime}(-1) = \lambda$. Hence $\psi = z^{(1)}/\lambda$, and the lemma follows directly from Theorem 1.

LEMMA 1.2. *There exists a denumerable set of eigenvalues λ_n and corresponding eigenfunctions ψ_n satisfying* (5) *and* (6).

Proof. Any solution of (17) satisfying (18) will be an eigenfunction of the problem if, in addition, ψ satisfies the equation

$$\psi(1, \lambda_n) = 0$$

for some $\lambda = \lambda_n$. The corresponding λ_n are the eigenvalues. Since both (17) and its initial conditions (18) are entire functions of the complex variable λ, it follows that ψ itself will be entire in λ for each x in I (see [2, p. 72]). In particular, $\psi(1, \lambda)$ is an entire function of λ. If $\psi(1, \lambda)$ has no zeros in the complex λ-plane, then

$$\psi(1, \lambda) = e^{h(\lambda)}, \tag{20}$$

where

$$h(\lambda) = h_0 + h_1\lambda + h_2\lambda^2 + \cdots$$

is an entire function of λ. From Lemma 1.1 it is obvious that

$$\psi(1, \lambda) = \frac{1}{\lambda}\sinh(2\lambda + c)\left\{1 + O\left(\frac{1}{|\lambda|}\right)\right\} = O\left(\frac{e^{2\lambda}}{|\lambda|}\right), \tag{21}$$

and hence, (17) can only hold if

$$h(\lambda) = h_0 + h_1\lambda.$$

However this last equation is impossible since (20) would then assert that $|\psi(1, \lambda)|$ is exponentially decreasing along one ray of the real axis, whereas (21) asserts that $|\psi(1, \lambda)|$ is exponentially increasing along both rays of the real λ-axis. Thus, $\psi(1, \lambda)$ has at least one zero, λ_0. If $\psi(1, \lambda)$ has only one zero, then it is of the form

$$\psi(1, \lambda) = (\lambda - \lambda_0)e^{h(\lambda)},$$

where h is again some entire function of λ. The previous argument may now be repeated to demonstrate the existence of another zero, λ_1, of $\psi(1, \lambda)$. Continuing in this manner, we may show the existence of an infinite set of eigenvalues. The eigenvalues are denumerable; otherwise they would have an accumulation point in the finite part of the plane and hence $\psi(1, \lambda)$ would vanish identically.

LEMMA 1.3. *If $p(x) \neq$ const. $\neq 0$ on I, then the eigenvalues λ_n of the previous lemma satisfy the inequalities*:

(a) $\operatorname{Re}(\lambda_n) < 0$,

(b) $-\frac{1}{2}\max_I p \leqq \operatorname{Re}(\lambda_n) < 0$ *if* $\operatorname{Im}(\lambda_n) \neq 0$,

(c) $-\max_I p < \operatorname{Re}(\lambda_n) < 0$ *if* $\operatorname{Im}(\lambda_n) = 0$.

Proof. Multiplying (5) by the complex conjugate of ψ_n, i.e., $\bar{\psi}_n$, and multiplying the complex conjugate of (5) by ψ_n and subtracting the two resulting equations, we obtain

$$\bar{\psi}_n\psi_n'' - \psi_n\bar{\psi}_n'' = \{\lambda_n^2 - \bar{\lambda}_n^2 + (\lambda_n - \bar{\lambda}_n)p\}\psi_n\bar{\psi}_n,$$

which may be rewritten as

$$\frac{d}{dx}[\bar{\psi}_n\psi_n' - \psi_n\bar{\psi}_n'] = (\lambda_n - \bar{\lambda}_n)\{\lambda_n + \bar{\lambda}_n + p\}|\psi_n|^2.$$

Integrating the above equation over the interval I and utilizing the boundary conditions for ψ_n and $\bar{\psi}_n$ yields, after division by $2i$,

$$\operatorname{Im}(\lambda_n)\int_{-1}^{+1}\{2\operatorname{Re}(\lambda_n)+p\}|\psi_n|^2\,dx=0, \tag{22}$$

which can be used to obtain several results:

(i) If $\operatorname{Im}(\lambda_n)\neq 0$, then $\operatorname{Re}(\lambda_n)<0$. This statement follows since $2\operatorname{Re}(\lambda_n)+p$ should be positive on I and hence the integral of (22) would be positive instead of zero.

(ii) If $\operatorname{Im}(\lambda_n)\neq 0$, then $\operatorname{Re}(\lambda_n)>-\frac{1}{2}\max_I p$. This implication follows in a similar manner since its denial would imply that the integral of (22) is strictly negative instead of zero.

If we repeat the above multiplication process and add instead of subtracting the resulting equations, we derive the equation

$$\bar{\psi}_n\psi_n''+\psi_n\bar{\psi}_n''=\{\lambda_n^2+\bar{\psi}_n^2+(\lambda_n+\bar{\lambda}_n)p\}|\psi_n|^2,$$

which may be rewritten more compactly in the form

$$\operatorname{Re}(\psi_n\psi_n'')=\{\operatorname{Re}(\lambda_n^2)+\operatorname{Re}(\lambda_n)p\}|\psi_n|^2.$$

Integration and application of the boundary conditions yield

$$\begin{aligned}\int_{-1}^{+1}\{\operatorname{Re}(\lambda_n^2)+\operatorname{Re}(\lambda_n)-(\operatorname{Im}(\lambda_n))^2\}|\psi_n|^2\,dx&\\ =-\int_{-1}^{+1}|\psi_n'|^2\,dx<0.&\end{aligned} \tag{23}$$

From (23) the two following implications may be obtained in a manner similar to that previously used:

(iii) If $\operatorname{Im}(\lambda_n)=0$, then $\operatorname{Re}(\lambda_n)<0$.

(iv) If $\operatorname{Im}(\lambda_n)=0$, then $\operatorname{Re}(\lambda_n)>-\max_I p$.

From (i) and (iii) follows (a) of the lemma, and thence from (a) and (ii) follows (b). Similarly, (c) follows from (a) and (iv).

Thus, all the λ_n lie in the strip

$$-\max_I\ p<\operatorname{Re}(\lambda_n)<0$$

of the complex λ-plane. We are thereby assured that the terms of the proposed series solution (4) do not grow exponentially with time and will, in fact, represent damped oscillatory behavior.

LEMMA 1.4. *For $|n|$ large enough, there is exactly one eigenvalue of* (5) *in the rectangle R_n of the complex λ-plane defined by the inequalities*

$$-\max_I p<\operatorname{Re}(\lambda)<0,\qquad \tfrac{1}{2}(n-\tfrac{1}{2})\pi\leqq\operatorname{Im}(\lambda)\leqq\tfrac{1}{2}(n+\tfrac{1}{2})\pi.$$

Proof. Equation (21) may be written in the form

$$\lambda\psi(1,\lambda)=\sinh(2\lambda+c)+O(e^{2|\lambda|}/|\lambda|).$$

As previously noted, $\psi(1,\lambda)$ is an entire function of λ, as is the function $\sinh(2\lambda+c)$.

Let $E(\lambda) = \lambda\psi(1, \lambda)$, $E_1(\lambda) = \sinh(2\lambda + c)$, and let $E_2(\lambda)$ denote the function represented by $O(e^{2|\lambda|}/|\lambda|)$. E and E_1 are entire functions of λ, and from (24) we see that

$$E(\lambda) - E_1(\lambda) = E_2(\lambda);$$

hence, E_2 is also an entire function of λ. The entire function

$$E(\lambda) = E_1(\lambda) + E_2(\lambda)$$

is such that, for $|\lambda|$ large enough,

$$|E_2(\lambda)| < |E_1(\lambda)|;$$

hence, by Rouche's theorem, E has the same number of zeros in the domain enclosed as E_1. These zeros are the eigenvalues of (5) (excluding $\lambda = 0$). (This provides yet another proof of the existence of the eigenvalues.) The zeros of E_1 are at

$$\lambda = \frac{n\pi i}{2} - \frac{c}{2};$$

thus, for $|\lambda|$ large enough, or equivalently $|n|$ large enough, it is easy to see that there must be exactly one eigenvalue such that $\frac{1}{2}(n - \frac{1}{2})\pi \leqq \operatorname{Im}(\lambda_n) \leqq \frac{1}{2}(n + \frac{1}{2})\pi$, and this fact along with the previous lemma establishes the result.

Although we have shown that the set of eigenvalues of (5) is denumerable, we have not discussed any specific way of enumerating them. It would seem natural to do this such that

$$\lambda_n \sim \frac{n\pi i}{2} - \frac{c}{2}$$

for $n = \pm 1, \pm 2, \cdots$.

To ultimately accomplish the above result, we consider the circles in the complex λ-plane defined by

$$C_N : |\lambda| < \frac{N\pi}{2},$$

where N is an integer. For large enough N, the number of eigenvalues inside C_N is not greater than $2(N - 1)$. This last statement follows since, for N large enough, the number of eigenvalues in C_N is the same as the number of zeros of the function $\sinh(2\lambda + c)$; this number is found if we determine how large $|n|$ can be and satisfy the inequality

$$|\lambda| = \sqrt{\left(\frac{n\pi}{2}\right)^2 + \left(\frac{c}{2}\right)^2} < \frac{N\pi}{2}.$$

Manipulation of this inequality yields

$$n < \frac{1}{\pi}\sqrt{(N\pi)^2 - c^2} < \frac{1}{\pi}\sqrt{(N\pi)^2} = N,$$

which shows that the number of eigenvalues is not greater than $2(N - 1)$ since

n may assume the values $n = \pm 1, \pm 2, \cdots, \pm N - 1$. We now find N large enough so that:

(a) no more than $2(N-1)$ eigenvalues lie in C_N;

(b) the $2(N-1)$ rectangles R_n, $n = \pm 1, \pm 2, \cdots, \pm N - 1$, lie in C_n;

(c) every eigenvalue exterior to C_N lies alone in a rectangle R_n for $|n| \geqq N$.

The desired enumeration is then accomplished by numbering the eigenvalues inside C_n in any manner using the subscripts $1, 2, \cdots, N-1$ and voiding any extra subscripts and denoting the eigenvalues exterior to C_N lying in R_n by λ_n, $n = \pm N, \pm(N+1), \cdots$.

LEMMA 1.5. *The eigenvalues of* (5) *are such that*

$$\lambda_n = \frac{c}{2} + \frac{n\pi i}{2} + O\left(\frac{1}{|n|}\right).$$

Proof. This lemma follows directly from Lemma 1.4 and the system of enumeration discussed above if we substitute into (21) the expression

$$\lambda_n = \frac{c}{2} + \frac{n\pi i}{2} + \delta_n$$

and thereby verify that

$$|\delta_n| = O\left(\frac{1}{|n|}\right).$$

The theorem and lemmas proved so far allow us to assert via the transformation equations

$$z_n^{(1)} = \psi_n, \qquad z_n^{(2)} = \psi_n'/\lambda_n$$

the existence of solutions of (10) and to obtain estimates of the solution eigenvectors when needed. Due to the singularity of the transformation equations at $\lambda = 0$, the system (10) possesses the extraneous eigenvalue $\lambda = 0$ and the corresponding eigenvector $(0, 1)$. This fact does not influence the solution series (4) since the term corresponding to the eigenvalue 0 is zero.

3. Orthogonality and completeness of the eigenvectors.

THEOREM 2. *Let* $\mathbf{z}_n$ *and* $\mathbf{z}_m$ *be solutions of* (10) *corresponding to different eigenvalues,* λ_n *and* λ_m. *Then*

$$\int_{-1}^{+1} \mathbf{z}_n \cdot M\mathbf{z}_m \, dx = 0,$$

where

$$M = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

and the dot indicates the usual scalar product.

Proof. $\mathbf{z}_m$ satisfies the equation

$$\mathbf{z}_m' = \lambda_m A\mathbf{z}_m + B\mathbf{z}_m. \tag{24}$$

Let J denote the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Taking the scalar product of (10) with $J\mathbf{z}_m$ and the scalar product of (26) with $J\mathbf{z}_n$ and subtracting, we obtain the equation

$$\begin{aligned} &J\mathbf{z}_m \cdot \mathbf{z}_n' - J\mathbf{z}_n \cdot \mathbf{z}_m' \\ &\quad = \lambda_n J\mathbf{z}_n \cdot A\mathbf{z}_m - \lambda_m J\mathbf{z}_n \cdot A\mathbf{z}_m + J\mathbf{z}_m \cdot B\mathbf{z}_n - J\mathbf{z}_n \cdot B\mathbf{z}_m. \end{aligned} \tag{25}$$

Since $J^T = -J$, $J^T A = (J^T A)^T$, $J^T B = (J^T B)^T$ and $J^T A = -M$, we arrive at the following equations:

$$\begin{aligned} J\mathbf{z}_m \cdot \mathbf{z}_n' - J\mathbf{z}_n \cdot \mathbf{z}_m' &= \{\mathbf{z}_m \cdot J\mathbf{z}_n' + \mathbf{z}_m' \cdot J\mathbf{z}_n\} \\ &= -(\mathbf{z}_m \cdot J\mathbf{z}_n)'; \\ \lambda_n J\mathbf{z}_m \cdot A\mathbf{z}_n - \lambda_m J\mathbf{z}_n \cdot A\mathbf{z}_m &= -(\lambda_n - \lambda_m)\mathbf{z}_m \cdot M\mathbf{z}_n; \\ J\mathbf{z}_m \cdot B\mathbf{z}_n - J\mathbf{z}_n \cdot B\mathbf{z}_m &= 0. \end{aligned}$$

Hence, we may write (25) in the form

$$(\mathbf{z}_m \cdot J\mathbf{z}_n)' = -(\lambda_m - \lambda_n)\mathbf{z}_m \cdot M\mathbf{z}_n.$$

The theorem may now be obtained from this last equation by integration and application of the boundary conditions.

THEOREM 3. *Let* $\mathbf{F} = (F^{(1)}(x), F^{(2)}(x))$ *be a vector such that*

$$\int_{-1}^{+1} \mathbf{z}_n \cdot J\mathbf{F}\, dx = 0, \tag{26}$$

where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

for all eigenvectors of (10). *Then* $\mathbf{F} \equiv 0$ *on* I.

Proof. Consider the equation

$$\mathbf{\Phi}' = \lambda A\mathbf{\Phi} + B\mathbf{\Phi} + \mathbf{F} \tag{27}$$

with boundary conditions

$$\Phi^{(1)}(\pm 1) = 0.$$

Eliminating the second component of $\mathbf{\Phi}$, i.e. $\Phi^{(2)}$, from the two scalar equations of (27), we find that $\Phi^{(1)}$ satisfies the equation

$$\Phi^{(1)''} = (\lambda^2 + \lambda p)\Phi^{(1)} = \lambda F^{(2)} + F^{(1)'}$$

with

$$\Phi^{(1)}(\pm 1) = 0.$$

Thus, $\Phi^{(1)}$ is an analytic function of λ for each x in I except possibly at the eigenvalues of the homogeneous equation (5); at these exceptional points, a necessary and sufficient condition for the analyticity of $\Phi^{(1)}$ is that

$$\int_{-1}^{+1} \psi_n(\lambda_n F^{(2)} + F^{(1)\prime})\, dx = 0$$

(see [2, p. 266]). Integrating the above equation by parts, we arrive at the equations

$$\int_{-1}^{+1} \lambda_n(\psi_n F^{(2)} - \psi_n' F^{(1)})\, dx = -\lambda_n \int_{-1}^{+1} \mathbf{z}_n \cdot J\mathbf{F}\, dx,$$

which shows that this condition is equivalent to (26), $\lambda \neq 0$. Therefore, $\Phi^{(1)}$ is an entire function of λ.

The first component of (27) can be written in the form

$$\Phi^{(2)}(x,\lambda) = \frac{1}{\lambda}(\Phi^{(1)\prime} - F^{(1)}),$$

which demonstrates that the function $\lambda\Phi^{(2)}$ is also an entire function of λ. We have thereby shown that the vector $\lambda\boldsymbol{\Phi}$ has components which are entire functions of λ for every x in I. To obtain estimates of $\|\lambda\boldsymbol{\Phi}\|$ we now form an integral representation of the vector $\boldsymbol{\Phi}$ using the two auxiliary equations

$$-\mathbf{w}_i' = \lambda A^T \mathbf{w}_i + B^T \mathbf{w}_i, \qquad i = 1, 2,$$

which satisfy the initial value problems

$$\mathbf{w}_1(1,\lambda) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \mathbf{w}_2(-1,\lambda) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

Using the equation for the $\mathbf{w}_i$ and (27) we form the usual cross inner products and subtract the resulting equations to obtain the equations

$$(\mathbf{w}_i \cdot \boldsymbol{\Phi})' = \mathbf{w}_i \cdot \mathbf{F}, \qquad i = 1, 2,$$

which may then be integrated so that we derive the two equations

$$\mathbf{w}_1 \cdot \boldsymbol{\Phi} = -\int_x^1 \mathbf{w}_1 \cdot \mathbf{F}\, dt,$$

$$\mathbf{w}_2 \cdot \boldsymbol{\Phi} = \int_{-1}^x \mathbf{w}_2 \cdot \mathbf{F}\, dt.$$

Solution of the last two equations algebraically for the vector $\boldsymbol{\Phi}$ yields

$$\boldsymbol{\Phi} = \frac{1}{\delta}\int_x^1 N_1 \mathbf{F}\, dt + \frac{1}{\delta}\int_{-1}^x N_2 \mathbf{F}\, dt, \tag{28}$$

where

$$N_1 = \begin{pmatrix} -w_2^{(2)}(x) w_1^{(1)}(t) & -w_2^{(2)}(x) w_1^{(2)}(t) \\ w_2^{(1)}(x) w_1^{(1)}(t) & w_2^{(1)}(x) w_1^{(2)}(t) \end{pmatrix},$$

$$N_2 = \begin{pmatrix} -w_1^{(2)}(x) w_2^{(1)}(t) & -w_1^{(2)}(x) w_2^{(2)}(t) \\ w_1^{(1)}(x) w_2^{(1)}(t) & w_1^{(1)}(x) w_2^{(2)}(t) \end{pmatrix}$$

and

$$\delta = w_1^{(1)}(x)w_2^{(2)}(x) - w_1^{(2)}(x)w_2^{(1)}(x).$$

From the equation for $\mathbf{\Phi}$ we deduce, as previously,

$$\begin{aligned}\|\lambda\mathbf{\Phi}\| &\leqq \left\|\frac{\lambda}{\delta}\int_x^1 N_1\mathbf{F}\,dt\right\| + \left\|\frac{\Delta}{\delta}\int_{-1}^x N_2\mathbf{F}\,dt\right\| \\ &\leqq \sqrt{2}\left|\frac{\lambda}{\delta}\right|\sqrt{\int_x^1 \|N_1\|^2\|\mathbf{F}\|^2\,dt} + \sqrt{2}\left|\frac{\Delta}{\delta}\right|\sqrt{\int_{-1}^x \|N_2\|^2\|\mathbf{F}\|^2\,dt}.\end{aligned} \tag{29}$$

To estimate $\|\lambda\mathbf{\Phi}\|$ as $|\lambda| \to \infty$, asymptotic estimates for the $\mathbf{w}_i$ are necessary. Since $\mathbf{w}_2$ satisfies the equation

$$-\mathbf{w}_2' = \lambda A^T\mathbf{w}_2 + B^T\mathbf{w}_2,$$

we find that

$$-(J\mathbf{w}_2)' = \lambda JA^T\mathbf{w}_2 + JB^T\mathbf{w}_2 = -\lambda A(J\mathbf{w}_2) - B(J\mathbf{w}_2)$$

and

$$J\mathbf{w}_2(-1) = \begin{pmatrix}0\\1\end{pmatrix}.$$

Comparing this result with (12) we conclude that $\mathbf{z} = J\mathbf{w}_2$; hence, from Theorem 1 we have

$$\mathbf{w}_2 = \begin{pmatrix} -\dfrac{Q'}{\lambda}\cosh Q \\ \sinh Q \end{pmatrix} + O\left(\frac{e^{|\lambda|(x-1)}}{|\lambda|}\right).$$

The same equation applies to $\mathbf{w}_1$ with boundary conditions at $x = 1$; a simple change of variable in Theorem 1 yields the obvious result

$$\mathbf{w}_1 = \begin{pmatrix} \dfrac{R'}{\lambda}\cosh R \\ -\sinh R \end{pmatrix} + O\left(\frac{e^{|\lambda|(x-1)}}{|\lambda|}\right),$$

where

$$R(x, \lambda) = \lambda(x - 1) + \frac{1}{2}\int_1^x p\,ds.$$

We therefore have the asymptotic relations

$$\begin{aligned} w_1^{(1)} &= O(e^{-|\lambda|(x-1)}), & w_1^{(2)} &= O(e^{-|\lambda|(x-1)}), \\ w_2^{(1)} &= O(e^{|\lambda|(x+1)}), & w_2^{(2)} &= O(e^{|\lambda|(x+1)}), \end{aligned}$$

from which it is easy to verify that

$$\delta = O(e^{2|\lambda|}) \quad \text{as } |\lambda| \to \infty.$$

The function δ is actually independent of x as is indicated from this last equation;

in fact,

$$\begin{aligned}\frac{d}{dx}(\delta) &= -\mathbf{w}_2 \cdot J\mathbf{w}_1' - \mathbf{w}_2' \cdot J\mathbf{w}_1 \\ &= -\mathbf{w}_2 \cdot J\mathbf{w}_1' + \mathbf{w}_1 \cdot J\mathbf{w}_2' \\ &= \lambda(\mathbf{w}_2 \cdot JA\mathbf{w}_1 - \mathbf{w}_1 \cdot JA\mathbf{w}_2) + \mathbf{w}_1 \cdot JB^T\mathbf{w}_2 - \mathbf{w}_2 \cdot JB^T\mathbf{w}_1 = 0.\end{aligned}$$

The vector $\mathbf{F}$ in (28) is chosen so that the zeros of δ are removable singularities of the quotient terms for each component equation (except possibly $\lambda = 0$ for the second component). Since all of the functions involved on the left side of (28) are exponential in character, so will be the resultant entire functions $\lambda\Phi^{(1)}$ and $\lambda\Phi^{(2)}$ (see [3, Ex. iv, p. 255]). From our previous results, we have

$$\begin{aligned}\|N_1\|^2 &= O(e^{2|\lambda|(x-t)+4}), \\ \|N_2\|^2 &= O(e^{2|\lambda|(t-x)+4}).\end{aligned}$$

Thus,

$$\sqrt{2}|\lambda|\sqrt{\int_x^1 \frac{\|N_1\|^2}{|\delta|}\|\mathbf{F}\|^2\,dt} = |\lambda| O\left(\frac{1}{|\lambda|}\right) = O(1);$$

and, similarly,

$$\sqrt{2}|\lambda|\sqrt{\int_{-1}^x \frac{\|N_2\|^2}{|\delta|}\|\mathbf{F}\|^2\,dt} = O(1),$$

so that, from the inequality (29), we find

$$\|\lambda\mathbf{\Phi}\| = O(1).$$

Since both components of $\lambda\mathbf{\Phi}$ are entire functions, both components must be constant by Liouville's theorem. The function $\Phi^{(1)}$ itself is entire; hence, $\Phi^{(1)} = 0$, which leaves us with the reduced equations

$$\lambda\Phi^{(2)} = -F^{(1)}(x), \qquad \Phi^{(2)\prime} = F^{(2)}(x),$$

which immediately yield the fact that $-F^{(1)\prime}(x) = \lambda F^{(2)}(x)$. These equations can only hold provided $F^{(2)} = 0$ and $F^{(1)\prime} = 0$. Thus $\mathbf{F}$ must be of the form $(c_0, 0)$, where c_0 is a constant. This is impossible unless $c_0 = 0$ because, for the eigenvalue $\lambda = 0$ of (10), condition (26) becomes

$$\int_{-1}^{+1} z_n \cdot J\mathbf{F}\,dx = \int_{-1}^{+1} \begin{pmatrix}0\\1\end{pmatrix} \cdot \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\begin{pmatrix}c_0\\0\end{pmatrix} dx = 2c_0 = 0.$$

The theorem is proved.

Using Theorem 3 we can orthogonalize the eigenvectors $\mathbf{z}_n$ with respect to the indefinite bilinear form

$$\int_{-1}^{+1} (\cdots) \cdot M(\cdots)\,dx.$$

LEMMA 3.1.

$$\int_{-1}^{+1} \mathbf{z}_n \cdot M\mathbf{z}_n \, dx \neq 0$$

for all n.

Proof. Suppose that there exists some $n = n'$ such that

$$\int_{-1}^{+1} \mathbf{z}_{n'} \cdot M\mathbf{z}_{n'} \, dx = 0. \tag{30}$$

Then, since

$$M\mathbf{z}_{n'} = -J^T A\mathbf{z}_{n'} = J(A\mathbf{z}_{n'}),$$

we have for (26) and (30) that the vector $A\mathbf{z}_{n'} \equiv \mathbf{F}$ satisfies the equation

$$\int_{-1}^{+1} \mathbf{z}_n \cdot J\mathbf{F} \, dx = 0$$

for all n; therefore, by Theorem 3, $A\mathbf{z}_{n'} = 0$ which implies $\mathbf{z}_n = \mathbf{0}$, a contradiction.

Since the integral in the previous lemma never vanishes, we may define

$$\gamma_n^2 = 1 \Big/ \int_{-1}^{+1} \mathbf{z}_n \cdot M\mathbf{z}_n \, dx$$

and the set of vectors $\mathbf{Z}_n = \gamma_n \mathbf{z}_n$ satisfies the orthonormality conditions

$$\int_{-1}^{+1} \mathbf{Z}_n \cdot M\mathbf{Z}_n \, dx = \delta_{nm}.$$

If we make the identification $\mathbf{z}_n = \mathbf{z}(x, \lambda_n)$, it follows from Theorem 1 and Lemma 1.5 that

$$\mathbf{z}_n = \begin{pmatrix} \sinh(c + n\pi i/2)(x+1) + \int_{-1}^{x} p \, ds \\ (1 + p(x)/(c + n\pi i/2))\cosh(c + n\pi i/2)(x+1) + \int_{-1}^{x} p \, ds \end{pmatrix} + O\left(\frac{1}{|n|}\right)$$

if we notice that the $O(\exp|\lambda|(x+1))$ terms in the asymptotic expressions actually depend only on $\operatorname{Re}(\lambda)$ which is uniformly bounded for $\lambda = \lambda_n$. This last expression for $\mathbf{z}_n$ shows that both components of the vector are bounded as $|\lambda_n| \to \infty$. The identification of $\mathbf{z}(x, \lambda)$ and $\mathbf{z}_n$ is certainly permissible since $\mathbf{z}_n$ has thus far only been determined up to an arbitrary constant multiple.

4. Convergence of the eigenvector expansion.

LEMMA 3.2.

$$\gamma_n = \frac{1}{\sqrt{2}} + \varepsilon_n,$$

where

$$|\varepsilon_n| \to 0 \quad as \ |n| \to \infty.$$

Proof. We find that

$$\frac{1}{\gamma_n^2} = \int_{-1}^{+1} \mathbf{z}_n \cdot M\mathbf{z}_n \, dx = \int_{-1}^{+1} [-(z_n^{(1)})^2 + (z_n^{(2)})^2] \, dx$$

$$= \int_{-1}^{+1} \left\{ -\left[\sinh Q(x, \lambda_n) + O\left(\frac{1}{|n|}\right) \right]^2 \right.$$

$$\left. + \left[\left(1 + O\left(\frac{1}{|n|}\right)\right) \cosh Q(x, \lambda_n) + O\left(\frac{1}{|n|}\right) \right]^2 \right\} dx$$

$$= \int_{-1}^{+1} \cosh^2 Q(x, \lambda_n) - \sinh^2 Q(x, \lambda_n) + O\left(\frac{1}{|n|}\right) dx$$

$$= \int_{-1}^{+1} \left(1 + O\left(\frac{1}{|n|}\right)\right) dx = 2 + \int_{-1}^{+1} O\left(\frac{1}{|n|}\right) dx.$$

This last integral I_n can be made as small as desired by taking $|n|$ large enough. Thus,

$$\gamma_n = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{1 + I_n/2}} = \frac{1}{2}(I + O(\sqrt{I_n}))$$

from which the lemma follows.

The last lemma demonstrates that the components of the set of eigenvectors $\mathbf{Z}_n = \gamma_n \mathbf{Z}_n$ are bounded as $|n| \to \infty$. The expansion problem embodied in (11) can now be solved. From the previous orthogonality equation, the coefficients c_n of (11) should be given by

$$c_n = \int_{-1}^{+1} \begin{pmatrix} g \\ f' \end{pmatrix} \cdot M\mathbf{Z}_n \, dx. \tag{31}$$

THEOREM 4. *Let f be trice and g be twice continuously differentiable on I with $g(\pm 1) = 0$. The series $\sum_n c_n \mathbf{Z}_n$, where c_n is given by* (31), *converges uniformly and absolutely to the vector* (g, f').

Proof.

$$c_n = \int_{-1}^{+1} \begin{pmatrix} g \\ f' \end{pmatrix} \cdot M\mathbf{Z} \, dx = \int_{-1}^{+1} \begin{pmatrix} g \\ f' \end{pmatrix} \cdot M\{\lambda_n A + B\}^{-1}\{\lambda_n A + B\}\mathbf{Z}_n \, dx$$

$$= \gamma_n \int_{-1}^{+1} \begin{pmatrix} g \\ f' \end{pmatrix} \cdot M \frac{\{\lambda_n A + B\}}{\lambda_n^2 + \lambda_n p} \{\lambda_n A + B\}\mathbf{Z}_n \, dx$$

$$= \gamma_n \int_{-1}^{+1} \begin{pmatrix} g \\ f' \end{pmatrix} \cdot M \frac{\{\lambda_n A + B\}}{\lambda_n^2 + \lambda_n p} \mathbf{Z}_n' \, dx.$$

We now integrate this last equation twice by parts to obtain

$$c_n = -\gamma_n \begin{pmatrix} g' \\ f'' \end{pmatrix} \cdot \frac{M\mathbf{Z}_n}{\lambda_n^2 + \lambda_n p} \Bigg|_{-1}^{+1} + \gamma_n \int_{-1}^{+1} \frac{d}{dt} \left\{ \begin{pmatrix} g' \\ f'' \end{pmatrix} \cdot \frac{M}{\lambda_n^2 + d_n p} \right\} \mathbf{Z}_n \, dx$$

$$= -\gamma_n \int_{-1}^{+1} \begin{pmatrix} g \\ f' \end{pmatrix} \cdot \frac{MB'}{\lambda_n^2 + \lambda_n p} \mathbf{Z}_n \, dx + \lambda_n \gamma_n \int_{-1}^{+1} \begin{pmatrix} g \\ f' \end{pmatrix} \cdot \frac{M(\lambda_n A + B)}{(\lambda_n^2 + \lambda_n p)^2} p' \mathbf{Z}_n \, dx.$$

The last equation, the boundedness of the components of $\mathbf{z}_n$, and the estimate for γ_n yield

$$c_n = O\left(\frac{1}{|\lambda_n|^2}\right) = O\left(\frac{1}{n^2}\right).$$

Hence, the series converges absolutely and uniformly. To show that the series does indeed converge to the desired vector (g, f'), we set

$$\begin{pmatrix} r^{(1)} \\ r^{(2)} \end{pmatrix} = \sum_n c_n \mathbf{Z}_n$$

and note that

$$\int_{-1}^{+1} \begin{pmatrix} r^{(1)} \\ r^{(2)} \end{pmatrix} \cdot M\mathbf{Z}_n \, dx = \sum_n \int_{-1}^{+1} \begin{pmatrix} g \\ f' \end{pmatrix} \cdot M\mathbf{Z}_n \, dx \delta_{nm} = \int_{-1}^{+1} \begin{pmatrix} g \\ f' \end{pmatrix} \cdot M\mathbf{Z}_n \, dx,$$

which implies

$$\int_{-1}^{+1} \left[\begin{pmatrix} r^{(1)} \\ r^{(2)} \end{pmatrix} - \begin{pmatrix} g \\ f' \end{pmatrix} \right] \cdot M\mathbf{Z}_n \, dx = 0$$

for all n, so, $r^{(1)} = g$ and $r^{(2)} = f'$.

Since the coefficients c_n may be found using (31) and conditions asserting the convergence of the eigenvector expansion have been found, the final solution may now be formulated.

5. Solution and summary.

THEOREM 5. *The solution of* (1) *satisfying the boundary conditions in* (3a)–(3d) *is given by the series*

$$\varphi(x, t) = \sum_n \frac{c_n}{\lambda_n} \psi_n(x) e^{\lambda_n t},$$

where the λ_n *and* ψ_n *are defined through the equations*

(i) $\psi_n'' - (\lambda_n^2 + \lambda_n p)\psi_n = 0$,

(ii) $\displaystyle\int_{-1}^{+1} \left[-\psi_n^2 + \left(\frac{\psi_n'}{\lambda_n}\right)^2 \right] dx = 1$,

(iii) $\psi_n(\pm 1) = 0$

and

$$c_n = \int_{-1}^{+1} \left[-g\psi_n + f' \frac{\psi_n'}{\lambda_n} \right] dx = -\int_{-1}^{+1} \left(g + \frac{f''}{\lambda_n} \right) \psi_n \, dx.$$

The series solution is absolutely and uniformly convergent provided that f and g are 3 and 2 times continuously differentiable respectively and $g(\pm 1) = 0$.

As we have seen, certain nonseparable partial differential equations of the form (1) can be solved in terms of eigenfunction expansions as in the separable case. The major disadvantage in assuming a specific form for the solution, in order to separate it, arises in the nonlinearity of the eigenvalue problem obtained.

In spite of this consideration, the ordinary differential equation obtained in the separation process may be transformed to a system of two differential equations in which the eigenvalue parameter occurs linearly. The eigenvectors of the resulting system are orthogonal with respect to an indefinite bilinear form, and the boundary conditions for the original problem become expressed as a simple vector expansion in which formulas for the coefficients in the expansion can be found. Thus, even though the solution is obtained in terms of nonorthogonal eigenfunctions, the coefficients of the solution expansion are known exactly without resorting to the use of infinite systems of algebraic equations which would generally have to be solved approximately.

Acknowledgments. The author wishes to thank Professor G. E. Latta for suggesting this line of work and the referee for his many valuable improvements in the paper.

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SIAM J. MATH. ANAL.
Vol. 2, No. 2, May 1971

THEORETICAL PROPERTIES OF BEST POLYNOMIAL APPROXIMATION IN $W^{1,2}[-1, 1]$*

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1. Introduction. The purpose of this paper is to present several theorems concerning the behavior of best polynomial fits to absolutely continuous functions $f(x)$ with square-summable derivatives in the norm

$$\|f\|_1^2 = \int_{-1}^{1} f^2\,dx + \alpha \int_{-1}^{1} f'^2\,dx. \tag{1}$$

Here α is any given positive number. The properties of the best fits in this norm are also contrasted with those for the topologically equivalent norm

$$\|f\|_2^2 = f^2(-1) + \int_{-1}^{1} f'^2\,dx. \tag{2}$$

2. Analysis. There is a certain interesting connection between best approximation by polynomials of a preassigned degree in the Sobolev norm (1) and best approximation by polynomials of the same degree in the topologically equivalent, but somewhat simpler-looking norm (2). The following theorem summarizes this result. The proof is due, in part, to the work of Theodore A. Orlow.

THEOREM 1. *Suppose f is an absolutely continuous function on $[-1, 1]$ and that the derivative of f is square-summable. Assume that $k \geqq 2$. Then, if $P_{k,\alpha}(x)$ is the best polynomial approximation of degree at most k in* (1) *for f, then* $\lim_{\alpha\to\infty} P_{k,\alpha}$ *exists. Furthermore, this limit is the best polynomial of degree at most k for f in the norm* (2).

Proof. To find the best fit $P_{k,\alpha}$ in (1) of degree at most k, the following expression is to be minimized over all polynomials $R_k(x) = \sum_{j=0}^{k} a_j x^j$:

$$S_{k,\alpha}(a_0, a_1, \cdots, a_k) = \int_{-1}^{1} (f - R_k)^2\,dx + \alpha \int_{-1}^{1} (f' - R_k')^2\,dx. \tag{3}$$

Equation (3) will be minimized if the following conditions are met by the a_j:

$$-\tfrac{1}{2}\partial S_{k,\alpha}/\partial a_0 = \int_{-1}^{1} \left[f(x) - \sum_{j=0}^{k} a_j x^j\right] dx = 0, \tag{4}$$

$$\begin{aligned} -\tfrac{1}{2}\partial S_{k,\alpha}/\partial a_i = &\int_{-1}^{1} \left[f(x) - \sum_{j=0}^{k} a_j x^j\right] x^i\,dx \\ &+ \alpha i \int_{-1}^{1} \left[f'(x) - \sum_{j=1}^{k} j a_j x^{j-1}\right] x^{i-1}\,dx = 0, \quad 1 \leqq i \leqq k. \end{aligned} \tag{5}$$

If (4) and (5) are explicitly solved by Cramer's rule for the a_j as functions of α, it is found that, for every j, $a_j(\alpha)$ is a quotient of two polynomials of degree k in α. After dividing both numerator and denominator by α^k, we see that $a_j(\alpha)$ tends to a limit as $\alpha \to \infty$, since the limit of the denominator is a nonzero constant times

* Received by the editors March 10, 1970, and in revised form September 17, 1970.

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the Grammian in $L^2[-1,1]$ of the set $\{x^i\}_{i=0}^{k-1}$. This Grammian is well-known to be positive [1, p. 178]. Now, if $P_{k,\alpha}$ minimizes (3), it is clear that it also minimizes

$$(1/\alpha)\int_{-1}^{1}(f-R_k)^2\,dx+\int_{-1}^{1}(f'-R_k')^2\,dx.$$

Let $Q_{k-1}(x)$ be that polynomial of degree at most $k-1$ such that

$$\int_{-1}^{1}(f'-Q_{k-1})^2\,dx \quad \text{is minimum} \tag{6}$$

and define

$$I_k(x)\equiv\int_{-1}^{x}Q_{k-1}(t)\,dt$$

as its integral. Then, by definition of $P_{k,\alpha}$,

$$\begin{aligned}(1/\alpha)&\int_{-1}^{1}(f-P_{k,\alpha})^2\,dx+\int_{-1}^{1}(f'-P_{k,\alpha}')^2\,dx\\ &\leqq(1/\alpha)\int_{-1}^{1}(f-I_k)^2\,dx+\int_{-1}^{1}(f'-Q_{k-1})^2\,dx.\end{aligned}$$

Letting $\alpha\to\infty$, we have

$$\int_{-1}^{1}(f'-P_k')^2\,dx\leqq\int_{-1}^{1}(f'-Q_{k-1})^2\,dx,$$

where $P_k(x)\equiv\lim_{\alpha\to\infty}P_{k,\alpha}(x)$. However, by (6), $Q_{k-1}(x)$ is the best fit to $f'(x)$, so that $P_k'(x)=Q_{k-1}(x)$. Therefore, it is necessary that the following conditions be satisfied:

$$\frac{\partial}{\partial a_i}\int_{-1}^{1}(f'-P_k')^2\,dx=0,\quad 1\leqq i\leqq k. \tag{7}$$

Since f is absolutely continuous, we have, upon setting $i=1$ in (7),

$$f(1)-f(-1)=P_k(1)-P_k(-1). \tag{8}$$

So the average slope of both f and P_k are the same. From (4), we see, upon letting $\alpha\to\infty$,

$$\int_{-1}^{1}f(x)\,dx=\int_{-1}^{1}P_k(x)\,dx, \tag{9}$$

so that the areas are the same, also. Letting $i=2$ in (7), one sees, by integration by parts, that

$$f(1)+f(-1)-\int_{-1}^{1}f(x)\,dx=P_k(1)+P_k(-1)-\int_{-1}^{1}P_k(x)\,dx,$$

so that, using (9),

$$f(-1)+f(1)=P_k(-1)+P_k(1). \tag{10}$$

From (10) and (8), one deduces that $f(\pm 1) = P_k(\pm 1)$ for the same sign. Thus P_k is just the best fit in the norm (2).

It is well known [1, pp. 236–237] that if $f(x)$ is continuous on $[a, b]$, then the best approximation to $f(x)$ in the least squares sense over all polynomials of degree not exceeding n coincides with f in at least $n + 1$ points of the open interval (a, b). The following theorem is a generalization of this result for $\|f\|_2$.

THEOREM 2. *Suppose that $k \geqq 2$. If $P_k(x)$ is the best polynomial of degree at most k for f in* (2), *then the deviation $D_k = f - P_k$ changes sign in at least $k - 1$ points in the open interval $(-1, 1)$ unless f itself is a polynomial of degree not exceeding k. Furthermore, if $f'(x)$ exists everywhere on $(-1, 1)$, either finite or infinite, then D_k' changes sign in at least k points of $(-1, 1)$.*

Proof. It is clear that the best polynomial approximation $P_k(x)$ of maximum degree k is just that polynomial whose derivative is the best least squares fit to $f'(x)$ and whose value at $x = -1$ is $f(-1)$. For $i \neq 0$, one sees from (7) that

$$[x^i f(x)]_{-1}^{1} - \int_{-1}^{1} i x^{i-1} f(x)\, dx = [x^i P_k(x)]_{-1}^{1} - \int_{-1}^{1} i x^{i-1} P_k(x)\, dx,$$

$1 \leqq i \leqq k - 1$, from which it follows that

$$\int_{-1}^{1} x^{i-1} D_k(x)\, dx = 0, \quad 1 \leqq i \leqq k - 1. \tag{11}$$

If $k = 2$, it is clear from (11) that $D_k(x)$ must change sign at least once unless f is a polynomial of degree at most k. If $k > 2$, suppose that f is not a polynomial of degree less than or equal to k and that there are at most r changes of sign of $D_k(x) = f(x) - P_k(x)$ in $(-1, 1)$, say at $x_1, x_2, \cdots, x_r$, where $r \leqq k - 2$. Then the product $D_k(x) \prod_{i=1}^{r} (x - x_i)$ is of one sign throughout $(-1, 1)$, and we have

$$\int_{-1}^{1} D_k(x) \prod_{i=1}^{r} (x - x_i)\, dx \neq 0. \tag{12}$$

However, this is impossible, since (11) implies that the left member of (12) is zero. Therefore, there are at least $k - 1$ such points. Also, we know that $D_k(-1) = D_k(1) = 0$. So, if $f'(x)$ exists everywhere on $(-1, 1)$, either finite or infinite, we may apply Rolle's theorem to assert that $D_k'(x)$ changes sign in at least k points of $(-1, 1)$.

Note. If instead of using $x = -1$ or $x = 1$ in the norm, we choose any point a in the open interval $(-1, 1)$, the same kind of result is not in general obtainable for the norm

$$\|f\|^2 = f^2(a) + \int_{-1}^{1} f'^2\, dx.$$

One can see this immediately if a is close to the absolute minimum of the deviation curve for (2), for all the deviation curves obtained for various values of a are merely translates of each other, each being 0 for $x = a$.

Let $D_{k,\alpha}(x) = f(x) - P_{k,\alpha}(x)$, and define the moments

$$M_{i,\alpha} = \int_{-1}^{1} D_{k,\alpha}(x) x^i\, dx, \quad 0 \leqq i \leqq k.$$

Then we have the following theorem.

THEOREM 3. *For the Sobolev norm* (1), *the moments* $M_{i,\alpha}$ *can always be expressed in terms of the deviations* $D_{k,\alpha}$ *at the endpoints. Furthermore, if f is an even function, the odd moments vanish; and, if f is an odd function, the even moments vanish.*

Proof. First of all, we know from (4) that

$$\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} P_{k,\alpha}(x)\,dx.$$

Therefore $M_{0,\alpha} = \int_{-1}^{1} D_{k,\alpha}\,dx = 0$. By setting $i = 1$ in (5),

$$\int_{-1}^{1} D_{k,\alpha}x\,dx + \alpha\int_{-1}^{1} D'_{k,\alpha}\,dx = 0.$$

We thus have

$$M_{1,\alpha} = -\alpha[D_{k,\alpha}(1) - D_{k,\alpha}(-1)].$$

By setting $i = 2$,

$$\int_{-1}^{1} D_{k,\alpha}x^2\,dx + 2\alpha\int_{-1}^{1} D'_{k,\alpha}x\,dx = 0,$$

from which one deduces that

$$M_{2,\alpha} = -2\alpha[D_{k,\alpha}(1) + D_{k,\alpha}(-1)].$$

With $i = 3$,

$$\int_{-1}^{1} D_{k,\alpha}x^3\,dx + 3\alpha\int_{-1}^{1} D'_{k,\alpha}x^2\,dx = 0,$$

and

$$M_{3,\alpha} = -3\alpha(2\alpha + 1)[D_{k,\alpha}(1) - D_{k,\alpha}(-1)].$$

One may continue this process to obtain similar results for the other moments. In particular, for example, if i is odd and $i \leqq k$, it can be shown by mathematical induction that

$$M_{i,\alpha} = -i\alpha(D_{k,\alpha}(1) - D_{k,\alpha}(-1))\sum_{j=0}^{(i-1)/2} (i-1)^{(2j)}\alpha^j,$$

where, as is usual in the calculus of finite differences, we define

$$i^{(0)} = 1,$$

$$(i-1)^{(k)} = \prod_{j=1}^{k} (i-j).$$

A similar result can be established by induction when i is even. If f is even, we know that $P_{k,\alpha}$ is also an even function. Therefore, $D_{k,\alpha}(1) = D_{k,\alpha}(-1)$ and

$$M_{2i-1,\alpha} = 0, \qquad 1 \leqq i \leqq k/2.$$

On the other hand, if f is odd, so is $P_{k,\alpha}$, and we see that

$$M_{2i,\alpha} = 0, \qquad 1 \leqq i \leqq (k-1)/2.$$

It is known [1, pp. 165–167] that there exists an orthogonal set $\{p_{k,\alpha}(x)\}$ of polynomials, each of degree k, in $\|f\|_1$. These are used in the following theorem. As is customary, also, we define

$$(f, g)_1 = \int_{-1}^{1} fg\, dx + \alpha \int_{-1}^{1} f'g'\, dx.$$

THEOREM 4. *Suppose that $k \geqq 2$. If $P_{k,\alpha}(x)$ is the best polynomial approximation of degree at most k to $f(x)$ in the Sobolev norm* (1) *and $P_k(x)$ is the best approximant of degree at most k in the norm* (2), *then*

$$P_{k,\alpha}(x) - P_k(x) = \beta_1 p_{k,\alpha}(x) + \beta_2 p_{k-1,\alpha}(x),$$

where

$$\beta_1 = \int_{-1}^{1} x^k(f(x) - P_k(x))\, dx/(p_{k,\alpha}, x^k)_1$$

and

$$\beta_2 = \int_{-1}^{1} x^{k-1}(f(x) - P_k(x))\, dx/(p_{k-1,\alpha}, x^{k-1})_1.$$

Proof. Let $P_{k,\alpha}(x) = P_k(x) + q_k(x)$, where $q_k(x)$ is to be determined. The first orthogonality condition is obtained as before, by differentiating partially with respect to the constant term. We have

$$\int_{-1}^{1} (f(x) - P_k(x) - q_k(x))\, dx = 0.$$

But using (9), we have

$$\int_{-1}^{1} q_k(x)\, dx = 0.$$

From (5), we see that

$$\int_{-1}^{1} x^i(f(x) - P_k(x) - q_k(x))\, dx + \alpha i \int_{-1}^{1} x^{i-1}(f'(x) - P_k'(x) - q_k'(x))\, dx = 0,$$

which, on substitution of (7) and (11), yields

$$\int_{-1}^{1} x^i q_k(x)\, dx + \alpha i \int_{-1}^{1} x^{i-1} q_k'(x)\, dx = 0, \qquad 1 \leqq i \leqq k-2,$$

i.e.,

$$(x^i, q_k(x))_1 = 0, \qquad 1 \leqq i \leqq k-2.$$

From these conditions, it follows easily that $q_k(x)$ is a linear combination of $p_{k,\alpha}(x)$ and $p_{k-1,\alpha}(x)$ alone, i.e.,

$$q_k(x) = \beta_1 p_{k,\alpha}(x) + \beta_2 p_{k-1,\alpha}(x).$$

One can verify that

$$(x^{k-1}, q_k(x))_1 = \int_{-1}^{1} x^{k-1}(f(x) - P_k(x))\, dx$$

and that

$$(x^k, q_k(x))_1 = \int_{-1}^{1} x^k(f(x) - P_k(x))\,dx$$

by using the fact that

$$P_{k,\alpha}(x) - P_k(x) = D_k(x) - D_{k,\alpha}(x)$$

and by using (5) and (7). Therefore,

$$(x^{k-1}, q_k(x))_1 = \beta_2(p_{k-1,\alpha}, x^{k-1})_1 = \int_{-1}^{1} x^{k-1}(f(x) - P_k(x))\,dx,$$

so

$$\beta_2 = \int_{-1}^{1} x^{k-1}(f(x) - P_k(x))\,dx/(p_{k-1,\alpha}, x^{k-1})_1.$$

Upon taking the inner product with x^k, we find

$$\beta_1(p_{k,\alpha}, x^k)_1 = (q_k, x^k)_1 = \int_{-1}^{1} x^k(f(x) - P_k(x))\,dx,$$

so

$$\beta_1 = \int_{-1}^{1} x^k(f(x) - P_k(x))\,dx/(p_{k,\alpha}, x^k)_1.$$

The result in Theorem 4 is important because it has been demonstrated more explicitly how $P_{k,\alpha}(x)$, the best approximant in (1), differs from $P_k(x)$, the best approximant in (2). It turns out that, for any α, $P_{k,\alpha}(x)$ and $P_k(x)$, when expanded upon the basis $\{p_{k,\alpha}\}$, differ only in the last two Fourier coefficients.

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SIAM J. MATH. ANAL.
Vol. 2, No. 2, May 1971

THE ABSTRACT BACKWARD BEAM EQUATION*

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Abstract. We consider a two-point boundary value problem for the equation $u_{tt} - A(t)u = f(t, u)$ where, for each t, $A(t)$ is an unbounded operator in Hilbert space whose domain varies with t. We establish the existence (but not the uniqueness) of "weak" solutions of this problem by using finite difference methods. When $A(t)$ represents a differential operator in the space variables, the method of proof can be converted into a convergent numerical procedure.

1. Introduction. Let H be a separable Hilbert space with scalar product $\langle\cdot,\cdot\rangle$ and corresponding norm $\|\cdot\|_H$. For each t in the finite interval $[0, T]$ let $A(t)$ be a closed linear operator with (variable) domain $D_A(t)$ dense in H, and let $A(t)$ have the following property: For every complex λ with positive real part, $[A(t) + \lambda]^{-1}$ exists and is a bounded operator with domain all of H, and

$$\|[A(t) + \lambda]^{-1}\|_H \leqq \frac{1}{\operatorname{Re}\lambda}, \qquad \operatorname{Re}\lambda > 0. \tag{1.1}$$

It then follows (see [5, p. 279]) that

$$\operatorname{Re}\langle A(t)v, v\rangle \geqq 0 \quad \text{for all } v \in D_A(t). \tag{1.2}$$

We note that according to the Hille–Yosida theorem, (1.1) is a necessary and sufficient condition in order that the closed densely defined operator $-A(t)$ be the generator of a strongly continuous semigroup of contraction operators.

In this study we are concerned with the mildly nonlinear two-point boundary value problem

$$\begin{gathered} u_{tt} - A(t)u = f(t, u), \qquad 0 < t < T, \\ u(0) = g_1, \qquad u(T) = g_2, \end{gathered} \tag{1.3}$$

where u is an H-valued function and g_1, g_2 are given vectors in H.

Although *initial value* problems for ordinary differential equations in Banach space have been extensively studied in the literature, two-point problems were apparently first considered by S. Krein and G. Laptev in [7] and [8]. Previously, in [6], the same authors considered the related "eigenvalue problem" in connection with a problem of M. Krein's arising in wave-guide theory. (See also [9].) In [7] and [8] the *linear* problem is considered, and A is assumed *independent of t*. The differential equation in (1.3) is then replaced by an equivalent first order system involving $A^{1/2}$, and an important role is played by the semigroup of operators generated by $-A^{1/2}$. More recently, the *fixed domain*, time-dependent linear problem has been considered by Sobolevskii in [12]. The latter author announces results on the solvability of the problem in certain Lipschitz spaces with weight. The method consists in relating the time-dependent case to the case with constant A by means of a partition of unity argument. It is assumed

* Received by the editors June 11, 1970.

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that $-A^{1/2}$ generates an analytic semigroup and that $A(t)A^{-1}(0)$ satisfies a Lipschitz condition.

Our motivation for studying this problem is that we view the equation

$$u_{tt} - A(t)u = 0 \tag{1.4}$$

as an abstract version of

$$\frac{\partial^2 v}{\partial t^2} = \frac{\partial^4 v}{\partial x^4}, \tag{1.5}$$

and our interest in (1.4) stems from the fact that a solution of the heat equation $v_t = v_{xx}$ also satisfies (1.5). In the near future we hope to make use of this observation in connection with "boundary value techniques" for the numerical computation of parabolic problems with a known steady state solution (cf. [1]). For want of a better name, we call (1.5) the *backward beam equation* because of its apparent similarity with the equation of the vibrating beam, $v_{tt} + v_{xxxx} = 0$ (see [2, p. 295]).

In the present paper we establish the existence of "weak" solutions to the problem (1.3) using techniques borrowed from numerical analysis. We discretize the t-variable and replace u_{tt} by a centered second difference quotient to obtain an inhomogeneous system of equations with coefficients which are linear operators in H. This approximate problem is simpler than (1.3). In the linear case, we show that this system always has a unique solution, and we develop an algorithm for finding this solution. As a matter of fact, this algorithm (which is the basis for "Gaussian elimination" in numerical analysis) is the key to many of our results; it seemed interesting to us that it could be implemented even though the domain of $A(t)$ varies with t. In the nonlinear case we assume that $f(t, u)$ is monotone and that, in addition to (1.1), $[A(t) + \lambda]^{-1}$ is compact for $\operatorname{Re} \lambda > 0$. Existence of solutions for the finite difference equations then follows from the results for the linear case and the Schauder fixed-point theorem, while uniqueness is a consequence of the monotonicity of f. As a by-product of our analysis, it follows that if (1.3) has a (unique) strong solution $u(t)$ satisfying a relatively weak smoothness assumption, namely (3.3) below, then the solution of the finite difference equations converges *uniformly* to that of (1.3) at the rate of $O(\Delta t^2)$. Finally, we associate a "weak" two-point problem with (1.3). Using a priori estimates for the solutions of the difference equations, we study their behavior as $\Delta t \to 0$, and we prove convergence of the solutions of a sequence of discrete problems to a solution of the weak problem; moreover we estimate this solution in the *uniform* norm in terms of the boundary data and the L^2-norm of the inhomogeneous term. We do not prove uniqueness in the weak problem. On the other hand, it is easily seen by means of an integration by parts that there is at most one solution to (1.3) possessing two continuous derivatives. Hence, uniqueness follows upon proving that weak solutions are strong solutions. Such "regularity" results can easily be obtained for the case where the domain of $A(t)$ is independent of t, and $g_1, g_2 \in D_A(t)$. We do not know how to handle the regularity theory for the variable domain problem.

In conclusion we would like to make some remarks about possible alternative ways of treating the existence question for (1.3). Consider the linear problem for simplicity. Unlike the case of scalar two-point boundary value problems (see e.g. [13, p. 68]) there seems to be no simple way of reducing the problem to the case where g_1 and g_2 are zero. Such a reduction would be attractive provided the operator $\mathscr{L}$, in $L^2(0, T; H)$, defined by

$$\mathscr{L}[u] \equiv u'' - A(t)u, \qquad 0 < t < T, \\ u(0) = u(T) = 0, \tag{1.6}$$

could be shown to have an inverse which is bounded on the whole space. However, even if it were possible, such a reduction would still necessitate proving the existence of solutions of certain *initial value* problems for $u'' - A(t)u = 0$, with the domain of $A(t)$ variable. Thus our method is more direct and more elementary. In the case where we have uniqueness, it is also constructive. Finally, if $A(t)$ represents a differential operator in the space variables, our method of proof can form the basis for a numerical computation of the solution. It is only necessary to do the space-differencing in such a way that the discretized operator $A(\Delta x, t)$ has nonnegative eigenvalues. Moreover, our analysis makes it clear that the "block tridiagonal algorithm" can then be used to solve the resulting system of finite difference equations.

2. Notation. Let $\Delta t = T/(N + 1)$ where N is a positive integer and let $H^N(\Delta t)$ be the complex vector space of all N-tuples $\{v^1, v^2, \cdots, v^N\}$ where $v^k \in H$ for each $k = 1, \cdots, N$. Elements of H^N will be denoted by capital letters and represented as column vectors, i.e., as

$$V = \begin{bmatrix} v^1 \\ v^2 \\ \cdot \\ \cdot \\ \cdot \\ v^N \end{bmatrix}. \tag{2.1}$$

We equip H^N with the scalar product

$$(V, W) = \Delta t \sum_{n=1}^{N} \langle v^n, w^n \rangle, \tag{2.2}$$

and we write

$$\|V\|_{H^N} = (V, V)^{1/2}. \tag{2.3}$$

Evidently, H^N is also a separable Hilbert space under (2.3). We also consider $N \times N$ matrices whose entries will be linear operators in H. Such matrices define linear operators in H^N and we write

$$\|P\|_{H^N} = \sup_{\|V\|_{H^N} = 1} \{\|PV\|_{H^N}\}, \tag{2.4}$$

the supremum being taken over all V in the domain of P.

3. Discrete approximation to the linear problem. In this section we examine the case where $f(t, u)$ is independent of u. The results obtained here form the basis for a discussion of the nonlinear problem in § 5.

With $T = (N + 1)\Delta t$ consider the system of linear equations

$$\begin{aligned} &\frac{v^{n+1} - 2v^n + v^{n-1}}{\Delta t^2} - A^n v^n = f^n, \qquad n = 1, 2, \cdots, N, \\ &v^0 = g_1, \quad v^{N+1} = g_2, \end{aligned} \tag{3.1}$$

where we use the notation A^n, f^n for $A(n\Delta t)$ and $f(n\Delta t)$. The above system is an approximation to

$$\begin{aligned} &u_{tt} - A(t)u = f(t), \qquad 0 < t < T, \\ &u(0) = g_1, \quad u(T) = g_2 \end{aligned} \tag{3.2}$$

in the following sense: If (3.2) has a solution $u(t)$ and if u^n denotes $u(n\Delta t)$, then $\{u^n\}$ satisfies (3.1) provided we add a "truncation error term" τ^n to the right of (3.1). In Theorem 4 below we assume that (3.2) has a unique solution smooth enough that

$$\left\{\Delta t \sum_{n=1}^{N} \|\tau^n\|_H^2\right\}^{1/2} \leqq K\Delta t^2, \tag{3.3}$$

where K is a constant independent of Δt and N.

We write the system (3.1) in matrix vector notation as a single linear operator equation in H^N, i.e.,

$$PV = F + G, \tag{3.4}$$

where V is given by (2.1) and P is the tridiagonal $N \times N$ matrix

$$P = \frac{1}{\Delta t^2}\begin{bmatrix} (2 + A^1 \Delta t^2) & -I & & & 0 \\ -I & (2 + A^2\Delta t^2) & -I & & \\ & -I & \cdot & \cdot & \\ & & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot & -I \\ 0 & & & & -I & (2 + A^N\Delta t^2) \end{bmatrix}, \tag{3.5}$$

where I is the identity operator on H. F and G are defined as follows:

$$F = \begin{bmatrix} -f^1 \\ -f^2 \\ \vdots \\ -f^N \end{bmatrix}, \tag{3.6}$$

$$G = \frac{1}{\Delta t^2}\begin{bmatrix} g_1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ g_2 \end{bmatrix}. \tag{3.7}$$

Clearly $V \in H^N$ is in the domain of P if and only if its jth component v^j is in the domain of $A(j\Delta t)$. We now show the following theorem.

THEOREM 1. *P^{-1} exists and is a bounded operator defined on all of H^N; hence* (3.4) *has a unique solution V for arbitrary $F, G \in H^N$. Moreover, one can solve* (3.4) *by Gaussian elimination.*

Remark. We show later that in fact P^{-1} is bounded uniformly in Δt as $\Delta t \to 0$.

It is convenient to break up the proof of Theorem 1 into several steps by means of the following lemmas.

LEMMA 1. *Let $\{b_k\}_{k=1}^{\infty}$ be a sequence of nonnegative real numbers such that*

$$b_1 \leqq \frac{1}{2} \quad \text{and} \quad b_{k+1} \leqq \frac{1}{2 - b_k}, \qquad k = 1, 2, \cdots.$$

Then $b_m \leqq m/(m+1)$ for each $m = 1, 2, \cdots$.

Proof. Use induction on m.

LEMMA 2. *Let T and B be linear operators in H with T closed and B bounded. Let T^{-1} exist and be a bounded operator with domain H. Then, if*

$$\|B\|_H \|T^{-1}\|_H < 1, \tag{3.8}$$

$S = T + B$ is closed, invertible, and S^{-1} is bounded with domain H. Moreover,

$$\|S^{-1}\|_H \leqq \frac{\|T^{-1}\|_H}{1 - \|B\|_H \|T^{-1}\|_H} \tag{3.9}$$

and S^{-1} is compact if T^{-1} is.

Proof. See [5, p. 196].

LEMMA 3. *Let $\Lambda_1 = [2/\Delta t^2 + A^1]$ and $\Gamma_1 = -(\Lambda_1)^{-1}$. For each $n = 2, 3, \cdots, N$ define*

$$\Lambda_n = \left[\frac{2}{\Delta t^2} + A^n\right] + \frac{1}{\Delta t^4}\Gamma_{n-1}, \tag{3.10}$$

where

$$\Gamma_n = -(\Lambda_n)^{-1}, \qquad n = 2, 3, \cdots, N. \tag{3.11}$$

Then, for each $n = 1, 2, \cdots, N$, Γ_n exists and is a bounded operator with domain H and

$$\frac{1}{\Delta t^2}\|\Gamma_n\|_H \leqq \frac{n}{n+1}. \tag{3.12}$$

Proof. By our hypothesis (1.1) concerning $A(t)$, $[2/\Delta t^2 + A^n]$ is a closed invertible operator whose inverse has domain H and

$$\left\|\left[\frac{2}{\Delta t^2} + A^n\right]^{-1}\right\|_H \leqq \frac{\Delta t^2}{2}, \qquad n = 1, 2, \cdots, N. \tag{3.13}$$

Consequently

$$\frac{1}{\Delta t^2}\|\Gamma_1\|_H \leqq \frac{1}{2}. \tag{3.14}$$

Now suppose that for a positive integer $k < N$, Γ_k exists and is bounded with

$$\frac{1}{\Delta t^2}\|\Gamma_k\|_H < 2. \tag{3.15}$$

In that case, since then

$$\frac{1}{\Delta t^4}\|\Gamma_k\|_H\left\|\left(\frac{2}{\Delta t^2} + A^{k+1}\right)^{-1}\right\|_H < 1, \tag{3.16}$$

we obtain from Lemma 2 that Γ_{k+1} exists and is a bounded operator on H, and from (3.9),

$$\frac{1}{\Delta t^2}\|\Gamma_{k+1}\|_H \leqq \frac{1}{2 - \|\Gamma_k\|_H/\Delta t^2}. \tag{3.17}$$

Since $\|\Gamma_1\|_H/\Delta t^2 \leqq \frac{1}{2}$, the conclusion of Lemma 3 now follows from Lemma 1.

LEMMA 4. *Let Λ_k, Γ_k be as in Lemma* 3 *and let $Z = \{z^n\}$ be a given vector in H^N. Then $X = \{x^n\} \in H^N$ is a solution of $PX = Z$ if and only if*

$$\begin{aligned} x^N &= w^N, \\ x^{N-1} + \frac{1}{\Delta t^2}\Gamma_{N-1}x^N &= w^{N-1}, \\ &\vdots \\ x^1 + \frac{1}{\Delta t^2}\Gamma_1 x^2 &= w^1, \end{aligned} \tag{3.18}$$

where the $\{w^j\}$ are defined by

$$\begin{aligned} \Lambda_1 w^1 &= z^1, \\ \Lambda_2 w^2 &= z^2 + \frac{1}{\Delta t^2}w^1, \\ &\vdots \\ \Lambda_N w^N &= z^N + \frac{1}{\Delta t^2}w^{N-1}. \end{aligned} \tag{3.19}$$

Proof. The system (3.19) always has a unique solution $\{w^n\}$ since the $\{\Lambda_n\}$ are invertible and their inverses are bounded operators on H. Let X be a solution of $PX = Z$. We shall show that if the $\{w^j\}$ are defined by (3.18) then each w^j is in the domain of Λ_j and the equations of (3.19) are satisfied. Indeed, X is a solution of $PX = Z$ only if each x^j is in the domain of A^j. On the other hand, x^{j+1} is in the domain of Γ_j since Γ_j is defined everywhere. Hence x^j is in the domain of Λ_j. Also $\Gamma_j x^{j+1}$ is in the domain of Λ_j since $\Gamma_j = -(\Lambda^j)^{-1}$, and therefore from (3.18) it follows that w^j is in the domain of Λ_j. To verify (3.19), define

$x^0 = w^0 = x^{N+1} = 0$ and note that, for each $j = 1, \cdots, N$,

$$\begin{aligned}\Lambda_j w^j &= \Lambda_j x^j - \frac{1}{\Delta t^2}x^{j+1} \\ &= \left[\frac{2}{\Delta t^2} + A^j\right]x^j + \frac{1}{\Delta t^4}\Gamma_{j-1}x^j - \frac{1}{\Delta t^2}x^{j+1} \\ &= \left[\frac{2}{\Delta t^2} + A^j\right]x^j - \frac{(x^{j+1} + x^{j-1})}{\Delta t^2} - \frac{w^{j-1}}{\Delta t^2} \\ &= z^j + \frac{w^{j-1}}{\Delta t^2}.\end{aligned}$$

Conversely, let the $\{w^n\}$ be the solution of (3.19) and define the $\{x^j\}$ by (3.18). Then each x^j is seen to be in the domain of Λ_j and hence of A^j. Applying Λ_j to the jth equation in (3.18) for each j now shows that $PX = Z$.

Proof of Theorem 1. Since each $A^n, n = 1, \cdots, N$, is closed in H, it easily follows that P is closed in H^N. On the other hand, Lemma 4 shows that given any $Z \in H^N$ there exists a unique X in the domain of P such that $PX = Z$. Hence P is invertible and the range of P is all of H^N so that P^{-1} is a closed operator on H^N. This implies that P^{-1} is bounded in view of the closed-graph theorem.

THEOREM 2. *In addition to hypothesis* (1.1) *let* $[A(t) + \lambda]^{-1}$ *be compact for* $\operatorname{Re}\lambda > 0$; *then* P^{-1} *is compact on* H^N.

Proof. In that case it follows from Lemma 2 that the Γ_n in Lemma 3 are compact operators on H. Let $\{Z_k\}_{k=1}^{\infty}$ be a bounded sequence in H^N and let

$$X_k = P^{-1}Z_k, \qquad k = 1, 2, \cdots. \tag{3.20}$$

We claim that $\{X_k\}_{k=1}^{\infty}$ contains a convergent subsequence. A glance at the system (3.19) of Lemma 4 shows that there exists a sequence of positive integers $\{k_p\}$ tending to infinity and such that $\{w_{k_p}^n\}$ converges for every n. Using the continuity of the Γ_n in (3.18) it then follows that $\{x_{k_p}^n\}$ converges for each n.

LEMMA 5. *Let* $T = (N + 1)\Delta t$ *and let* $\hat{M}$ *be the tridiagonal* $N \times N$ *matrix of real numbers given by*

$$\hat{M} = \frac{1}{\Delta t^2}\begin{bmatrix} 2 & -1 & & & & 0 \\ -1 & \cdot & \cdot & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & -1 \\ 0 & & & & -1 & 2 \end{bmatrix}.$$

Let $\xi = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y^N \end{bmatrix}$ *be an arbitrary N-component column vector of complex numbers* *and let* ξ^* *be the conjugate transpose of* ξ. *Then with* y^0, y^{N+1} *defined to be zero we have*

$$\xi^*\hat{M}\xi = \sum_{k=0}^{N} \frac{|y^{k+1} - y^k|^2}{\Delta t^2} \geqq \frac{4}{T^2}\xi^*\xi. \tag{3.21}$$

Proof. It is easily verified that the first equality in (3.21) is true. To show that

$$\xi^*\hat{M}\xi \geqq \frac{4}{T^2}\xi^*\xi$$

we need only observe that $\hat{M}$ is real symmetric and that its smallest eigenvalue is given by

$$\lambda_1 = \frac{4}{\Delta t^2}\sin^2\frac{\pi}{2(N+1)}. \tag{3.22}$$

The result now follows from the inequality

$$\frac{\sin^2\theta}{\theta^2} \geqq \frac{4}{\pi^2}, \qquad 0 \leqq \theta \leqq \frac{\pi}{2},$$

on putting $T = (N+1)\Delta t$.

LEMMA 6. *Let* M *be the linear operator on* H^N *defined by the tridiagonal* $N \times N$ *matrix*

$$M = \frac{1}{\Delta t^2}\begin{bmatrix} 2I & -I & & & & \\ -I & \cdot & \cdot & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & & & & \cdot & \cdot & -I \\ & & & & & -I & 2I \end{bmatrix}. \tag{3.23}$$

Let $V \in H^N$ *and let* v^0, v^{N+1} *be the zero vector in* H; *then*

$$(V, MV) = \Delta t \sum_{k=0}^{N} \frac{\|v^{k+1} - v^k\|_H^2}{\Delta t^2} \geqq \frac{4}{T^2}(V, V). \tag{3.24}$$

Proof. It is easily verified that

$$(V, MV) = \Delta t \sum_{k=0}^{N} \frac{\|v^{k+1} - v^k\|_H^2}{\Delta t^2}. \tag{3.25}$$

Since H is separable there exists an orthonormal basis $\{\phi_m\}_{m=1}^{\infty}$ for H. Consider the Fourier expansion of the vector $v^j \in H$ with respect to this basis, i.e., let

$$v^j = \sum_{m=1}^{\infty} c_m^j \phi_m, \qquad j = 0, 1, \cdots, N+1, \tag{3.26}$$

where

$$c_m^j = \langle v^j, \phi_m \rangle. \tag{3.27}$$

Then by Parseval's relation,

$$\|v^{k+1} - v^k\|_H^2 = \sum_{m=1}^{\infty} |c_m^{k+1} - c_m^k|^2, \tag{3.28}$$

and hence

$$\Delta t \sum_{k=0}^{N} \frac{\|v^{k+1} - v^k\|_H^2}{\Delta t^2} = \sum_{m=1}^{\infty} \Delta t \sum_{k=0}^{N} \frac{|c_m^{k+1} - c_m^k|^2}{\Delta t^2}. \tag{3.29}$$

By Lemma 5, we have, for each $m = 1, 2, \cdots,$

$$\Delta t \sum_{k=0}^{N} \frac{|c_m^{k+1} - c_m^k|^2}{\Delta t^2} \geqq \frac{4}{T^2} \Delta t \sum_{k=1}^{N} |c_m^k|^2, \tag{3.30}$$

and thus (3.24) follows upon inserting (3.30) into (3.29).

LEMMA 7. *Let $G \in H^N$ be the vector*

$$G = \frac{1}{\Delta t^2} \begin{bmatrix} g_1 \\ 0 \\ \vdots \\ 0 \\ g_2 \end{bmatrix} \tag{3.31}$$

and let $\{x^n\} \equiv X = P^{-1}G$; then for each $n = 1, 2, \cdots, N$ we have

$$\|x^n\|_H \leqq \frac{n}{N+1} \|g_2\|_H + \frac{N+1-n}{N+1} \|g_1\|_H. \tag{3.32}$$

Proof. By Lemma 4, $PX = G$ if and only if

$$\begin{aligned} &x^N = w^N, \\ &x^{N-1} + \frac{1}{\Delta t^2} \Gamma_{N-1} x^N = w^{N-1}, \\ &\vdots \\ &x^1 + \frac{1}{\Delta t^2} \Gamma_1 x^2 = w^1, \end{aligned} \tag{3.33}$$

where the w^j are given by

$$\begin{aligned} &\Lambda_1 w^1 = \frac{1}{\Delta t^2} g_1, \\ &\Lambda_2 w^2 = 0 + \frac{1}{\Delta t^2} w^1, \\ &\vdots \\ &\Lambda_j w^j = 0 + \frac{1}{\Delta t^2} w^{j-1}, \\ &\vdots \\ &\Lambda_N w^N = \frac{1}{\Delta t^2} g_2 + \frac{1}{\Delta t^2} w^{N-1}. \end{aligned} \tag{3.34}$$

By (3.12), we have $\|\Gamma_k\|_H = \|\Lambda_k^{-1}\|_H \leqq k\Delta t^2/(k+1)$, and hence it follows from (3.34) that

$$\|w^k\|_H \leqq \frac{1}{k+1}\|g_1\|_H, \qquad k = 1, \cdots, N-1, \tag{3.35}$$

$$\|w^N\|_H \leqq \frac{1}{N+1}\|g_1\|_H + \left(\frac{N}{N+1}\right)\|g_2\|_H. \tag{3.36}$$

If these estimates are inserted into (3.33) it follows by induction that, for $k = 0, 1, \cdots, N-1$,

$$\|x^{N-k}\|_H \leqq \frac{k+1}{N+1}\|g_1\|_H + \frac{N-k}{N+1}\|g_2\|_H, \tag{3.37}$$

which is equivalent to (3.32).

The next theorem gives an a priori estimate which will be used in establishing the existence of weak solutions.

THEOREM 3. *P^{-1} is bounded uniformly as $\Delta t \to 0$; we have*

$$\|P^{-1}\|_{H^N} \leqq \frac{T^2}{4}. \tag{3.38}$$

Moreover, if $V = \{v^n\}$ is the solution of (3.4), *we have, independently of Δt,*

$$\max_n \|v^n\|_H \leqq \frac{T^{3/2}}{2}\|F\|_{H^N} + \|g_1\|_H + \|g_2\|_H. \tag{3.39}$$

Proof. Let M be the matrix (3.23) of Lemma 6; then for W in the domain of P we have

$$(PW, W) = (MW, W) + \Delta t \sum_{n=1}^{N} \langle A^n w^n, w^n\rangle. \tag{3.40}$$

By (1.2),

$$\operatorname{Re}\langle A^n w^n, w^n\rangle \geqq 0 \tag{3.41}$$

and hence

$$\operatorname{Re}(PW, W) \geqq \operatorname{Re}(MW, W) = (MW, W) \geqq \frac{4}{T^2}(W, W) \tag{3.42}$$

on using (3.24) since (MW, W) is real. Hence

$$|(PW, W)| \geqq \operatorname{Re}(PW, W) \geqq \frac{4}{T^2}(W, W) \tag{3.43}$$

and so

$$\|P^{-1}\|_{H^N} \leqq \frac{T^2}{4}. \tag{3.44}$$

Now let $PV = G + F$ with F and G given by (3.6), (3.7). Then, defining

$$W = V - P^{-1}G, \tag{3.45}$$

we have

$$PW = F \tag{3.46}$$

and

$$\|W\|_{H^N} \leqq \frac{T^2}{4}\|F\|_{H^N}. \tag{3.47}$$

On the other hand, since

$$\|W\|_{H^N}\|PW\|_{H^N} \geqq |(PW, W)| \geqq (MW, W), \tag{3.48}$$

it follows from (3.24), (3.46) and (3.47) that

$$\Delta t \sum_{k=0}^{N} \frac{\|w^{k+1} - w^k\|_H^2}{\Delta t^2} \leqq \frac{T^2}{4}\|F\|_{H^N}^2. \tag{3.49}$$

Now for any n with $1 \leqq n \leqq N$, we have

$$w^n = \Delta t \sum_{k=0}^{n-1} \frac{(w^{k+1} - w^k)}{\Delta t} \tag{3.50}$$

and, on using Schwarz's inequality,

$$\begin{aligned} \|w^n\|_H &\leqq \Delta t \sum_{k=0}^{n-1} \frac{\|w^{k+1} - w^k\|_H}{\Delta t} \\ &\leqq \left\{\Delta t \sum_{k=0}^{N} 1\right\}^{1/2} \left\{\Delta t \sum_{k=0}^{N} \frac{\|w^{k+1} - w^k\|_H^2}{\Delta t^2}\right\}^{1/2} \end{aligned} \tag{3.51}$$

so that

$$\max_n \|w^n\|_H \leqq \frac{T^{3/2}}{2}\|F\|_{H^N}. \tag{3.52}$$

Since $V = W + P^{-1}G$, we have from Lemma 7,

$$\max_n \|v^n\|_H \leqq \tfrac{1}{2}T^{3/2}\|F\|_{H^N} + \|g_1\|_H + \|g_2\|_H. \tag{3.53}$$

THEOREM 4 (Corollary). *Let the problem* (3.2) *have a solution $u(t)$ smooth enough that* (3.3) *is satisfied, and let $U \in H^N$ be the vector $\{u^n\}_{n=1}^N$. Let V be the unique solution of* (3.4). *Then with K defined as in* (3.3),

$$\max_n \|v^n - u^n\|_H \leqq \tfrac{1}{2}KT^{3/2}\Delta t^2. \tag{3.54}$$

Proof. Let $W = V - U$; then with $\tau = \{\tau^n\}_{n=1}^N$ we have

$$PW = \tau, \tag{3.55}$$

and the conclusion follows from (3.3) and the a priori estimate (3.52) for the solution of (3.46).

4. Weak solutions of the linear problem. In this section we demonstrate the existence of solutions to a weak two-point problem associated with the equation $u_{tt} - A(t)u = f(t)$, $0 < t < T$, making use of the a priori estimates of § 3. We assume $f(t)$ to be a continuous function from $[0, T]$ to H, and we phrase our regularity assumptions on $A(t)$ in terms of the smoothness in t of the adjoint family $A^*(t)$. In posing the weak problem we follow Lions [10, Chap. I]. It will be necessary to operate within the framework of an L^2-space of H-valued functions on $[0, T]$. We summarize below certain results that will be needed referring the reader to [4] and [10] for details and proofs. Since H is assumed to be separable, weak and strong measurability are equivalent.

DEFINITION. Let $1 \leqq p < \infty$. $L^p(0, T; H)$ is the vector space of weakly measurable functions $f(t)$ from $[0, T]$ to H (i.e., for every $h \in H$, the scalar function $\langle f(t), h \rangle$ is Lebesgue measurable on $[0, T]$) for which

$$\int_0^T \|f(t)\|_H^p \, dt < \infty. \tag{4.1}$$

$L^p(0, T; H)$ is a Banach space under the norm

$$|||f|||_p = \left\{ \int_0^T \|f(t)\|_H^p \, dt \right\}^{1/p}, \tag{4.2}$$

and $L^2(0, T; H)$ is a Hilbert space with scalar product

$$[f, g] = \int_0^T \langle f(t), g(t) \rangle \, dt. \tag{4.3}$$

$L^\infty(0, T; H)$ is the space of weakly measurable functions for which $\|f(t)\|_H$ is bounded except on a set of measure zero. It is a Banach space with the norm

$$|||f|||_\infty = \operatorname*{ess\,sup}_{t \in [0,T]} \|f(t)\|_H. \tag{4.4}$$

LEMMA 8. *Let $1 < p < \infty$. Then $L^p(0, T; H)$ is reflexive and its conjugate space is $L^q(0, T; H)$, where $1/p + 1/q = 1$.*

Proof. See [11, Theorem 5.7].

DEFINITION (Weak L^2-derivatives). Let $u \in L^2(0, T; H)$; then $u' = du/dt$ exists and belongs to $L^2(0, T; H)$ if and only if there exists a (unique) $v \in L^2(0, T; H)$ such that

$$\int_0^T \langle u(t), \psi'(t) \rangle \, dt = - \int_0^T \langle v(t), \psi(t) \rangle \, dt \tag{4.5}$$

for every strongly once continuously differentiable function $\psi(t)$ from $[0, T]$ to H such that $\psi(0) = \psi(T) = 0$.

LEMMA 9. *Let u and u' belong to $L^2(0, T; H)$; then (if necessary after modification on a set of measure zero) u is a continuous function from $[0, T]$ to H.*

Proof. See [10, Chap. I].

The weak problem. Let Ω be the family of functions $\phi(t)$ from $[0, T]$ which satisfy the following conditions:

(a) $\phi(t)$ is four times strongly continuously differentiable from $[0, T]$ to H;
(b) $\phi(0) = \phi(T) = 0$;
(c) for each $t \in [0, T]$, $\phi(t)$ belongs to the domain of $A^*(t)$;
(d) the function $A^*(t)\phi(t)$ is continuous from $[0, T]$ to H.

Without any assumptions on $A(t)$, Ω need contain only the identically zero function. We assume that the family $\{A^*(t)\}$ is smooth enough that Ω is dense in $L^2(0, T; H)$ and we pose the following problem.

Given $f(t)$, continuous from $[0, T]$ to H, and given g_1, g_2 arbitrary vectors in H, find a function $u(t) \in L^2(0, T; H)$ such that

$$\int_0^T \langle u(t), \phi''(t)\rangle\, dt - \int_0^T \langle u(t), A^*\phi(t)\rangle\, dt = \int_0^T \langle f(t), \phi(t)\rangle\, dt + \langle g_2, \phi'(T)\rangle - \langle g_1, \phi'(0)\rangle \tag{4.6}$$

for every $\phi(t) \in \Omega$.

Remark. If we assume further that the collection of functions $\psi(t)$ of the form

$$\psi(t) = \phi''(t) - A^*\phi(t) \quad \text{for } \phi \in \Omega$$

is *dense* in $L^2(0, T; H)$, then clearly there is at most one solution to (4.6).

We now show the following theorem.

THEOREM 5. *There exists a solution to problem* (4.6) *which is a limit as* $\Delta t \to 0$ *of solutions of the finite difference equations* (3.4). *Moreover, this solution belongs to* $L^\infty(0, T; H)$ *and satisfies the estimate*

$$|||u|||_\infty \leqq 2\left(\frac{T^{3/2}}{2} |||f|||_2 + \|g_1\|_H + \|g_2\|_H\right). \tag{4.7}$$

Proof. We need to refer to § 3. Let $\{v^k\}_{k=1}^N \equiv V \in H^N(\Delta t)$ be the unique solution of (3.4) and define the *continuous* function $v(t; \Delta t)$ by means of

$$v(t; \Delta t) \equiv v^k + \frac{(t - k\Delta t)}{\Delta t}[v^{k+1} - v^k], \qquad k\Delta t \leqq t \leqq (k+1)\Delta t, \tag{4.8}$$

for $k = 0, 1, \cdots, N + 1$ with v^0, v^{N+1} respectively defined to be g_1 and g_2.

For each $\Delta t > 0$, clearly $v(t; \Delta t) \in L^2(0, T; H)$; and, in fact, using (3.39) in (4.8), we have

$$\begin{aligned} \max_{t\in[0,T]} \|v(t; \Delta t)\|_H &\leqq \max_k \{\|v^k\|_H + \|v^{k+1}\|_H\} \\ &\leqq 2\left(\frac{T^{3/2}}{2} \|F\|_{H^N} + \|g_1\|_H + \|g_2\|_H\right), \end{aligned} \tag{4.9}$$

where F is defined in (3.6).

Moreover, since $\|f(t)\|_H$ is continuous on $[0, T]$,

$$\int_0^T \|f(t)\|_H^2\, dt = \|F\|_{H^N}^2 + o(1) \quad \text{as } \Delta t \to 0 \tag{4.10}$$

(weaker assumptions on $f(t)$ would clearly suffice for (4.10)), so that for all sufficiently small Δt,

$$\|F\|_{H^N(\Delta t)} \leqq \text{const.} \tag{4.11}$$

Hence the family of continuous functions $\{v(t; \Delta t)\}$, indexed by Δt, is uniformly bounded in $L^\infty(0, T; H)$ and we have

$$|||v(\Delta t)|||_\infty \leqq 2\left(\frac{T^{3/2}}{2} |||f|||_2 + \|g_1\|_H + \|g_2\|_H\right) + o(1). \tag{4.12}$$

By Lemma 8, each $L^p(0, T; H)$, $1 < p < \infty$, is a reflexive Banach space and hence [3, p. 68] the family $\{v(t; \Delta t)\}$ is weakly compact in any given $L^p(0, T; H)$. Choosing $p = 2$, it follows that there exists a sequence $\{\Delta t_m\}$ tending to zero as $m \to \infty$ such that

$$v(t; \Delta t_m) \xrightarrow[\text{weakly}]{} w(t) \in L^2(0, T; H). \tag{4.13}$$

Next fix a $p_0 > 2$. Then there exists a subsequence $\{\Delta t_{m_k}\}$ of $\{\Delta t_m\}$ such that

$$v(t; \Delta t_{m_k}) \xrightarrow[\text{weakly}]{} h(t) \in L^{p_0}(0, T; H); \tag{4.14}$$

it then follows that $h(t) = w(t)$. Indeed, since the conjugate of $L^{p_0}(0, T; H)$ is $L^{q^0}(0, T; H)$, where $q_0 < 2$, and since $L^2 \subset L^{q^0}$, it follows that

$$v(t; \Delta t_{m_k}) \xrightarrow[\text{weakly}]{} h(t) \in L^2(0, T; H) \tag{4.15}$$

and therefore $h(t) = w(t)$. Moreover, the entire sequence $v(t; \Delta t_m)$ converges weakly to $w(t)$ in each $L^p(0, T; H)$ with $2 \leqq p < \infty$. Furthermore, according to [3, p. 68],

$$\begin{aligned} |||w|||_p &\leqq \liminf_{m \to \infty} |||v(\Delta t_m)|||_p \\ &\leqq 2T^{1/p}\left[\frac{T^{3/2}}{2} |||f|||_2 + \|g_1\|_H + \|g_2\|_H\right] \end{aligned} \tag{4.16}$$

on using (4.12). Since $|||w|||_\infty = \lim_{p \to \infty} |||w|||_p$, it follows that $w(t) \in L^\infty(0, T; H)$ and

$$|||w|||_\infty \leqq 2\left[\frac{T^{3/2}}{2} |||f|||_2 + \|g_1\|_H + \|g_2\|_H\right]. \tag{4.17}$$

We now show that $w(t)$ satisfies (4.6) for every $\phi(t) \in \Omega$. Write $v_m(t)$ for $v(t; \Delta t_m)$. By (4.13),

$$\int_0^T \langle w(t), \phi''(t)\rangle \, dt = \int_0^T \langle v_m(t), \phi''(t)\rangle \, dt + o(1) \tag{4.18}$$

as $\Delta t_m \to 0$; and since the integrand on the right of (4.18) is continuous on $[0, T]$,

$$\int_0^T \langle v_m(t), \phi''(t)\rangle \, dt = (V_m, \Phi'') + o(1), \tag{4.19}$$

where V_m, Φ'' denote the vectors $\in H^N(\Delta t_m)$ obtained by evaluating the continuous functions $v_m(t), \phi''(t)$ at the "mesh points" $k\Delta t_m$, $k = 1, 2, \cdots, N$, with $T = (N + 1)\Delta t_m$. As in § 3, $(\cdot, \cdot)$ denotes the scalar product in $H^N(\Delta t_m)$.

Now, by hypothesis, $\phi(t)$ has four strong continuous derivatives on $[0, T]$, and hence

$$(V_m, \Phi'') = -(V_m, M\Phi) + o(1), \tag{4.20}$$

where M is the matrix of (3.23).

Furthermore, by our construction, V_m is the unique solution in $H^N(\Delta t_m)$ of

$$PV = F + G, \tag{4.21}$$

where P is the matrix in (3.5).

Since $\phi(t) \in D_{A^*}(t)$, we have

$$(4.22)\qquad \begin{aligned}(PV_m, \phi) &= (V_m, P^*\Phi)\\ &= (V_m, M\Phi) + \Delta t_m \sum_{k=1}^{N} \langle v_m^k, A^{*k}\phi^k\rangle;\end{aligned}$$

and, since the scalar function $\langle v_m(t), A^*(t)\phi(t)\rangle$ is continuous on $[0, T]$,

$$(4.23)\qquad (PV_m, \Phi) = (V_m, M\Phi) + \int_0^T \langle v_m(t), A^*(t)\phi(t)\rangle\, dt + o(1).$$

Putting together (4.19), (4.20), (4.21) and (4.22) we now see that

$$(4.24)\ \int_0^T \langle v_m(t), \phi''(t)\rangle\, dt = \int_0^T \langle v_m(t), A^*(t)\phi(t)\rangle\, dt + o(1) + (F, \Phi) - (G, \Phi).$$

Finally,

$$(4.25)\qquad \begin{aligned}(G, \Phi) &= \left\langle g_1, \frac{\phi(\Delta t_m)}{\Delta t_m}\right\rangle + \left\langle g_2, \frac{\phi(T - \Delta t_m)}{\Delta t_m}\right\rangle\\ &= \left\langle g_1, \frac{\phi(\Delta t_m) - \phi(0)}{\Delta t_m}\right\rangle - \left\langle g_2, \frac{\phi(T) - \phi(T - \Delta t_m)}{\Delta t_m}\right\rangle\end{aligned}$$

since $\phi(T) = \phi(0) = 0$ for $\phi \in \Omega$. On using the continuity of $f(t)$ it follows from (4.24), (4.25) that

$$(4.26)\qquad \begin{aligned}&\int_0^T \langle v_m(t), \phi''(t)\rangle\, dt - \int_0^T \langle v_m(t), A^*\phi(t)\rangle\, dt\\ &\qquad = \int_0^T \langle f(t), \phi(t)\rangle\, dt - \langle g_1, \phi'(0)\rangle + \langle g_2, \phi'(T)\rangle\end{aligned}$$

and the theorem follows by letting $m \to \infty$ in (4.26).

Remark 1. In the case where we have uniqueness in the weak problem, the entire family $\{v(t; \Delta t)\}$ defined by (4.8) must converge and we may dispense with the selection of subsequences. In that case Theorem 5 and the results of § 3 amount to a "constructive" proof of existence.

Remark 2. If $g_1 = g_2 = 0$, then the unique solution of (3.4) satisfies the estimate (3.49) for the solution of (3.46). In that case the family $\{v(t; \Delta t)\}$ of (4.8) has a weak derivative in $L^2(0, T; H)$, bounded uniformly in Δt in the $L^2(0, T; H)$-norm. It follows that a $w(t)$ can be obtained as in (4.13) with $w(t)$ having a weak derivative in $L^2(0, T; H)$. Then, using Lemma 9, $w(t)$ is a continuous function on $[0, T]$, which satisfies the estimate (4.7) and equation (4.6) with $g_1 = g_2 = 0$.

5. The nonlinear problem. We consider now the problem $u_{tt} - A(t)u = f(t, u(t))$ under the following assumptions:

(a) in addition to (1.1), $[A(t) + \lambda]^{-1}$ is compact for $\operatorname{Re}\lambda > 0$;

(b) $f(t, w)$ is a continuous function from $[0, T] \times H$ into H;

(c) $f(t, w)$ is monotone, i.e., for all $u, v \in H$, $t \in [0, T]$,

$$(5.1)\qquad \operatorname{Re}\langle f(t, u) - f(t, v), u - v\rangle \geqq 0;$$

(d) if $\{v_m(t)\}$ is a sequence of continuous functions from $[0, T]$ to H such that $v_m(t) \xrightarrow[\text{weakly}]{} v(t)$ in $L^2(0, T; H)$, then also $f(t, v_m(t)) \xrightarrow[\text{weakly}]{} f(t, v(t))$ in $L^2(0, T; H)$;

(e) given any $\rho > 0$, there exists a continuous, monotone in the sense of (c) above, and *bounded* function $f_\rho(t, w)$ from $[0, T] \times H$ into H such that

$$f_\rho(t, v(t)) = f(t, v(t)) \tag{5.2}$$

whenever $v(t)$ is a continuous function from $[0, T]$ to H satisfying

$$\left\{\int_0^T \|v(t)\|_H^2 \, dt\right\}^{1/2} \leqq \rho. \tag{5.3}$$

In the notation of § 3, let V be a vector $\{v^n\} \in H^N(\Delta t)$ and define $F(V)$ by

$$F(V) = -\begin{bmatrix} f^1(v^1) \\ f^2(v^2) \\ \vdots \\ f^N(v^N) \end{bmatrix}, \tag{5.4}$$

where $f^n(v^n)$ denotes $f(n\Delta t, v^n)$.

As a discrete analogue of the problem

$$\begin{aligned} u_{tt} - A(t)u &= f(t, u(t)), \qquad 0 < t < T, \\ u(0) &= g_1, \qquad u(T) = g_2, \end{aligned} \tag{5.5}$$

we consider the equation

$$PV = F(V) + G \tag{5.6}$$

with P and G as in (3.5), (3.7).

Using only the monotonicity of $f(t, w)$ we have the following lemma.

LEMMA 10. *There is at most one solution to the discrete problem* (5.6).

Proof. Let U, V be any two solutions. Then,

$$(PU - PV, U - V) = (F(U) - F(V), U - V). \tag{5.7}$$

From Lemma 6, it then follows that

$$\frac{4}{T^2}\|U - V\|_{H^N}^2 \leqq \operatorname{Re}(F(U) - F(V), U - V) \leqq 0, \tag{5.8}$$

and hence $U = V$.

LEMMA 11. *Let $Q(W)$ be a continuous function from $H^N(\Delta t)$ into itself and let $Q(W)$ be bounded, i.e., let there exist $M_0 < \infty$ such that*

$$\|Q(W)\|_{H^N} \leqq M_0 \quad \textit{for all } W \in H^N(\Delta t); \tag{5.9}$$

then there exists a solution to

$$PV = Q(V). \tag{5.10}$$

Proof. Let $\Gamma \subset H^N$ be the closed ball

$$\Gamma = \left\{W \in H^N \,|\, \|W\|_{H^N} \leqq \frac{T^2}{4} M_0\right\}, \tag{5.11}$$

and let $\phi : H^N \to H^N$ be the map

$$\phi(V) = P^{-1}Q(V). \tag{5.12}$$

By Theorem 2, P^{-1} is compact, and by Theorem 3, $\|P^{-1}\|_{H^N} \leqq T^2/4$. Hence ϕ is a continuous function which maps the closed convex set Γ into a precompact subset $\Gamma_0 \subset \Gamma$, and so ϕ has a fixed point by the Schauder fixed-point theorem.

THEOREM 6. *There exists a unique solution to the discrete problem*

$$PV = F(V) + G. \tag{5.13}$$

Moreover,

$$\max_n \|v^n\|_H \leqq \left\{\frac{T^{3/2}}{2}\|F(P^{-1}G)\|_{H^N} + \|g_1\|_H + \|g_2\|_H\right\}. \tag{5.14}$$

Proof. Let

$$\rho \geqq \left\{\frac{T^2}{4}\|F(P^{-1}G)\|_{H^N} + \|P^{-1}G\|_{H^N}\right\}$$

and let $S_\rho \subset H^N(\Delta t)$ be the ball

$$S_\rho = \{Z \in H^N | \, \|Z\|_{H^N} \leqq \rho\}. \tag{5.15}$$

For any given $Z \in S_\rho$, let $z(t)$ be the continuous function on $[0, T]$ obtained from Z by linear interpolation of the components $\{z^n\}$ of Z. For fixed Δt sufficiently small,

$$\left\{\int_0^T \|z(t)\|_H^2 \, dt\right\}^{1/2} = \|Z\|_{H^N} + o(1) \leqq \rho_1. \tag{5.16}$$

By hypothesis (e) there exists a bounded monotone and continuous function $f_{\rho_1}(t, w)$ such that

$$f_{\rho_1}(t, z(t)) = f(t, z(t)) \tag{5.17}$$

for every $z(t)$ satisfying (5.16).

Hence if $F_{\rho_1}(W)$ is the vector in H^N whose components are $f_{\rho_1}(n\Delta t, w^n)$, $n = 1, \cdots, N$, we have that F_{ρ_1} is a bounded continuous and monotone function from $H^N(\Delta t)$ into itself. Moreover,

$$F_{\rho_1}(Z) = F(Z) \quad \text{for all } Z \in S_\rho. \tag{5.18}$$

Define $\tilde{F}_{\rho_1}(W) = F_{\rho_1}(W + P^{-1}G)$ for every $W \in H^N$. Then from Lemma 11 it follows that there exists a solution W_0 to the equation

$$PW = \tilde{F}_{\rho_1}(W) = F_{\rho_1}(W + P^{-1}G). \tag{5.19}$$

We claim that W_0 is also a solution of

$$PW = F(W + P^{-1}G). \tag{5.20}$$

Indeed,

$$\begin{aligned} (PW_0, W_0) &= (F_{\rho_1}(W_0 + P^{-1}G), W_0) \\ &= (F_{\rho_1}(W_0 + P^{-1}G) - F_{\rho_1}(P^{-1}G), W_0) + (F_{\rho_1}(P^{-1}G), W_0). \end{aligned} \tag{5.21}$$

Hence taking the real part of both sides of (5.21) and using Lemma 6 and the monotonicity of F_{ρ_1} we obtain

$$\frac{4}{T^2}\|W_0\|_{H^N}^2 \leqq \|F_{\rho_1}(P^{-1}G)\|_{H^N}\|W_0\|_{H^N}, \tag{5.22}$$

i.e.,

$$\|W_0\|_{H^N} \leqq \frac{T^2}{4}\|F_{\rho_1}(P^{-1}G)\|_{H^N}. \tag{5.23}$$

Since $\|P^{-1}G\|_{H^N} \leqq \rho$, we have $F_{\rho_1}(P^{-1}G) = F(P^{-1}G)$; furthermore, from (5.23),

$$\|W_0 + P^{-1}G\|_{H^N} \leqq \left\{\frac{T^2}{4}\|F(P^{-1}G)\|_{H^N} + \|P^{-1}G\|_{H^N}\right\} \leqq \rho, \tag{5.24}$$

and hence $F_{\rho_1}(W_0 + P^{-1}G) = F(W_0 + P^{-1}G)$. Therefore

$$PW_0 = F(W_0 + P^{-1}G). \tag{5.25}$$

Now define $V_0 = W_0 + P^{-1}G$. Then V_0 is in the domain of P and, from (5.25),

$$PV_0 = F(V_0) + G. \tag{5.26}$$

This proves the existence of a solution to the discrete problem; the uniqueness follows from Lemma 10. We now establish the estimate (5.14). First consider the vector W_0 defined above. We have, from (5.23),

$$\|W_0\|_{H^N} \leqq \frac{T^2}{4}\|F(P^{-1}G)\|_{H^N}. \tag{5.27}$$

Since by Lemma 6,

$$\Delta t \sum_{k=0}^{N} \frac{\|w_0^{k+1} - w_0^k\|_H^2}{\Delta t^2} \leqq \operatorname{Re}(PW, W), \tag{5.28}$$

we have, from (5.21) and (5.27),

$$\Delta t \sum_{k=0}^{N} \frac{\|w_0^{k+1} - w_0^k\|_H^2}{\Delta t^2} \leqq \frac{T^2}{4}\|F(P^{-1}G)\|_{H^N}^2; \tag{5.29}$$

and hence, by the same device as in (3.51),

$$\max_k \|w_0^k\|_H \leqq \frac{T^{3/2}}{2}\|F(P^{-1}G)\|_{H^N}. \tag{5.30}$$

Finally, putting $V = W_0 + P^{-1}G$ and using Lemma 7 we obtain (5.14) from (5.30).

Existence of solutions in the weak problem. Let Ω be the family of functions described in § 4, and consider the following problem:

Find $u(t) \in L^2(0, T; H)$ such that

$$\begin{aligned} &\int_0^T \langle u(t), \phi''(t)\rangle\, dt - \int_0^T \langle u(t), A^*(t)\phi(t)\rangle\, dt \\ &\qquad = \int_0^T \langle f(t, u(t)), \phi(t)\rangle\, dt + \langle g_2, \phi'(T)\rangle - \langle g_1, \phi'(0)\rangle \end{aligned} \tag{5.31}$$

for every $\phi(t) \in \Omega$.

Let $X = P^{-1}G \in H^N(\Delta t)$ and let $\{x^n\}$ be the components of X. Define the continuous function $x(t)$ from $[0, T]$ to H by linear interpolation of the x^n. Then for all sufficiently small Δt,

$$\|F(P^{-1}G)\|_{H^N}^2 = \int_0^T \|f(t, x(t))\|_H^2 \, dt + o(1). \tag{5.32}$$

Put

$$\omega = \left\{\int_0^T \|f(t, x(t))\|_H^2 \, dt\right\}^{1/2} \tag{5.33}$$

and note that ω can be estimated in terms of $\|g_1\|_H$, $\|g_2\|_H$, by Lemma 7, whenever f is given explicitly. Using the a priori estimate (5.14) and hypothesis (d) we have the following theorem, whose proof is essentially the same as that of Theorem 5 of § 4.

THEOREM 7. *There exists a solution to problem* (5.31) *which is a limit as* $\Delta t \to 0$ *of solutions of the finite difference equations* (5.6). *Moreover, this solution belongs to* $L^\infty(0, T; H)$ *and satisfies the estimate*

$$|||u|||_\infty \leqq 2\left(\frac{\omega T^{3/2}}{2} + \|g_1\|_H + \|g_2\|_H\right). \tag{5.34}$$

Convergence to a smooth solution of the strong problem. Let us define a *strong* solution of the problem (5.5) to be a function $u(t)$ which is continuous on $[0, T]$, twice continuously differentiable, with $u(t) \in D_A(t)$ on $(0, T)$, and which satisfies the differential equation and boundary conditions in (5.5). It is easily seen that strong solutions are unique: if $u(t), v(t)$ are any two strong solutions and if $w(t) = u(t) - v(t)$, we then have $w(0) = w(T) = 0$ and

$$w_{tt} - A(t)w = f(t, u) - f(t, v), \tag{5.35}$$

and hence

$$\langle w_{tt}, w\rangle - \langle A(t)w, w\rangle = \langle f(t, u) - f(t, v), u - v\rangle. \tag{5.36}$$

Since $\operatorname{Re}\langle A(t)w, w\rangle \geqq 0$ and $\operatorname{Re}\langle f(t, u) - f(t, v), u - v\rangle \geqq 0$, we have

$$\operatorname{Re}\int_0^T \langle w_{tt}, w\rangle \, dt \geqq 0. \tag{5.37}$$

Hence on integrating by parts,

$$\int_0^T \|w'(t)\|_H^2 \, dt \leqq 0, \tag{5.38}$$

and therefore $w(t) \equiv 0$ since $w(0) = 0$.

Suppose now that (5.5) has a strong solution smooth enough that if U is the vector in $H^N(\Delta t)$ obtained from evaluating $u(t)$ at the mesh points $t = n\Delta t$, $n = 1, \cdots, N$, then

$$PU = F(U) + G + \tau, \tag{5.39}$$

where τ satisfies

$$\|\tau\|_{H^N} \leqq K\Delta t^2 \tag{5.40}$$

for some positive constant K independent of Δt and N. We then have the following convergence theorem, which is analogous to Theorem 4 of § 3.

THEOREM 8. *Let V be the unique solution of* (5.6) *and let $u(t)$ be a solution of* (5.5) *satisfying* (5.40). *Then with K as in* (5.40),

$$\max_n \|v^n - u^n\|_H \leqq \tfrac{1}{2}KT^{3/2}\Delta t^2. \tag{5.41}$$

Proof. We have

$$(PU - PV, U - V) = (F(U) - F(V), U - V) + (\tau, U - V). \tag{5.42}$$

Using Lemma 6 and the monotonicity of $F(W)$, taking real parts in (5.42), we have

$$\frac{4}{T^2}\|U - V\|_{H^N} \leqq \|\tau\|_{H^N}; \tag{5.43}$$

and in the usual way, if $w^k = u^k - v^k$,

$$\Delta t \sum_{k=0}^{N} \frac{\|w^{k+1} - u^{k+1}\|_H^2}{\Delta t^2} \leqq \frac{T^2}{4}\|\tau\|_{H^N}^2. \tag{5.44}$$

The last inequality implies (5.41).

Acknowledgments. It is a pleasure to acknowledge several stimulating conversations with B. Epstein and R. Hersh of the University of New Mexico.

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SIAM J. MATH. ANAL.
Vol. 2, No. 2, May 1971

SOME INEQUALITIES CONCERNING JACOBI POLYNOMIALS*

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Abstract. For polynomials $P(z) = c\prod_{k=1}^{n}(z - a_k)$ with $a_1, a_2, \cdots, a_n$ real, we write $|P(x + iy)|^2 = \sum_{k=0}^{n} L_k(P;x)y^{2k}$, where

$$L_k(P;x) = \sum_{j=0}^{2k} \frac{(-1)^{j+k}}{(2k)!}\binom{2k}{j} P^{(j)}(x)P^{(2k-j)}(x).$$

We show for $k = 1$, 2 and 3 that the functions $L_k(P^{(m)};x)$, $m \geqq 0$, for Jacobi polynomials $P(x) = P_n^{(\alpha,\beta)}(x)$ and their derivatives satisfy the inequalities $L_k(P^{(m)};x) \leqq L_k(P^{(m)};1)$ for $-1 \leqq x \leqq 1$ and $-1 < \alpha = \beta$; $L_k(P^{(m)};x) \leqq L_k(P^{(m)};1)$ for $0 \leqq x \leqq 1$ and $-1 < \beta < \alpha$; and $L_k(P^{(m)};x) \leqq L_k(P^{(m)};-1)$ for $-1 \leqq x \leqq 0$ and $-1 < \alpha < \beta$.

1. Introduction. Suppose $P(x)$ is a polynomial of degree n of the form $c\prod_{k=1}^{n}(x - a_k)$, where $a_1, a_2, a_3, \cdots, a_n$ are real numbers. It can be shown that

$$|P(x + iy)|^2 = \sum_{k=0}^{n} L_k(P;x)y^{2k},$$

where

$$L_k(P;x) = \sum_{j=0}^{2k} \frac{(-1)^{j+k}}{(2k)!}\binom{2k}{j} P^{(j)}(x)P^{(2k-j)}(x)$$

with $P^{(l)}(x)$ denoting the lth derivative of $P(x)$. In this paper we consider $L_k(P;x)$ when $P(x) = P_n^{(\alpha,\beta)}(x)$, $\alpha > -1$, $\beta > -1$, $-1 \leqq x \leqq 1$, is the Jacobi polynomial of degree n. It is a consequence of [15, Theorem 7.32.1] by Szegö that if $-\frac{1}{2} \leqq \beta \leqq \alpha$ then $L_0(P;x) \leqq L_0(P;1)$ for $-1 \leqq x \leqq 1$, and if $-\frac{1}{2} \leqq \alpha \leqq \beta$ then $L_0(P;x) \leqq L_0(P;-1)$ for $-1 \leqq x \leqq 1$. Note that $P^{(0)}(x) = P(x)$ and $L_0(P;x) = [P(x)]^2$. In § 3 we show that in the case $-1 < \alpha = \beta$, $-1 \leqq x \leqq 1$, $L_k(P;x) \leqq L_k(P;1)$ for $k = 1, 2$ and 3. We also show that if $-1 < \beta < \alpha$, $0 \leqq x \leqq 1$, $L_k(P;x) \leqq L_k(P;1)$, and if $-1 < \alpha < \beta$, $-1 \leqq x \leqq 0$, $L_k(P;x) \leqq L_k(P;-1)$ for $k = 1, 2$ and 3.

R. J. Duffin and A. C. Schaeffer [6] showed that the Chebyshev polynomials, $T_n(z)$, satisfy the inequality $|T_n(x + iy)| \leqq |T_n(1 + iy)|$, $-1 \leqq x \leqq 1$, $-\infty < y < \infty$. They then use this inequality to prove W. Markoff's theorem under a weaker hypothesis than that used by Markoff [9]. Markoff's theorem gives the best possible bounds for the higher derivatives of a polynomial $f(x)$ of degree n such that $|f(x)| \leqq 1$ in the interval $(-1, 1)$. Duffin and Schaeffer's proof requires that $f(x)$ be bounded by 1 only at the $n + 1$ points $x = \cos(\nu\pi/n)$, $\nu = 0, 1, 2, \cdots, n$.

The function $L_k(P;x)$ arose while we were attempting to obtain inequalities similar to those for Chebyshev polynomials for other Jacobi polynomials, in particular, for ultraspherical polynomials. In § 2 of this paper we obtain inequalities of the Duffin and Schaeffer type with restricted values of y for polynomials with real roots which are symmetric about the origin.

We note that the function $L_1(P;x) = [P'(x)]^2 - P(x)P''(x)$ and, more generally,

$$L_1(P^{(m)};x) = [P^{(m+1)}(x)]^2 - P^{(m)}(x)P^{(m+2)}(x)$$

have been studied for many different functions. In [2] it is shown that the Laguerre

* Received by the editors December 24, 1969, and in revised form September 1, 1970.

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inequality

$$L_1(P^{(m)}; z) = [P^{(m+1)}(z)]^2 - P^{(m)}(z)P^{(m+2)}(z) > 0, \qquad -\infty < z < \infty, \quad m \geqq 0,$$

holds when $P(z)$ is an entire function of a special type. Skovgaard [13] shows that the Laguerre inequality for a certain type $P(z)$ can be used to show that the Turán inequality $[u_n(x)]^2 - u_{n-1}(x)u_{n+1}(x) \geqq 0$, $n \geqq 1$, holds for functions $u_n = u_n(x)$ which have $P(z)$ as a generating function. The Turán inequality and other Turán-like expressions for various special functions have been studied by Szegö [14]; Karlin and Szegö [8]; Carlitz [3]; Forsythe [7]; Beckenbach, Seidel and Szász [1]; Danese [4], [5]; Szász [11], [12]; Nanjundiah [10]; Thiruvenkatachar and Nanjundiah [16]; and Venkatachaliengar and Rao [17]. Webster [18] develops relations between the Laguerre expression $[P_n'(x)]^2 - P_n(x)P_n''(x)$ and the Turán expression $[P_n(x)]^2 - P_{n-1}(x)P_{n+1}(x)$ for ultraspherical, Hermite, and generalized Laguerre polynomials and the Bessel functions of the first kind.

We show that the Laguerre expression

$$L_1(P^{(m)}; x) = [P^{(m+1)}(x)]^2 - P^{(m)}(x)P^{(m+2)}(x), \qquad m \geqq 0,$$

$$P(x) = P_n^{(\alpha,\beta)}(x)$$

satisfies the inequalities $L_1(P^{(m)}; x) \leqq L_1(P^{(m)}; 1)$ for $-1 \leqq x \leqq 1$ and $-1 < \alpha = \beta$; $L_1(P^{(m)}; x) \leqq L_1(P^{(m)}; 1)$ for $0 \leqq x \leqq 1$ and $-1 < \beta < \alpha$; and $L_1(P^{(m)}; x) \leqq L_1(P^{(m)}; -1)$ for $-1 \leqq x \leqq 0$ and $-1 < \alpha < \beta$. In addition, we show that similar inequalities hold for the functions $L_2(P^{(m)}; x)$ and $L_3(P^{(m)}; x)$.

2. Some properties of polynomials with real roots. In this section we consider polynomials of degree n in the complex plane with n real roots.

A polynomial of this type can be written in the factored form

$$P(z) = c \prod_{k=1}^{n} (z - a_k),$$

where c is a constant and a_k, $k = 1, 2, \cdots, n$, are real and not necessarily distinct.

THEOREM 2.1. *Let $P(z) = c \prod_{k=1}^{n} (z - a_k)$ be a polynomial with real roots only. Then we can write*

$$|P(x + iy)|^2 = \sum_{k=0}^{n} L_k(P; x)y^{2k}, \tag{2.1}$$

where

$$L_k(P; x) = \sum_{j=0}^{2k} \frac{(-1)^{j+k}}{(2k)!} \binom{2k}{j} P^{(j)}(x)P^{(2k-j)}(x). \tag{2.2}$$

Proof. This theorem may be proved by using induction on the degree of P. The details of the proof are omitted here.

THEOREM 2.2. *Let $P(z) = P(x + iy)$ be a polynomial of degree n with n real roots. Then for all x, $L_k(P; x) \geqq 0$, $k = 0, 1, \cdots, n$.*

Proof. We have that

$$P(z) = P(x + iy) = c \prod_{k=1}^{n} (z - a_k),$$

and it follows that

$$|P(x+iy)|^2 = |c|^2 \prod_{k=1}^{n} [(x-a_k)^2 + y^2].$$

This can be written in the form

$$|P(x+iy)|^2 = |c|^2 \sum_{k=0}^{n} C_k(x) y^{2k}, \tag{2.3}$$

where the $C_k(x)$ are products and sums of products of the $(x - a_k)^2$. Therefore each $C_k(x) \geqq 0$ for all x, and since (2.1) and (2.3) are identically equal we have that each $L_k(P; x) \geqq 0$ for all x.

THEOREM 2.3. *Let $P(z)$ be a polynomial of degree n with all real roots which are symmetric about the origin in the interval $-1 \leqq x \leqq 1$. Then for $-1 \leqq x \leqq 1$ and y with $x^2 + y^2 \geqq \max_k \{a_k^2\}$,*

$$|P(x+iy)| \leqq |P(1+iy)|,$$

where a_k, $k = 1, 2, \cdots, l$, are the positive roots (not necessarily distinct) of $P(z)$.

Proof. If $l = 0$, i.e., if $P(z)$ has no positive roots, then $P(z) = cz^n$ and $|P(x+iy)|^2 = c(x^2+y^2)^n$. In this case it is easy to see that the theorem holds, i.e., $|P(x+iy)| \leqq |P(1+iy)|$ for $-1 \leqq x \leqq 1$ and y with $x^2 + y^2 \geqq 0$.

If $l > 0$, then since $P(z)$ has real roots which are symmetric about the origin we can write

$$P(z) = cz^r \prod_{k=1}^{l} (z - a_k)(z + a_k),$$

where $r + 2l = n$ and a_k, $k = 1, 2, \cdots, l$, are the positive roots of $P(z)$ and c is a constant. From this we have, for $z = x + iy$,

$$\begin{aligned} |P(x+iy)|^2 &= |c|^2(x^2+y^2)^r \prod_{k=1}^{l} [(x-a_k)^2 + y^2][(x+a_k)^2 + y^2] \\ &= |c|^2(x^2+y^2)^r \prod_{k=1}^{l} [(x^2 - a_k^2)^2 + 2(x^2 + a_k^2)y^2 + y^4]. \end{aligned}$$

For a fixed $y \neq 0$ let $f(x) = |P(x+iy)|^2$. Then for $0 \leqq x \leqq 1$,

$$\begin{aligned} \frac{f'(x)}{f(x)} &= \frac{2rx}{x^2+y^2} + 4x \sum_{k=1}^{l} \frac{x^2 - a_k^2 + y^2}{[(x-a_k)^2 + y^2][(x+a_k)^2 + y^2]} \\ &\geqq 4x \sum_{k=1}^{l} \frac{x^2 - a_k^2 + y_2}{[(x-a_k)^2 + y^2][(x+a_k)^2 + y^2]}. \end{aligned}$$

Since $f(x) = f(-x)$ we need only consider the functions for $0 \leqq x \leqq 1$. It is easy to see that if $0 \leqq x \leqq 1$ and $x^2 + y^2 \geqq \max_k \{a_k^2\}$ then $f'(x)/(4f(x)) \geqq 0$. But for fixed $y \neq 0$, $f(x) > 0$, which means that $f'(x) \geqq 0$ for $0 \leqq x \leqq 1$ and $x^2 + y^2 \geqq \max_k \{a_k^2\}$. In other words, in this case, $f(x)$ is a nondecreasing function of x in $0 \leqq x \leqq 1$. Since $f(x) = f(-x)$ and $f(x) = |P(x+iy)|^2$, it follows that $|P(x+iy)| \leqq |P(1+iy)|$ for $-1 \leqq x \leqq 1$ and y with $x^2 + y^2 \geqq \max_k \{a_k^2\}$.

3. Some inequalities for functions derived from Jacobi polynomials. We recall some of the basic properties of Jacobi polynomials which are found in [15, Chap. 4].

The Jacobi polynomials $P(x) = P_n^{(\alpha,\beta)}(x)$ are orthogonal on the interval $[-1, 1]$ with respect to the weight function $w(x) = (1 - x)^\alpha(1 + x)^\beta$, where $\alpha > -1$, $\beta > -1$. The normalization of $P_n^{(\alpha,\beta)}(x)$ is effected by

$$P_n^{(\alpha,\beta)}(1) = \binom{n + \alpha}{n}. \tag{3.1}$$

Combining this identity with

$$P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(-x),$$

we have

$$P_n^{(\alpha,\beta)}(-1) = (-1)^n \binom{n + \beta}{n}. \tag{3.2}$$

The Jacobi polynomials satisfy the linear homogeneous differential equation of the second order

$$(1 - x^2)P'' + [\beta - \alpha - (\alpha + \beta + 2)x]P' + n(n + \alpha + \beta + 1)P = 0. \tag{3.3}$$

They also have the property that

$$\frac{d}{dx}(P_n^{(\alpha,\beta)}(x)) = \tfrac{1}{2}(n + \alpha + \beta + 1)P_{n-1}^{(\alpha+1,\beta+1)}(x). \tag{3.4}$$

From this it follows by induction that

$$\begin{aligned} \frac{d^k}{dx^k}(P_n^{(\alpha,\beta)}(x)) = {} & \frac{1}{2^k}(n + \alpha + \beta + 1)(n + \alpha + \beta + 2) \cdots \\ & \cdot (n + \alpha + \beta + k)P_{n-k}^{(\alpha+k,\beta+k)}(x). \end{aligned} \tag{3.5}$$

Writing (3.3) for $P_{n-k}^{(\alpha+k,\beta+k)}(x)$ and substituting from (3.5) we have the following equation for $P(x) = P_n^{(\alpha,\beta)}(x)$:

$$\begin{aligned} & (1 - x^2)P^{(k+2)} + [\beta - \alpha - (\alpha + \beta + 2k + 2)x]P^{(k+1)} \\ & + [n(n + \alpha + \beta + 1) - k(\alpha + \beta + k + 1)]P^{(k)} = 0. \end{aligned} \tag{3.6}$$

We now consider the function

$$L_1(P; x) = [P'(x)]^2 - P(x)P''(x) \tag{3.7}$$

defined by (2.1), where

$$P(x) = P_n^{(\alpha,\beta)}(x), \quad \alpha > -1, \quad \beta > -1, \quad -1 \leqq x \leqq 1,$$

and prove the following theorems.

THEOREM 3.1. *Let* $P(x) = P_n^{(\alpha,\beta)}(x)$, $\alpha = \beta > -1$, $n \geqq 1$, $-1 \leqq x \leqq 1$, *and let* $L_1(P; x) = [P'(x)]^2 - P(x)P''(x)$. *Then* $0 \leqq L_1(P; x) \leqq L_1(P; 1)$.

Proof. For the proof we need the function

$$F_1(x) = L_1(P; x) + a_1(1 - x^2)L_1(P'; x), \tag{3.8}$$

where

$$a_1 = \frac{\alpha + \beta + 2}{(\alpha + \beta + 4)n(n + \alpha + \beta + 1)}, \quad -1 \leqq x \leqq 1, \quad \alpha > -1, \quad \beta > -1,$$

and $L_1(P'; x) = [P''(x)]^2 - P'(x)P'''(x)$. Differentiating (3.8) with respect to x we have

$$\begin{aligned} F_1'(x) = P'P'' - PP''' + a_1(1 - x^2)[P''P''' - P'P^{(4)}] \\ -2a_1x[P'']^2 + 2a_1xP'P'''. \end{aligned} \tag{3.9}$$

Using (3.6) with $k = 1$ and $k = 2$ and substituting for $(1 - x^2)P'''$, $(1 - x^2)P^{(4)}$ and a_1 gives

$$\begin{aligned} F_1'(x) &= \frac{n(n + \alpha + \beta + 1) - (\alpha + \beta + 2)}{n(n + \alpha + \beta + 1)}P'P'' - PP''' \\ &\quad + \frac{(\alpha + \beta + 2)}{(\alpha + \beta + 4)n(n + \alpha + \beta + 1)}[\beta - \alpha - (\alpha + \beta + 4)x]P'P''' \\ &\quad - \frac{(\alpha + \beta + 2)}{(\alpha + \beta + 4)n(n + \alpha + \beta + 1)}[\beta - \alpha - (\alpha + \beta + 2)x][P'']^2. \end{aligned} \tag{3.10}$$

By using (3.3) the second and third terms of (3.10) yield

$$\begin{aligned} \frac{(1 - x^2)P''P'''}{n(n + \alpha + \beta + 1)} + \left\{\frac{\alpha + \beta + 2}{\alpha + \beta + 4}[\beta - \alpha - (\alpha + \beta + 4)x]\right. \\ \left. + [\beta - \alpha - (\alpha + \beta + 2)x]\right\}\frac{P'P'''}{n(n + \alpha + \beta + 1)}. \end{aligned}$$

The first term of this expression can then be combined with the first term of (3.10) by using (3.6) with $k = 1$ to yield

$$\frac{[\alpha - \beta + (\alpha + \beta + 4)x]}{n(n + \alpha + \beta + 1)}[P'']^2.$$

Using these two expressions to rewrite (3.10) we have

$$\begin{aligned} F_1'(x) &= \frac{4x}{(\alpha + \beta + 4)n(n + \alpha + \beta + 1)}L_0(P''; x) \\ &\quad + 2\frac{[(\alpha - \beta)(\alpha + \beta + 3) + (\alpha + \beta + 2)(\alpha + \beta + 4)x]}{(\alpha + \beta + 4)n(n + \alpha + \beta + 1)}L_1(P'; x), \end{aligned} \tag{3.11}$$

where $L_0[P''; x] = [P''(x)]^2$ as defined by (2.2).

When $\alpha = \beta$ we have from (3.11) that

$$\begin{aligned} F_1'(x) &= \frac{2xL_0(P''; x)}{(\alpha + 2)n(n + 2\alpha + 1)} \\ &\quad + \frac{4(\alpha + 1)(\alpha + 2)xL_1(P'; x)}{(\alpha + 2)n(n + 2\alpha + 1)}. \end{aligned} \tag{3.12}$$

Again by Theorem 2.2, $L_0(P'';x) \geqq 0$ and $L_1(P';x) \geqq 0$. Since $\alpha > -1$ and $n \geqq 1$ the coefficients in (3.12) involving α and n are positive. Therefore the sign of $F_1'(x)$ depends only on x. It follows then that for $-1 \leqq x \leqq 0$, $F_1'(x) \leqq 0$, and for $0 \leqq x \leqq 1$, $F_1'(x) \geqq 0$. This means that $F_1(x)$ is a nonincreasing function on $[-1, 0]$ and a nondecreasing function on $[0, 1]$.

By Theorem 2.2, $L_1(P;x) \geqq 0$ and $L_1(P';x) \geqq 0$ for all x since the zeros of Jacobi polynomials are real. With $-1 < \alpha = \beta$ and $n \geqq 1$ it follows that $a_1 > 0$ in (3.8). Therefore, for $-1 \leqq x \leqq 1$, $0 \leqq L_1(P;x) \leqq F_1(x)$ with $L_1(P;\pm 1) = F_1(\pm 1)$. It follows now that $L_1(P;x)$ takes its absolute maximum at $x = 1$ or $x = -1$. Since P is either odd or even, it follows that $L_1(P;1) = L_1(P;-1)$. Hence $L_1(P;x) \leqq L_1(P;1)$ for $-1 \leqq x \leqq 1$ which proves the theorem.

COROLLARY 3.1. *Let* $P(x) = P_n^{(\alpha,\beta)}(x)$, $-1 < \beta = \alpha$, $n \geqq 1$, *and*

$$L_1(P^{(m)};x) = [P^{(m+1)}(x)]^2 - P^{(m)}(x)P^{(m+2)}(x).$$

Then for $-1 \leqq x \leqq 1$ *and* $m = 1, \cdots, n-1$,

$$0 \leqq L_1(P^{(m)};x) \leqq L_1(P^{(m)};1).$$

Proof. From (3.5),

$$\begin{aligned} L_1(P^{(m)};x) &= \frac{1}{2^{2m}}(n+\alpha+\beta+1)^2(n+\alpha+\beta+2)^2 \cdots (n+\alpha+\beta+m)^2 \\ &\quad \cdot \{[\bar{P}'(x)]^2 - \bar{P}(x)\bar{P}''(x)\}, \end{aligned} \tag{3.13}$$

where $\bar{P}(x) = P_{n-m}^{(\alpha+m,\beta+m)}(x)$. Letting $\alpha = \beta$ and applying Theorem 3.1 to $\bar{P}(x)$ we have

$$0 \leqq [\bar{P}'(x)]^2 - \bar{P}(x)\bar{P}''(x) \leqq [\bar{P}'(1)]^2 - \bar{P}(1)\bar{P}''(1).$$

Using this inequality with (3.13) we have $0 \leqq L_1(P^{(m)};x) \leqq L_1(P^{(m)};1)$ which proves the corollary.

THEOREM 3.2. *Let* $P(x) = P_n^{(\alpha,\beta)}(x)$, $-1 < \beta < \alpha$, $n \geqq 1$, *and let* $L_1(P;x)$ *be defined as in Theorem* 3.1. *Then for* $0 \leqq x \leqq 1$, $0 \leqq L_1(P;x) \leqq L_1(P;1)$.

Proof. Consider $F_1(x)$ defined previously. Evidently the terms in (3.11) are nonnegative for $x \geqq 0$, $-1 < \beta < \alpha$. Hence the previous considerations are valid for this interval and the assertion follows.

COROLLARY 3.2. *Let* $P(x) = P_n^{(\alpha,\beta)}(x)$, $-1 < \beta < \alpha$, $n \geqq 1$, *and let* $L_1(P^{(m)};x)$ *be defined as in Corollary* 3.1. *Then for* $0 \leqq x \leqq 1$ *and* $m = 1, \cdots, n-1$,

$$L_1(P^{(m)};x) \leqq L_1(P^{(m)};1).$$

Proof. The proof is identical to that of Corollary 3.1.

THEOREM 3.3. *Let* $P(x) = P_n^{(\alpha,\beta)}(x)$, $-1 < \alpha < \beta$, $n \geqq 1$, *and let* $L_1(P;x)$ *be defined as in Theorem* 3.1. *Then for* $-1 \leqq x \leqq 0$,

$$0 \leqq L_1(P;x) \leqq L_1(P;-1).$$

Proof. This theorem follows directly from Theorem 3.2 since $P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x)$.

We obtain a similar set of theorems and corollaries for the functions

$$L_2(P;x) = \frac{[P''(x)]^2}{(2!)^2} - \frac{2P'(x)P'''(x)}{3!} + \frac{2P(x)P^{(4)}x}{4!} \tag{3.14}$$

and

$$\begin{aligned} L_3(P;x) = {} & \frac{[P'''(x)]^2}{(3!)^2} - \frac{2P''(x)P^{(4)}(x)}{2!4!} \\ & + \frac{2P'(x)P^{(5)}(x)}{5!} - \frac{2P(x)P^{(6)}(x)}{6!} \end{aligned} \tag{3.15}$$

defined by (2.2), where $P(x) = P_n^{(\alpha,\beta)}(x)$, $\alpha > -1$, $\beta > -1$, $-1 \leqq x \leqq 1$. Because of the similarity of the proofs of these theorems and corollaries to the previous ones, we only outline the method of proof.

To obtain the similar set of inequalities for $L_2(P;x)$ we define a new function

$$F_2(x) = L_2(P;x) + a_2(1 - x^2)L_2(P';x),$$

where

$$a_2 = \frac{\alpha + \beta + 2}{(\alpha + \beta + 6)n(n + \alpha + \beta + 1)}$$

with $L_2(P';x)$ obtained by replacing P with P' in (3.14). We then show, using the differential equation (3.6) a number of times, that

$$\begin{aligned} F_2'(x) = {} & \frac{4x}{(\alpha + \beta + 6)n(n + \alpha + \beta + 1)} L_1(P'';x) \\ & + 2\frac{[(\alpha - \beta)(\alpha + \beta + 4) + (\alpha + \beta + 2)(\alpha + \beta + 7)x]}{(\alpha + \beta + 6)n(n + \alpha + \beta + 1)} L_2(P';x), \end{aligned}$$

where $L_1(P'';x)$ is obtained by replacing P by P'' in (3.7). We then use the fact that $L_2(P;x) \geqq 0$, $L_2(P';x) \geqq 0$, $L_1(P'';x) \geqq 0$, $L_2(P;x) \leqq F_2(x)$ and $L_2(P;1) = F_2(1)$ to obtain inequalities for $L_2(P;x)$ similar to those for $L_1(P;x)$.

When considering the function $L_3(P;x)$ given by (3.15) we define the function $F_3(x) = L_3(P;x) + a_3(1 - x^2)L_3(P';x)$, where

$$a_3 = \frac{\alpha + \beta + 2}{(\alpha + \beta + 8)n(n + \alpha + \beta + 1)}$$

with $L_3(P';x)$ obtained by replacing P with P' in (3.15). Again, using the differential equation (3.6) we show that

$$\begin{aligned} F_3'(x) = {} & \frac{4x}{(\alpha + \beta + 8)n(n + \alpha + \beta + 1)} L_2(P'';x) \\ & + \frac{2[(\alpha - \beta)(\alpha + \beta + 5) + (\alpha + \beta + 2)(\alpha + \beta + 10)x]}{(\alpha + \beta + 8)n(n + \alpha + \beta + 1)} L_3(P';x). \end{aligned}$$

As in the other cases, $L_3(P;x) \geqq 0$, $L_3(P';x) \geqq 0$, $L_2(P'';x) \geqq 0$, $L_3(P;x) \leqq F_3(x)$ and $L_3(P;\pm 1) = F_3(\pm 1)$ so we are able to obtain inequalities for $L_3(P;x)$ similar to those for $L_1(P;x)$ and $L_2(P;x)$.

4. Summary. We have shown for $k = 1, 2$ and 3 that the functions $L_k(P^{(m)}; x)$, $m \geqq 0$, defined by (2.2) for Jacobi polynomials $P(x) = P_n^{(\alpha,\beta)}(x)$ and their derivatives, satisfy the inequalities $L_k(P^{(m)}; x) \leqq L_k(P^{(m)}; 1)$ for $-1 \leqq x \leqq 1$ and $-1 < \alpha = \beta$; $L_k(P^{(m)}; x) \leqq L_1(P^{(m)}; 1)$ for $0 \leqq x \leqq 1$ and $-1 < \beta < \alpha$; and $L_k(P^{(m)}; x) \leqq L_k(P^{(m)}; -1)$ for $-1 \leqq x \leqq 0$ and $-1 < \alpha < \beta$. We note that $L_n(P; x)$ is a constant for all polynomials P of degree n, so $L_n(P; x) \leqq L_n(P; 1)$ obviously holds for all x.

We conjecture that similar inequalities hold for $L_k(P; x)$, $k = 4, 5, \cdots, n-1$, defined by (2.2) with $P(x) = P_n^{(\alpha,\beta)}(x)$, $-1 \leqq x \leqq 1$, $-1 < \alpha$, $-1 < \beta$. Our method of proof becomes too unwieldy for $k \geqq 4$.

Acknowledgment. The author is grateful to Professor R. J. Duffin for his guidance and direction in the preparation of the author's doctoral thesis, of which this paper is an excerpt.

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SIAM J. MATH. ANAL.
Vol. 2, No. 2, May 1971

ON THE BOUNDEDNESS AND THE STABILITY OF SOME DIFFERENTIAL EQUATIONS OF THE FOURTH ORDER*

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Abstract. The aim of this paper is to give sufficient conditions (Theorem 1) for the asymptotic stability (in the large) of the trivial solution $x = 0$ of the differential equation

$$D(x) \equiv x^{(4)} + f_1(\ddot{x})\dddot{x} + f_2(\dot{x}, \ddot{x}) + g(\dot{x}) + h(x, \dot{x}) = 0.$$

A result (Theorem 2) on the boundedness of the solutions of the differential equation $D(x) = p(t)$ is also established.

1. Introduction. In this paper we investigate certain fourth order nonhomogeneous differential equations. Ezeilo [2] established results for the equation

$$(1.1) \qquad x^{(4)} + f(\ddot{x})\dddot{x} + a_2\ddot{x} + g(\dot{x}) + a_4x = p(t).$$

Harrow [3], [4], [5] derived interesting results for the problem

$$(1.2) \qquad x^{(4)} + a\dddot{x} + f(\ddot{x}) + g(\dot{x}) + h(x) = p(t).$$

We establish similar results for

$$(1.3) \qquad x^{(4)} + f_1(\ddot{x})\dddot{x} + f_2(\dot{x}, \ddot{x}) + g(\dot{x}) + h(x, \dot{x}) = p(t).$$

The real-valued functions f_1, f_2, g, h and p depend (at most) on the arguments displayed explicitly. The functions $g'(y)$, $\partial f_2(y, z)/\partial y$, $\partial h(x, y)/\partial y$ exist and are continuous along with f_1 and f_2 for all values of their arguments. Moreover, the existence and the uniqueness of the solutions of (1.3) will be assumed. We write

$$\frac{dx}{dt} = \dot{x}, \quad \frac{d^2x}{dt^2} = \ddot{x}, \quad \frac{d^3x}{dt^3} = \dddot{x}, \quad \frac{d^4x}{dt^4} = x^{(4)}.$$

In what follows we use the following system which is equivalent to (1.3):

$$(1.4) \quad \dot{x} = y, \quad \dot{y} = z, \quad \dot{z} = w, \quad \dot{w} = -wf_1(z) - f_2(y, z) - g(y) - h(x, y) + p(t).$$

THEOREM 1. *Assume the following conditions hold*:

(i) $f_2(y, 0) = g(0) = h(0, y) = 0$, *and there exist finite positive constants* a_1, a_2, a_3, a_4, A_4 *and finite nonnegative constants* k_1, k_2, k_3, k_4 *such that*

$$f_1(z) \geqq a_1 + k_1 \quad \text{for all } z,$$

$$f_2(y, z)/z \geqq a_2 + k_2 \quad \text{for all } z \neq 0,$$

$$g(y)/y \geqq a_3 + k_3 + k_4 \quad \text{for all } y \neq 0,$$

$$a_4 \leqq h(x, y)/x \leqq (a_4A_4)^{1/2} \quad \text{for all } x \neq 0;$$

* Received by the editors July 14, 1970, and in revised form September 21, 1970.

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The research of the first author was supported in part by the National Research Council of Canada under Grant A5293 and in part by a special grant from the Principal's Research Fund.

(ii) $\partial f_2(y,z)/\partial y \leqq 0$ *for all* y *and* z, *and for* $y \neq 0$, $z \neq 0$, *the functions* $(1/y)\partial h(x,y)/\partial y$ *and* $f_2(y,z)/z$ *satisfy*

$$[G(x,y,z)]^2 \leqq 4a_3k_2k_4/A_4,$$
$$a_1[f_1(z) - a_1]^2 \leqq 4a_3k_1k_3/A_4,$$

where

$$G(x,y,z) = \frac{f_2(y,z)}{z} - a_2 - \int_0^x \frac{1}{y}\frac{\partial h(s,y)}{\partial y}\,ds - \frac{a_3}{A_4}\frac{\partial h(x,y)}{\partial y};$$

(iii) *a finite constant* Δ_0 *exists such that*

$$\{a_1a_2 - g'(y)\}a_3 - a_1A_4f_1(z) \geqq \Delta_0 > 0 \quad \textit{for all } y \textit{ and } z;$$

(iv) $g'(y) - \{g(y)/y\} \leqq \delta_1$ *for* $y \neq 0$, *where the constant* δ_1 *is such that* $\delta_1 < 2A_4\Delta_0/(a_1a_3^2)$;

(v) $\partial h(x,y)/\partial y = 0$ *at* $y = 0$, $\partial h(x,y)/\partial y < \Delta_0/(3a_3)$ *and* $A_4 - \delta_3 \leqq \partial h(x,y)/\partial x \leqq A_4$ *for all x and y, where*

$$\delta_3 < D_0 = \frac{2\Delta_0}{a_1a_3}\left(\frac{\Delta_0}{2a_1^2a_2a_3} + \frac{1}{a_1}\right)^2.$$

Then every solution of (1.3) *with* $p(t) = 0$ *satisfies*

$$x \to 0, \quad \dot{x} \to 0, \quad \ddot{x} \to 0, \quad \dddot{x} \to 0 \quad \textit{as } t \to \infty. \tag{1.5}$$

Remark. The conditions of the theorem agree with the Routh–Hurwitz criterion [1, p. 21] for asymptotic stability in the large for

$$x^{(4)} + a\dddot{x} + b\ddot{x} + c\dot{x} + dx = 0.$$

THEOREM 2. *If hypotheses* (i) *to* (v) *of Theorem* 1 *hold and if further*

$$\int_0^t |p(s)|\,ds \leqq A < \infty$$

for all $t \geqq 0$, *where A is some positive number, then given any finite numbers* x_0, y_0, z_0, w_0 *there exists a finite constant* $D = D(x_0, y_0, z_0, w_0)$ *such that the solution* $x(t)$ *of* (1.3) *determined by the initial conditions*

$$x(0) = x_0, \quad \dot{x}(0) = y_0, \quad \ddot{x}(0) = z_0, \quad \dddot{x}(0) = w_0$$

satisfies

$$|x(t)| \leqq D, \quad |\dot{x}(t)| \leqq D, \quad |\ddot{x}(t)| \leqq D, \quad |\dddot{x}(t)| \leqq D \tag{1.6}$$

for all $t \geqq 0$.

2. The function $V(x, y, z, w)$**.** The proofs of (1.5) and (1.6) depend on properties of the function V defined by

$$\begin{aligned} 2V(x,y,z,w) = 2d_2 &\int_0^x h(s,y)\,ds + (a_2d_2 - A_4d_1)y^2 \\ &+ 2\int_0^y g(s)\,ds + 2d_1\int_0^z f_2(y,s)\,ds - d_2z^2 \\ &+ 2\int_0^z sf_1(s)\,ds + d_1w^2 + 2h(x,y)y \\ &+ 2d_1h(x,y)z + 2a_1d_2yz + 2d_1g(y)z \\ &+ 2d_2yw + 2zw, \end{aligned} \tag{1.7}$$

where $d_1 = \varepsilon + (1/a_1)$, $d_2 = A_4/a_3$ and ε is a positive constant.

Defining $\gamma(y) = g(y)/y = g'(0)$ for $y = 0$, we obtain the following estimate:

$$\begin{aligned}2V(x,y,z) \geqq &\left\{a_4 d_2 - \frac{h^2(x,y)}{x^2}\frac{1}{\gamma(y)}\right\}x^2 + \{a_2 d_2 - A_4 d_1 - d_2^2 a_1\}y^2 \\ &+2\int_0^y g(s)\,ds - yg(y) + \{a_2 d_1 - d_2 - d_1^2\gamma(y)\}z^2 \\ &+2\int_0^z sf_1(s)\,ds - a_1 z^2 + \left\{d_1 - \frac{1}{a_1}\right\}w^2 + \frac{1}{a_1}\{w + a_1 z + a_1 d_2 y\}^2 \\ &+\frac{1}{\gamma(y)}\left\{\frac{h(x,y)}{x} + y\gamma(y) + d_1 z\gamma(y)\right\}^2.\end{aligned}$$

The following inequalities can be easily checked:

$$a_4 d_2 - h^2(x,y)/\{x^2\gamma(y)\} \geqq 0; \tag{1.8}$$

$$a_2 d_1 - d_2 - d_1^2\gamma(y) > \frac{1}{a_1}\left\{\frac{\Delta_0}{a_1 a_3} - a_1 a_2 \varepsilon\right\} > 0 \tag{1.9}$$

provided that $\varepsilon < \Delta_0/(a_1^2 a_2 a_3)$;

$$\begin{aligned}&\{a_2 d_2 - A_4 d_1 - d_2^2 a_1\}y^2 + 2\int_0^y g(s)\,ds - yg(y) \\ &\geqq \frac{A_4}{a_3}\left\{\frac{\Delta_0}{a_1 a_3} - a_1 a_2 \varepsilon\right\}y^2 + \int_0^y s\left\{\frac{g(s)}{s} - g'(s)\right\}ds \\ &\geqq \left\{\frac{\Delta_0 A_4}{a_1 a_3^2} - \frac{A_4}{a_3}a_1 a_2 \varepsilon - \frac{\delta_1}{2}\right\}y^2.\end{aligned} \tag{1.10}$$

By using the above inequalities it is possible to select a positive constant D_5 such that

$$2V(x,y,z,w) \geqq D_5(y^2 + z^2 + w^2),$$

provided that ε is sufficiently small. Since $V(0,0,0,0) = 0$ and $2V(x,0,0,0) \geqq a_4 d_2 x^2$, it follows that $V(x,y,z,w)$ is positive definite and $V(x,y,z,w) \to \infty$ as $x^2 + y^2 + z^2 + w^2 \to \infty$. We need the following lemma.

LEMMA. *Under the hypotheses* (i)–(v) *of Theorem* 1 *there exists a positive constant* D_6 *such that if* (x,y,z,w) *is any solution of* (1.4) *with* $p(t) = 0$, *then*

$$\dot V \equiv \frac{d}{dt}V(x,y,z,w) \leqq -D_6(z^2 + w^2). \tag{1.11}$$

Proof. A straightforward calculation gives

$$\begin{aligned}\dot V = &\left[\frac{\partial h}{\partial x}y^2 - d_2 g(y)y\right] - \left[A_4 d_1 - d_1\frac{\partial h}{\partial x}\right]yz \\ &+[a_2 d_2 yz - d_2 f_2(y,z)y] + \frac{\partial h}{\partial y}yz \\ &-\left[f_2(y,z)z - \left\{d_1 g'(y) + d_1\frac{\partial h}{\partial y} + a_1 d_2\right\}z^2\right] \\ &-d_2[f_1(z) - a_1]yw - [d_1 f_1(z) - 1]w^2 \\ &+d_2\int_0^x \frac{\partial h(s,y)}{\partial y}z\,ds + d_1\int_0^z \frac{\partial f_2(y,s)}{\partial y}ds.\end{aligned} \tag{1.12}$$

To establish (1.11) we consider four cases.

Case 1. $y = 0$ and $z = 0$. We have

$$\frac{d}{dt}V(x,0,0,w) = -f_1(0)\left[d_1 - \frac{1}{f_1(0)}\right]w^2 \leqq -a_1\varepsilon w^2.$$

Case 2. $y = 0$ and $z \neq 0$. In this case

$$\begin{aligned}\frac{dV}{dt} &\leqq -[a_2 - d_1 g'(0) - d_2 f_1(z)]z^2 - a_1\varepsilon w^2 \\ &\leqq -\left[\frac{\Delta_0}{a_1 a_3} - a_1 a_2 \varepsilon\right]z^2 - a_1 \varepsilon w^2.\end{aligned}$$

Case 3. $z = 0$ and $y \neq 0$. In this case

$$\begin{aligned}\frac{dV}{dt} &= -\left[\frac{d_2 g(y)}{y} - \frac{\partial h}{\partial x}\right]y^2 - d_2[f_1(0) - a_1]yw - [d_1 f_1(0) - 1]w^2 \\ &\leqq -\left[A_4 - \frac{\partial h}{\partial x}\right]y^2 - a_1\varepsilon w^2 - [d_2 k_3 y^2 \\ &\qquad + d_2\{f_1(0) - a_1\}yw + d_1 k_1 w^2] \leqq -a_1 \varepsilon w^2.\end{aligned}$$

Case 4. $y \neq 0$ and $z \neq 0$. In this case

$$\begin{aligned}\frac{dV}{dt} \leqq &-\left[A_4 - \frac{\partial h}{\partial x}\right]\left(y + \frac{d_1}{2}z\right)^2 - a_1\varepsilon w^2 \\ &-[d_2 k_3 y^2 + d_2\{f_1(z) - a_1\}yw + d_1 k_1 w^2] \\ &-[d_2 k_4 y^2 + d_2 G(x,y,z)yz + k_2 z^2] \\ &-\left[a_2 - d_1 g'(y) - a_1 d_2 - \frac{d_1^2}{4}\left(A_4 - \frac{\partial h}{\partial x}\right) - d_1\frac{\partial h}{\partial y}\right]z^2 \\ \leqq &-a_1\varepsilon w^2 - \frac{1}{12}\frac{\Delta_0}{a_1 a_3}z^2.\end{aligned}$$

A choice of D_6 is now obvious.

3. Proofs of Theorem 1 and Theorem 2. By using the function $V(x, y, z, w)$ and the lemma, a proof of Theorem 1 can be modeled on that of Harrow [3]. Alternatively, a standard theorem of La Salle and Lefschetz [6, Theorem 4, p. 58] could be applied.

For a proof of Theorem 2 let (x, y, z, w) be the solution of (1.4) with $x(0) = x_0$, $y(0) = y_0$, $z(0) = z_0$ and $w(0) = w_0$. With respect to the system (1.4) we have with V defined as in (1.7):

$$\frac{dV}{dt} = T + [d_1 w + d_2 y + z]p(t),$$

where T represents the expression on the right-hand side of (1.12). In view of

(1.11) we note that

$$\frac{dV}{dt} \leqq D_6[|w| + |y| + |z|]|p(t)|,$$

where $D_6 = \max(1, d_1, d_2)$. Since $|y| < 1 + y^2$, $|z| < 1 + z^2$ and $|w| < 1 + w^2$, we have

$$\begin{aligned}\frac{dV}{dt} &\leqq D_6(3 + y^2 + z^2 + w^2)|p(t)| \\ &= 3D_6|p(t)| + D_6(y^2 + z^2 + w^2)|p(t)| \\ &\leqq 3D_6|p(t)| + 2\frac{D_6}{D_5}V|p(t)|,\end{aligned}$$

since $2V \geqq D_5(y^2 + z^2 + w^2)$ $(D_5 > 0)$; or $dV/dt - \bar{D}V|p(t)| \leqq \bar{D}|p(t)|$, where $\bar{D} = \max(3D_6, 2D_6/D_5)$. Now the following inequality can be easily obtained:

$$V(t) = V(x(t), y(t), z(t), w(t)) \leqq [V(0) + A\bar{D}]e^{A\bar{D}} < \infty,$$

where $V(0) = V(x(0), y(0), z(0), w(0))$; hence (1.6) follows.

4. Summary. We have given some sufficient conditions for the asymptotic stability (in the large) of the trivial solution $x = 0$ of the differential equation (1.2) with $p(t) = 0$, and for the boundedness of the solutions of (1.2). In each case we have not been able to determine whether these conditions are also necessary.

The results obtained in Theorem 1 and Theorem 2 reduce to results which differ slightly from those obtained by Ezeilo [2] and Harrow [3]. The difference of results lies in the fact that the V functions constructed in each case are not identical. Also, although the corresponding V functions satisfy certain desirable properties which are similar in each case, the routes followed in establishing these properties are somewhat different.

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SIAM J. MATH. ANAL.
Vol. 2, No. 2, May 1971

NONCOMMUTATIVE CONTINUED FRACTIONS*

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Abstract. A number of theorems are proved concerning the convergence of continued fractions whose entries are linear operators on a Banach space. These theorems are analogues of some of the well-known results for ordinary continued fractions.

Introduction. Wynn [1] has discussed formal properties of formal continued fractions whose elements obey a noncommutative law of multiplication. He developed a number of identities, pointed out numerous applications and later gave two theorems concerning convergence of such expressions (see [2]).

This paper contains some new convergence theorems concerning noncommutative continued fractions, some of which are direct generalizations of well-known theorems in ordinary continued fractions. The reader is referred to Wall [3] and Khovanskii [4] and the references therein for these standard theorems.

1. Definitions and notation. We consider the formal expression

$$B_0 + \cfrac{A_1}{B_1 + \cfrac{A_2}{B_2 + \ddots}} = B_0 + \{B_1 + [B_2 + \cdots]^{-1}A_2\}^{-1}A_1, \tag{1.1}$$

where the A_n and B_n are, unless otherwise specified, bounded linear transformations on a Banach space X over the complex numbers. As usual, we denote this Banach space of bounded linear transformations by $[X]$, the identity element of which we denote by I. Lower-case letters always denote complex numbers.

In order to discuss convergence of (1.1) we associate with (1.1) a sequence of partial quotients $\{S_n\}$ defined by

$$S_0 = B_0,$$

$$S_1 = B_0 + B_1^{-1}A_1 = B_1^{-1}(B_1B_0 + A_1),$$

$$S_2 = B_0 + (B_1 + B_2^{-1}A_2)^{-1}A_1 = (B_2B_1 + A_2)^{-1}[(B_2B_1 + A_2)B_0 + B_2A_1],$$

and, in general, S_n is obtained from (1.1) by setting $A_{n+1} = 0$ and rationalizing the following expression. By induction, it can be shown that

$$S_n = Q_n^{-1}P_n,$$

where the expressions Q_n and P_n are computed by the recurrence relations

$$\begin{gathered} P_{n+1} = B_{n+1}P_n + A_{n+1}P_{n-1}, \\ Q_{n+1} = B_{n+1}Q_n + A_{n+1}Q_{n-1}, \\ P_0 = B_0, \quad P_1 = B_1B_0 + A_1; \quad Q_0 = I, \quad Q_1 = B_1. \end{gathered} \tag{1.2}$$

* Received by the editors May 20, 1969, and in final revised form June 10, 1970.

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For future use we record the identity

$$Q_{n+1}^{-1}P_{n+1} - Q_n^{-1}P_n = (-1)^{n+1}Q_{n+1}^{-1}A_{n+1}Q_{n-1}Q_n^{-1}A_nQ_{n-2}\cdots \cdot Q_2^{-1}A_2(B_1^{-1}A_1 - B_0). \tag{1.3}$$

See [1] for further formal properties of (1.1).

DEFINITION 1.1. Continued fraction (1.1) *converges* (A) if Q_n^{-1} exists for all n and the sequence $\{Q_n^{-1}P_n\}$ converges in the norm of $[X]$.

DEFINITION 1.2. Continued fraction (1.1) *converges* (B) if the continued fraction

$$\cfrac{A_n}{B_n + \cfrac{A_{n+1}}{B_{n+1} + \ddots}}$$

converges (A) for some positive integer n. Since B_0 in (1.1) does not affect convergence, we omit it from subsequent discussion.

2. Convergence theory. The first theorem is similar to one of Wynn [2]. Since we need some of the techniques and results of this theorem to prove Theorem 2.2, we outline the proof.

THEOREM 2.1. *In* (1.1) *if* $B_n = I$ *and* $\sum_{n=1}^{\infty} a_n < \infty$, *where* $a_n = \|A_n\|$, *then* (1.1) *converges* (B).

Proof. Without loss of generality, $\sum_{n=1}^{\infty} a_n < m = \ln 2$. Let $s_n = \sum_{i=1}^{n} a_i$. The sequence $\{\|P_n\|\}$ is bounded, for

$$\|P_1\| = \|A_1\| < e^{a_1} = e^{s_1},$$

$$\|P_2\| = \|A_1\| < e^{a_1+a_2} = e^{s_2}.$$

Use of the recurrence relations (1.2) and induction yield

$$\|P_n\| < e^{s_n} < e^m = 2.$$

Similarly, $\{\|Q_n\|\}$ is a bounded sequence. Now $\{P_n\}$ is a Cauchy sequence, for

$$\|P_{n+k} - P_n\| \leqq \sum_{i=1}^{k} \|P_{n+i} - P_{n+i-1}\| = \sum_{i=1}^{k} \|A_{n+i}P_{n+i-2}\|$$

$$\leqq \sum_{i=1}^{k} a_{n+i}\|P_{n+i-2}\| \leqq e^m \sum_{i=1}^{\infty} a_{n+i} < \varepsilon$$

for large n since $\sum_{i=1}^{\infty} a_i$ converges. In the same way, $\{Q_n\}$ is a Cauchy sequence, so there exist P, Q in $[X]$ such that $P_n \to P$ and $Q_n \to Q$.

We now show Q_n^{-1} exists for all n and that Q^{-1} exists. We have $Q_1 = I$, $Q_2 = I + A_2$ and $Q_3 = I + A_2 + A_3$, so that

$$\|Q_2 - I\| = \|A_2\| \leqq e^{s_2} - 1,$$

$$\|Q_3 - I\| \leqq \|A_2\| + \|A_3\| \leqq e^{s_3} - 1.$$

An easy induction argument shows that

$$\|Q_n - I\| \leqq e^{s_n} - 1 < e^m - 1 \leqq 2 - 1 = 1,$$

so that Q_n^{-1} exists for all n. Also, since $Q_n \to Q$,

$$\|Q - I\| = \lim_{n\to\infty} \|Q_n - I\| \leqq \lim_{n\to\infty} (e^{s_n} - 1) < e^m - 1 = 1,$$

and Q^{-1} exists. Consequently, $Q_n^{-1}P_n \to Q^{-1}P$ and (1.1) converges (B).

THEOREM 2.2. *If in* (1.1), $A_n = I$, *then a necessary condition for the convergence* (B) *of* (1.1) *is that* $\sum_{n=1}^{\infty} b_n$ *diverges, where* $b_n = \|B_n\|$.

Proof. Let Q_n^{-1} exist for all n and suppose $\sum_{n=1}^{\infty} b_n < m = \ln 2$. It is an easy induction argument to show that for all n,

$$\|Q_{2n} - I\| \leqq e^{s_{2n}} - 1 \quad \text{and} \quad \|Q_{2n+1}\| \leqq e^{s_{2n+1}}$$

simultaneously, where $s_n = \sum_{i=1}^{n} b_i$. In the same way, $\{\|P_n\|\}$ is a bounded sequence. Just as in Theorem 2.1, $\{\|Q_n^{-1}\|\}$ is a bounded sequence. Let M be a positive number such that $\|Q_n\| < M$ and $\|Q_n^{-1}\| < M$ for all n. By (1.3),

$$D_n = Q_{2n+1}^{-1}P_{2n+1} - Q_{2n}^{-1}P_{2n} = Q_{2n+1}^{-1}Q_{2n-1}Q_{2n}^{-1}Q_{2n-2} \cdots Q_3^{-1}Q_1Q_2^{-1}B_1^{-1}.$$

Thus, since

$$Q_{k-2}^{-1}Q_k = Q_{k-2}^{-1}B_kQ_{k-1} + I, \qquad \|Q_{k-2}^{-1}Q_k\| \leqq 1 + M^2b_k,$$

then

$$\begin{aligned} \|D_n\| \geqq \|D_n^{-1}\|^{-1} &= [\|B_1Q_2Q_1^{-1}Q_3Q_2^{-1}Q^4 \cdots \\ &\qquad Q_{2n-3}^{-1}Q_{2n-1}Q_{2n-2}^{-1}Q_{2n}Q_{2n-1}^{-1}Q_{2n+1}\|]^{-1} \\ &\geqq [b_1(1 + b_1b_2)\|Q_1^{-1}Q_3\|\,\|Q_2^{-1}Q_4\| \cdots \|Q_{2n-2}^{-1}Q_{2n}\|\,\|Q_{2n-1}^{-1}Q_{2n+1}\|]^{-1} \\ &\geqq [b_1(1 + b_1b_2)(1 + M^2b_3)(1 + M^2b_4) \cdots (1 + M^2b_{2n})(1 + M^2b_{2n+1})]^{-1}. \end{aligned}$$

Since $\{s_n\}$ converges, the infinite product

$$\prod_{n=3}^{\infty} (1 + M^2b_n)$$

converges to a finite positive number. Thus, $\lim_{n\to\infty} D_n \neq 0$, so that (1.1) diverges. Hence $\{s_n\}$ must diverge if (1.1) converges.

Notice that the proofs of Theorems 2.1 and 2.2 involve only elementary use of the property of norms in a Banach algebra, so that they are true if the elements of (1.1) are members of a Banach algebra.

In the scalar case (i.e., the A_n and B_n in (1.1) are complex numbers) in which $A_n = 1$ and $B_n = b_n$ are positive numbers, the divergence of $\sum_{n=1}^{\infty} b_n$ is a necessary and sufficient condition for the convergence (A) of (1.1). We remark that even in the case that $A_n = I$ and B_n in (1.1) are positive commuting self-adjoint operators on a Hilbert space, the condition of Theorem 2.2 is not sufficient for the convergence (B) of (1.1). For example, let

$$A_n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad B_n = \begin{pmatrix} 1 & 0 \\ 0 & 1/n^2 \end{pmatrix}$$

be 2×2 matrices. Now $\sum_{n=1}^{\infty} \|B_n\|$ diverges, but since all matrices are diagonal and commutative, the continued fraction (1.1) in this case is just the 2×2 matrix defined by

$$\begin{pmatrix} \cfrac{1}{1+\cfrac{1}{1+\ddots}} & 0 \\ 0 & \cfrac{1}{1+\cfrac{1}{\frac{1}{4}+\cfrac{1}{\frac{1}{9}+\ddots}}} \end{pmatrix}$$

in which the continued fraction on the lower right diverges.

THEOREM 2.3. *In* (1.1) *let* $B_i = I$, $A_i = k_i A$, k_i *complex numbers. Let* $\sigma(A)$ *denote the spectrum of* A.

(i) *If* $\lim_{n\to\infty} k_n = 0$, *continued fraction* (1.1) *converges* (B) *for all* A.

(ii) *If* $\lim_{n\to\infty} k_n = k \neq 0$, *continued fraction* (1.1) *converges* (B) *for all* A *such that* $\sigma(A) \cap S = \emptyset$ *where* S *is the ray* $[-1/(4k), \infty)$ *the direction of which is the direction of the ray* $(0, -1/(4k))$.

(iii) *If* $\limsup_{n\to\infty} |k_n| = k > 0$, *then* (1.1) *converges* (B) *as long as* $\sup\{|\lambda| : \lambda \in \sigma(A)\} < \delta < \frac{1}{4}k$.

Proof. (i) Let $A \in [X]$ and let V be a compact neighborhood of $\sigma(A)$. By [3, p. 138] there is an N such that for all $n \geqq N$,

$$\cfrac{k_n\lambda}{1+\cfrac{k_{n+1}\lambda}{1+\ddots}} \tag{2.1}$$

converges uniformly for $\lambda \in V$. Let $f_m(\lambda)$ be the approximants of (2.1). Then $f_m(\lambda)$ is analytic on V, $m = 1, 2, \cdots$, and $f_m(\lambda)$ converges uniformly on V to an analytic (on V) function, say $f(\lambda)$. Then, by Lemma 14 in [5, p. 271] $f_m(A)$ converges uniformly to $f(A)$ in the uniform topology of operators, and so (1.1) converges (B).

(ii) Let V be a compact neighborhood of $\sigma(A)$ such that $V \cap S = \emptyset$. Again, by [3, p. 138] there is an N such that for $n \geqq N$, (2.1) converges uniformly for $\lambda \in V$. Following the same argument as in (i), (1.1) converges (B).

(iii) There is a compact neighborhood V of $\sigma(A)$ such that $\sup\{|\lambda| : \lambda \in V\} < \rho < \frac{1}{4}k$. There is an N such that for all $n \geqq N$, and $\lambda \in V$, $|k_n\lambda| \leqq \frac{1}{4}k$. Thus, for $n \geqq N$ and $\lambda \in V$, (2.1) converges uniformly by Worpitzky's theorem. Again, use of the argument of (i) insures convergence (B) of (1.1).

The following remarks concern the case where the elements of (1.1) are positive self-adjoint commuting operators on a Hilbert space. We have, from (1.3),

$$Q_n^{-1}P_n - Q_{n-1}^{-1}P_{n-1} = (-1)^n(Q_{n-1}Q_n)^{-1}A_1A_2\cdots A_n, \tag{2.2}$$

and $Q_n^{-1}P_n$ is a positive self-adjoint operator. If we replace n by $2k+1$ in (2.2) and add for $k = 1, 2, \cdots, n$, we get

$$\begin{aligned} Q_{2n+1}^{-1}P_{2n+1} = Q_1^{-1}A_1 &- (Q_1Q_3)^{-1}A_1A_2B_3 - \cdots \\ &-(Q_{2n-1}Q_{2n+1})^{-1}A_1A_2\cdots A_{2n}B_{2n+1}, \end{aligned} \tag{2.3}$$

while the same technique gives

$$Q_{2n}^{-1}P_{2n} = Q_2^{-1}A_1B_2 + (Q_2Q_4)^{-1}A_1A_2A_3B_4 + \cdots + (Q_{2n-2}Q_{2n})^{-1}A_1A_2\cdots A_{2n-1}B_{2n}. \tag{2.4}$$

Thus $Q_{2n}^{-1}P_{2n}$ and $Q_{2n+1}^{-1}P_{2n+1}$ are increasing and decreasing sequences, respectively, and $Q_{2n-1}^{-1}P_{2n-1} > Q_{2n}^{-1}P_{2n}$. Now if (1.1) converges (A) in this case, say to F, then

$$\|F - Q_n^{-1}P_n\| \leqq \|(Q_{n-1}Q_n)^{-1}A_1A_2\cdots A_n\|. \tag{2.5}$$

THEOREM 2.4. *In* (1.1) *let* $A_n = I$ *and* B_n *be commuting positive self-adjoint operators on a Hilbert space such that* $B_n > \delta_n I > 0$ *and* $\sum_{n=1}^{\infty}\delta_n$ *diverges. Then* (1.1) *converges* (A).

Proof. Let $C = B_1$ if $\delta_1 - 1 < 0$ or $C = I$ if $\delta_1 - 1 > 0$. Then

$$Q_1 = B_1 \geqq C,$$

$$Q_2 = B_2B_1 + I \geqq (B_2 + I)C > (1 + \delta_2)C,$$

$$Q_3 = B_3B_2B_1 + B_3 + B_1 > (B_3B_2 + I)C + B_3 > (1 + \delta_3)C,$$

and by induction,

$$Q_{2n} > (1 + \delta_2 + \delta_4 + \cdots + \delta_{2n})C,$$

$$Q_{2n+1} > (1 + \delta_3 + \delta_5 + \cdots + \delta_{2n+1})C,$$

so that

$$Q_{2n}Q_{2n+1} > (1 + \delta_2 + \delta_3 + \cdots + \delta_{2n} + \delta_{2n+1})C^2.$$

Hence

$$(Q_{2n}Q_{2n+1})^{-1} < \frac{C^{-2}}{1 + \delta_2 + \delta_3 + \cdots + \delta_{2n} + \delta_{2n+1}}.$$

Making use of (2.2) and the preceding remarks, $Q_{2n+1}^{-1}P_{2n+1} - Q_{2n}^{-1}P_{2n} \to 0$ so that (1.1) converges (A).

3. Periodic continued fractions. Here we treat a certain periodic continued fraction whose entries are elements of a complex Banach algebra. Consider the quadratic equation

$$Y^2 - YB - A = 0, \tag{3.1}$$

where Y^{-1} exists. Rewriting we get

$$Y = B + Y^{-1}A,$$

and replacing Y on the right-hand side by $B + Y^{-1}A$, formally we have

$$Y = B + \frac{A}{B + Y^{-1}A} = B + \cfrac{A}{B + \cfrac{A}{B + \ddots}}. \tag{3.2}$$

Thus, the periodic continued fraction (3.2) arises quite naturally in the search for solutions of (3.1). Notice that if U is a solution to (3.1) then $U - B$ is a solution to

(3.3) $$Y^2 + BY - A = 0,$$

so that formally

(3.4) $$\cfrac{A}{B + \cfrac{A}{B + \ddots}}$$

is connected with a solution of (3.3). Convergence of either (3.2) or (3.4) implies convergence of the other.

The next theorem is similar to a theorem of McFarland [6], but, since the setting and the proof are different, we include it in this paper.

THEOREM 3.1. *If* (3.2) *converges* (A), *then it converges to a solution of* (3.1).

Proof. Let (3.2) converge (A), $Q_n^{-1}P_n$ be the nth approximant to (3.2) and $S_n^{-1}R_n$ be the nth approximant to (3.4). It is easily verified that $S_{n+1} = P_n$ and $R_{n+1} = Q_nA$, so that

$$\lim_{n\to\infty} Q_n^{-1}P_nS_n^{-1}R_n = \lim_{n\to\infty} Q_n^{-1}P_nS_{n+1}^{-1}R_{n+1} = A.$$

Thus if $Q_n^{-1}P_n \to F$, then $S_n^{-1}R_n \to F - B$; that is, $F(F - B) - A = 0$ or F is a solution to (3.1).

We define the formal transformation

(3.5) $$S = S(W) = \frac{A}{B + W} = (B + W)^{-1}A.$$

Y is called a *fixed point* of (3.5) if $S(Y) = Y$, that is,

$$Y^2 + BY - A = 0.$$

We can form a convergence criterion for (3.4) in terms of the fixed points of (3.5). The proof of the following theorem is patterned after a proof in Wall [3, p. 36]. We assume existence of required inverses, including those of the denominators of the approximants to (3.4).

THEOREM 3.2. *Let there be at least two distinct fixed points of* (3.5) *and let U be one such that $\|V^{-n}U^n\| \to 0$ for any fixed point V of* (3.5), $V \neq U$. *Further, suppose $\|V^{-n}W^n\| \nrightarrow 0$ for any distinct fixed points V, W of* (3.5), $V \neq U \neq W$. *Then* (3.4) *converges* (A) *to* U.

Proof. Let $V \neq U$ be a fixed point of (3.5). Then $\|V^{-n}U^n\| \to 0$ and it is easily verified that if $S = S^n(W)$,

(3.6) $$(S - V)^{-1}(S - U) = V^{-n}(W - V)^{-1}(W - U)U^n.$$

Let $F_n = Q_n^{-1}P_n$ be the nth approximant of (3.4). Then $F_n = S^n(0)$; that is, $S(W)$ is iterated n times and then W is set equal to zero. Thus by (3.6) with $W = 0$,

$$(F_n - V)^{-1}(F_n - U) = V^{-n}(V^{-1}U)U^n = V^{-n-1}U^{n+1}.$$

Since $\|V^{-n}U^n\| \to 0$,

$$F_n - U = (F_n - V)E_n,$$

where $\|F_n\| \to 0$, or

$$F_n(I - E_n) = U - VE_n, \qquad (F_n - U)(I - E_n) - UE_n = -VE_n.$$

Finally, for large n, $F_n - U = (U - V)E_n(I - E_n)$, so that $\|F_n - U\| \to 0$; that is, $F_n = Q_n^{-1}P_n$ converges (A) to U.

Conclusion. The applications of noncommutative continued fractions to problems in analysis, though numerous, have not yet been exhausted. Of course, the success with which these formal expressions may be used depends a great deal upon the state of development of convergence theory of such fractions. Thus the development of a sufficiently general convergence theory is of prime importance at this time.

Acknowledgment. The author wishes to thank the referees for their comments and suggestions.

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SIAM J. MATH. ANAL.
Vol. 2, No. 2, May 1971

GEOMETRIC THEORY OF DIFFERENTIAL EQUATIONS. I: SECOND ORDER LINEAR EQUATIONS*

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Abstract. In the first part of the paper, we develop a geometric theory of the equation $x'' + px = 0$ for p a generalized derivative and, simultaneously, the SL(2) differential geometry of curves with countably many cusps. After a discussion of Borůvka's dispersion function for generalized p we use the dispersion function to obtain majorizations of the Lyapunov integral $(b - a)\int_a^b p\,dt$ on intervals of disconjugacy. As an application we obtain that the lower endpoint λ_1 of the first interval of instability of a Hill equation $x'' + \lambda px = 0$ and period T is bounded above by $\pi^2(T\int_0^T p\,dt)^{-1}$. No other eigenvalue of the stability problem admits a similar upper bound.

In this note, we study several applications of the plane differential geometry of SL(2).

First, we give a geometric theory of the equation

$$x'' + Q'x = 0, \qquad a \leqq t \leqq b, \tag{1}$$

where Q' is the generalized derivative, in the sense of distributions, of a function defined on $a \leqq t \leqq b$. Such equations have been considered at least since 1892 [8]. If Q is of bounded variation, the differential operator defined by the left-hand side of (1) is a special case of the generalized second order operators whose theory has been developed by W. Feller [3]. Our standard assumption will be that Q is right (or left) continuous but not necessarily of bounded variation.

In a second part, we derive some upper bounds (mainly in terms of Borůvka's dispersion function) for

$$(b - a)(Q(b) - Q(a)) = (b - a)\int_a^b Q'\,dt$$

if $a \leqq t \leqq b$ is an interval of disconjugacy of (1). There exists an extensive literature about lower bounds for that expression but very little is known about upper bounds in special situations. We derive a universal upper bound for the first eigenvalue of the stability problem of a Hill equation.

1. We can give a sense to equation (1) for integrable Q by asking that $x(t)$ be absolutely continuous and

$$\int_a^b (D^2x + (DQ)x)f\,dt = 0$$

for all C^∞-functions f defined on $a \leqq t \leqq b$ with $f(a) = f(b) = 0$. Since

$$D(Qx) = (DQ)x + QDx$$

it is easily seen that $y = x' + Qx$ is an a.e. defined function and that (1) is equivalent to

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -Q & 1 \\ -Q^2 & Q \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \quad \text{a.e.} \tag{2}$$

* Received by the editors June 25, 1970.

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By Carathéodory's existence theorem we are assured of the existence and uniqueness of absolutely continuous solutions of (2). The information obtained in this way is not sufficient for many applications. We develop a theory in which (2) will be assumed to hold everywhere in the sense of one-sided derivatives.

2. Let $x(u) = (x_1(u), x_2(u))$ be a vector function

$$x : I \to \mathbb{R}^2 - 0$$

defined on some interval I of the real number line. We assume that $x(u)$ has everywhere a forward derivative x'_+ and a backward derivative x'_- that are bounded in length by an L^1- and L^2-function $m(u)$ and also that the forward derivative is everywhere linearly independent of $x(u)$. We fix an orthonormal frame in R^2 and express x in polar coordinates relative to this base,

$$x(u) = r(u)c(u),$$

where $c(u) = (\cos\alpha(u), \sin\alpha(u))$. The vector $c(u)$ forms an orthonormal base together with $n(u) = (-\sin\alpha(u), \cos\alpha(u))$. By Beppo Levi's theorem, the derivative x' exists except possibly at a countable set of points.

The determinant of two vectors a, b is denoted by $[a, b]$. The scalar product relative to the given basis is denoted by the usual dot.

The one-sided derivatives of $r(u)$ and $\alpha(u)$ exist:

$$r'_\pm = r^{-1} x \cdot x'_\pm,$$

$$\alpha'_\pm = r^{-2}[x, x'_\pm].$$

It follows that r and α are differentiable except at most at a countable set, continuous, and that

$$|r'_\pm| \leqq m(u), \quad |\alpha'_\pm| \leqq r^{-1}(u)m(u).$$

As long as $r(u)$ is bounded away from zero, $\alpha(u)$ is of bounded variation. Therefore, we may define the area function by

$$\begin{aligned} \tfrac{1}{2}t(u) &= \tfrac{1}{2}\int_{u_0}^{u} |[x, x']|\, du + \tfrac{1}{2}t_0 \\ &= \tfrac{1}{2}\int_{u_0}^{u} r^2|\alpha'|\, du + \tfrac{1}{2}t_0. \end{aligned}$$

Then

$$t'_\pm(u) = r^2|\alpha'_\pm(u)|$$

and the polar angle α is differentiable as a function of the parameter t except at points of discontinuity of $\operatorname{sgn}\alpha'_+$:

$$\frac{d\alpha}{dt} = \varepsilon_+ r^{-2}, \quad \varepsilon_+ = \operatorname{sgn}\alpha'_+.$$

From now on we shall use t as a parameter. This is admissible since $(dt/du)_+ > 0$. In particular, the "prime" symbol will be reserved for differentiation with respect to t.

We now add the requirement that ε_+ be constant. The reason for this condition is that the frame which we shall construct for $x(t)$ will be discontinuous at points of discontinuity of ε_+. We choose $\varepsilon_+ = 1$ $(= \varepsilon_-)$ for definiteness.

The definition of t amounts to

$$[x(t), x'_+(t)] = 1, \tag{3}$$

or that the hodograph curve $x'_+(t)$ is the envelope of the lines

$$r^{-1}(t)n(t) + \lambda c(t), \qquad -\infty < \lambda < +\infty.$$

The envelope is the image, in the rotation of center 0 and angle $\pi/2$, of the polar reciprocal of $x(t)$ with respect to the unit circle. If $x'_+(t)$ is differentiable, then $x'_+(t) = x'(t)$ everywhere and (3) implies $[x, x''] = 0$ or

$$x'' + q(t)x = 0,$$

where $q(t)\,dt = [x', x'']\,dt$ is twice the area element of the polar reciprocal (with an appropriate sign). In the general case, a cusp of $x(t)$ will appear as a straight segment on the polar reciprocal. It follows from our formulas that

$$x'_\pm = r'_\pm c + r^{-1}n. \tag{4}$$

The area of the triangle formed by the origin and the straight segment $x'_+x'_-$ is one half of

$$[x'_-(t), x'_+(t)] = \frac{1}{r}(r'_- - r'_+). \tag{5}$$

The differential geometry of curves is dominated by the Frenet equations. These differential equations determine a moving frame by functions ("curvatures") that are invariant under some transformation group. The determination of the curvatures always presupposes an elevated order of differentiability. We want to find a replacement of the Frenet equations of the geometry of SL(2) (area preserving linear transformations in the plane) adapted to our differentiability hypotheses. To this effect we have to find a frame (a column of vectors) that contains $x(t)$ as its first vector, is differentiable at least in the forward sense and whose forward derivatives satisfy an equation depending on integral invariants of x.

We assume without loss of generality that $x(t)$ is differentiable for $t = 0$. Since $x'_+(t)$ is linearly independent of $x(t)$, the second vector of our frame can be sought as

$$y(t) = x'_+(t) + Q(t)x(t).$$

The preceding argument suggests that Q should be the area function of the polar reciprocal, including the area generated by jumps, multiplied by a factor of 2. We write $x'_+(t) = \rho(t)c(\theta(t))$ and should obtain for Q a representation by some kind of Stieltjes integral

$$\begin{aligned} Q(t) &\overset{?}{=} \int_0^t \rho^2(\tau)\,d\theta(\tau) \\ &= \int_0^t (r'^2_+ + r^{-2})\,d(\alpha(\tau) + \operatorname{arc\,cot} rr'_+). \end{aligned}$$

Since our argument here is heuristic, we do not determine the sense in which the latter integral exists, nor do we investigate the validity of integration by parts. We only note that a formal manipulation by partial integration leads to the definition

$$Q(t) = \frac{r'(0)}{r(0)} - \frac{r'_+(t)}{r(t)} + \int_0^t (r^{-4} - r'^2 r^{-2})\, d\tau. \tag{6}$$

With this definition, we have

$$y(t) = r(t)\left\{\frac{r'(0)}{r(0)} + \int_0^t (r^{-4} - r'^2 r^{-2})\, dt\right\} c(t) + r(t)^{-1} n(t)$$

and

$$y'_+(t) = Qx'_+ = -Q^2 x + Qy.$$

The Frenet equation becomes

$$\frac{d}{dt_+}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -Q & 1 \\ -Q^2 & Q \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}. \tag{7}$$

The integral curvature Q is defined up to an additive constant and is invariant under the area preserving linear transformations of the plane. For monotone increasing Q the graph of x is convex, the polar reciprocal is convex and Q is the area function of the polar reciprocal. In general, it follows from the invariance under SL(2) and the homogeneity of degree zero that the product

$$(t - t_0)(Q(t) - Q(t_0))$$

is invariant under nonsingular linear transformations.

The constant in (6) can be replaced by an arbitrary one. The substitution $y(t) \to y(t) - ax(t)$ induces one $Q(t) \to Q(t) - a$.

Since the integral in (6) depends only on the a.e. defined function r', it follows that y depends on x but not on the direction of the differentiation:

$$\frac{d}{dt_-}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -Q_- & 1 \\ -Q_-^2 & Q_- \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \tag{8}$$

where

$$Q_-(t) = \frac{r'(0)}{r(0)} - \frac{r'_-(t)}{r(t)} + \int_0^t (r^{-4} - (r'/r)^2)\, d\tau$$

and

$$Q(t) - Q_-(t) = \frac{1}{r}(r'_- - r'_+)$$

is the quantity predicted by (5). We also see that the forward and backward derivatives of $x(t)$ are connected by

$$x'_+(t) = x'_-(t) - \{Q(t) - Q_-(t)\}x(t). \tag{9}$$

The matrix in (7) is of trace zero and therefore in the Lie algebra of SL(2). Hence, two admissible curves are images of one another in a unimodular linear map if and only if their functions Q differ at most by a constant.

For any solution vector x of the first row of (7), the Wronskian of the coordinate functions $x_1(t)$, $x_2(t)$ has an everywhere vanishing right derivative. By (8), it also has an everywhere vanishing left derivative. Hence, the Wronskian is constant and there always exists a pair of solutions of unit Wronskian. The condition (3) can be satisfied and the curves that we have studied in this section are the curves obtained as solutions of systems (7). We formulate this as a theorem on differential equations.

THEOREM. *If $Q(t)$ is right continuous for $a \leqq t \leqq b$, then*

$$\begin{pmatrix} x \\ y \end{pmatrix}'_+ = \begin{pmatrix} -Q & 1 \\ -Q^2 & Q \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

has solutions in $a \leqq t \leqq b$ that are uniquely determined by initial conditions and satisfy the equation for every value of t. In addition, for any pair $x_1(t)$, $x_2(t)$ of coordinate functions of a solution vector x of unit Wronskian,

$$Q(t) = c - r'_+(t)r^{-1}(t) + \int_a^t (r^{-4} - r'^2 r^{-2})\, d\tau,$$

where $r(t) = (x_1(t)^2 + x_2(t)^2)^{1/2}$.

3. It is reasonable to say that (1) has a periodic coefficient if

$$Q(t) = \lambda(t - t_0) - P(t)$$

with periodic P. For a study of equations with periodic coefficients one best uses the unimodular frame (x, z) defined by

$$\begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -\lambda(t - t_0) & 1 \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix}.$$

Then z remains bounded when x does and one easily proves all the usual theorems on Hill equations.

From the result of § 2 we may easily construct all equations (1) with periodic coefficients of period T and only periodic (i.e., $x(t + Y) = x(t)$) or semi-periodic ($x(t + T) = -x(t)$) solutions. For such an equation we have

$$\alpha(t + T) = \alpha(t) + k\pi, \quad k \text{ integer},$$

and, hence, *the equation* (1) *has all periodic solutions if and only if there exists a positive, periodic function $r(t)$ of period T that has square integrable one-sided derivatives everywhere for which*

$$\int_0^T r^{-2}\, dt = k\pi,$$

$$Q(t) = c - \frac{r'_+(t)}{r(t)} + \int_0^t (r^{-4} - r^{-2}r'^2)\, dt.$$

For continuous $Q'(t)$, an equivalent result is contained in the author's paper [4]. In the meantime, the author found that the same result is contained, in the form of a stability criterion, in the paper [5] by V. A. Jakubovič, whose priority is herewith acknowledged.

4. Following Borůvka [2], we define the dispersion function $\varphi(t)$ of a homogeneous linear differential equation $L[x] = 0$ defined on the real axis as the location of the zero following t of any solution of $L[x] = 0$ that vanishes at t but does not vanish identically. An alternative definition is

$$\varphi(a) = \sup\{b | a \leqq t \leqq b \text{ is an interval of disconjugacy for } L[x] = 0\}.$$

If $\varphi(t) < \infty$ for all t, $L[x] = 0$ is oscillatory.

For a solution curve of (1) of unit Wronskian, $\alpha'(t) > 0$. Hence, a third definition of the dispersion function is via the Abelian functional equation

$$\pi = \int_t^{\varphi(t)} d\alpha(\tau) = \int_t^{\varphi(t)} \frac{d\tau}{r^2(\tau)}$$

since the vector $x \cdot \varphi(t) = x(\varphi(t))$ points to the direction of $-x(t)$. From here we see that

$$\frac{d\varphi}{dt} = \frac{(r \circ \varphi(t))^2}{r(t)^2}. \tag{10}$$

By projection into the coordinate axes we obtain for any scalar solution $x(t)$ that does not vanish at t,

$$\frac{d\varphi}{dt} = \frac{x \circ \varphi(t)^2}{x(t)^2}.$$

In any case, *the dispersion function of an oscillatory equation defined by the generalized derivative of a right continuous function is differentiable.*

The second derivative does not exist in general; a simple computation shows that

$$\varphi''_+(t) - \varphi''_-(t) = 2\varphi'(t)\{Q(t) - Q_-(t) - \varphi'(t)[Q \circ \varphi(t) - Q_- \circ \varphi(t)]\}.$$

A necessary and sufficient condition for the existence of $\varphi''(t)$ is therefore

$$r(t)\{r'_+(t) - r'_-(t)\} = r \circ \varphi(t)\{r'_+ \circ \varphi(t) - r'_- \circ \varphi(t)\}.$$

5. The notion of dispersion is useful, for example, in some integral estimates. In order to follow the generally adopted notations, we consider an equation

$$x'' + p(t)x = 0, \qquad -\infty < t < +\infty, \tag{11}$$

where either p is a positive function, $p(t) > 0$, or p is the generalized derivative of an increasing one-sided continuous function. From Lyapunov's inequality it is known that

$$4 \leqq (\varphi(t) - t)\int_t^{\varphi(t)} p(\tau)\, d\tau \tag{12}$$

is best possible. It is also easy to construct oscillatory equations for which the right-hand side of (12) is arbitrarily big. In this framework, we prove the following theorem.

THEOREM. *If φ' is a nondecreasing function at $t = t_0$, then*

(i) $$(\varphi(t_0) - t_0)\int_{t_0}^{\varphi(t_0)} p(\tau)\, d\tau < 2(1 + \varphi'(t_0)^{1/2})^3 \varphi'(t_0)^\gamma,$$

where $\gamma = -1$ for $\varphi'(t_0) < 1$ and $\gamma = -1/2$ for $\varphi'(t_0) > 1$;

(ii) *if either* $\varphi'(t_0) = 1$ *or* $\int_{t_0}^{\varphi(t_0)} x_1\, dt = 0$ *for a solution* x_1 *of* (11) *that does not vanish identically but whose forward derivative vanishes at* t_0, *then*

$$(\varphi(t_0) - t_0) \int_{t_0}^{\varphi(t_0)} p(\tau)\, d\tau \leqq \pi^2.$$

The only known case of these inequalities seems to be inequality (ii) for $\varphi'(t) \equiv 1$ (Petty and Barry [7]). The inequality (i) is not best possible since it results from the combination of two inequalities with contradicting extremal cases.

We denote by μ (or μ_+, μ_-) the unoriented angle between the vectors x and x' (or x'_+, x'_-) for a solution vector x of unit Wronskian of (11). From (10) we have

$$\varphi''_\pm(t) = 2\varphi'(t) r^{-2}(t)\{\cot \mu_\pm \circ \varphi(t) - \cot \mu_\pm(t)\}.$$

By our hypotheses, the curve $x(t)$ is convex for $t_0 \leqq t \leqq \varphi(t_0)$ and $\mu_\pm \circ \varphi(t_0) \leqq \mu_\pm(t_0)$. We fix $x(t)$ by the initial conditions

$$x(t_0) = (1, 0), \quad x'_+(t_0) = (0, 1).$$

Then $x \circ \varphi(t_0) = (-\sqrt{\varphi'(t_0)}, 0)$ and $x'_+ \circ \varphi(t_0)$ points into the third quadrant (including the $-x_2$-axis).

If $\varphi'(t_0) = 1$, the curve together with its image in the reflection in the origin is a convex, closed curve. By a theorem of Blaschke [1], the product of the surface area of a plane convex set and the surface area of the polar reciprocal with respect to the centroid of the set is $\leqq \pi^2$. In our case the centroid is at the origin and the product of the surface areas for the closed curve is the left-hand side of the inequality (ii). This proves (ii) in the first case. In the second case, we consider the curve $x(t)$ on $t_0 \leqq t \leqq \varphi(t_0)$ and its reflection in the x_1-axis. The integral condition means that the centroid is at the origin and Blaschke's theorem applies again. Equality holds only for the solutions of $x'' + c^2 x = 0$.

In the general case (i), put $\eta = \max x_2(t)$, $t_0 \leqq t \leqq \varphi(t_0)$. Let (ξ, η) be a point $x(t)$ of ordinate η. The curve is contained in the rectangle of vertices $(1, 0)$, $(1, \eta)$, $(-\sqrt{\varphi'(t_0)}, \eta)$, $(-\sqrt{\varphi'(t_0)}, 0)$ and $\varphi(t_0) - t_0 \leqq 2\eta(1 + \varphi'(t_0)^{1/2})$. The curve contains the triangle of vertices $(1, 0)$, (ξ, η), $(-\sqrt{\varphi'(t_0)}, 0)$ and its polar reciprocal is contained in the polar reciprocal of the triangle. In order to avoid difficulties of definition, we consider the closed curve generated by reflection of $x(t)$ in the x_1-axis and the kite obtained by reflection of the triangle. The polar reciprocal of the kite has vertices

$$(1, \pm\eta^{-1}|\xi - 1|), \quad (-\varphi'(t_0)^{-1/2}, \pm\eta^{-1}|1 + \xi\varphi'(t_0)^{-1/2}|).$$

The surface area of the quadrilateral of the above vertices is

$$\frac{1 + \sqrt{\varphi'(t_0)}}{\varphi'(t_0)}\{|\xi + \sqrt{\varphi'(t_0)}| + \sqrt{\varphi'(t_0)}|\xi - 1|\} \tag{13}$$

and this majorizes $\int_{t_0}^{\varphi(t_0)} p\, dt$. The maximum of (13) on the interval

$$-\sqrt{\varphi'(t_0)} \leqq \xi \leqq 1$$

is attained at the right endpoint for $\varphi'(t_0) < 1$ and at the left endpoint for $\varphi'(t_0) > 1$. The inequality (i) is obtained by multiplication of the estimates of the areas of the curve and its polar reciprocal.

The general condition:

$$\varphi'(t) \text{ nondecreasing for } t = t_0$$

is necessary. If $r \circ \varphi(t)/r(t)$ is decreasing in some interval $t^* \leqq t \leqq t_0$, the convex curve $x(t)$ may have small surface area but points at great distance $r(t^*)$ from the origin. The curve then may have a given area even though it passes very near the origin and the surface area of its polar reciprocal is very big. For instance, for

$$p = \frac{1}{\varepsilon}\{\delta(t - \varepsilon) + \delta(t - (1 - \varepsilon))\},$$

$x(t)$ is the polygon of vertices

$$(1,0), \quad (1,\varepsilon), \quad (-\varepsilon^{-1} + 1, \varepsilon), \quad (-1,0).$$

We put $\alpha = \varepsilon^{-2} - 2\varepsilon^{-1}$. Then

$$\varphi(t) = 1 + \frac{t}{1 + \alpha t}, \qquad 0 \leqq t < \varepsilon,$$

and $\varphi''(0) = -2\alpha < 0$. We have $\varphi'(0) = 1$ but

$$(\varphi(0) - 0)\int_0^{\varphi(0)} p\, dt = \frac{2}{\varepsilon}$$

is arbitrarily big for suitably small ε.

6. We use the inequalities obtained in the preceding section to obtain a majorization in a more familiar setting. We consider a Hill equation

$$\begin{aligned} &x'' + \lambda p(t)x = 0, \qquad -\infty < t < \infty, \\ &p(t + T) = p(t) > 0, \quad p \in L^1(0, T). \end{aligned} \tag{14}$$

The dispersion function of this equation will be denoted by $\varphi(t, \lambda)$. Define $\lambda(u)$ as the first eigenvalue of the problem

$$x'' + \lambda px = 0, \quad u \leqq t \leqq u + T,$$

$$x(u) = x(u + T) = 0,$$

i.e., $\varphi(u, \lambda(u)) = u + T$. One shows in the theory of Hill equations [6] that

$$\lambda_1 = \min_{u \leqq t \leqq u+T} \lambda(t)$$

is the lower limit point of the first interval of instability of (14) and, therefore, there exist solutions of unit Wronskian that satisfy

$$x_1(t + T) = -x_1(t),$$

$$x_2(t + T) = -x_2(t) + x_1(t).$$

Let u^* be a zero of $x_1(t)$. Then $x(u^* + T) = -x(u^*)$, where we have put $x = (x_1, x_2)$ as always. This means that

$$\varphi'(u^*, \lambda_1) = 1.$$

By the minimum property of λ_1,

$$\varphi(u^* + \tau, \lambda_1) \geqq \varphi(u^*, \lambda_1), \qquad |\tau| < \varepsilon,$$

the function $\varphi(u, \lambda_1)$ has a minimum at u^* and the differentiable function $\varphi'(u, \lambda_1)$ is nondecreasing at u^*. (With a little more work the same result can be obtained for $p = Q'$.)

By inequality (ii),

$$(4 \leqq)\ \lambda_1 T \int_{u^*}^{u^* + T} p\, dt \leqq \pi^2,$$

or *the lower limit* λ_1 *of the first interval of instability of the Hill equation* (14) *satisfies*

$$\lambda_1 \leqq \frac{\pi^2}{T \int_0^T p\, dt}.$$

Similar necessary inequalities do not exist for higher eigenvalues. Counterexamples can be constructed as in § 6.

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SIAM J. MATH. ANAL.
Vol. 2, No. 2, May 1971

SMOOTHNESS OF SOLUTIONS OF VOLTERRA INTEGRAL EQUATIONS WITH WEAKLY SINGULAR KERNELS*

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Abstract. The purpose of this paper is to obtain results on the differentiability properties of solutions of nonlinear Volterra integral equations of the second kind with convolution kernels $a(t - s)$. It is assumed that $a(t)$ is continuous for $t > 0$ and integrable at the origin although $a(t)$ may become unbounded at $t = 0$. Solutions are known to be continuous for all $t \geqq 0$. The results in this paper prove that the solution $x(t)$ is smooth for $t > 0$. The existence and the possible nature of singularities in $x'(t)$ at $t = 0$ are studied for a large class of kernels. The special case $a(t) = t^{-p}$, $0 < p < 1$, is studied in particular detail.

1. Introduction. The purpose of this paper is to obtain some results on the differentiability properties of solutions of nonlinear integral equations of the form

$$x(t) = f(t) + \int_0^t a(t - s)g(s, x(s))\,ds, \qquad 0 \leqq t \leqq T, \tag{1}$$

when $f(t)$ and $g(t, x)$ are smooth functions, $a(t) \in C(0, T] \cap L^1(0, T)$ but $a(t)$ may become unbounded as $t \to 0$. Such results are necessary in order to estimate the error in numerical approximations of the solution of (1) (for example, see Linz [1, § II]). This type of result is also useful in proving the equivalence of certain nonlinear boundary value problems for the heat equation with a corresponding Volterra system (see [2, Theorem 3 and its proof]).

The general problem of determining the smoothness of solutions of (1) is rather complex. Suppose we fix a function g and a kernel $a(t) \in L^1(0, T)$. Then as f varies over the set $C[0, T]$ the solution $x(t) = x(t; f)$ will also vary over all possible functions in $C[0, T]$. To see this one has only to fix any $x^*(t) \in C[0, T]$ and then define

$$f^*(t) = x^*(t) - \int_0^t a(t - s)g(s, x^*(s))\,ds$$

on $0 \leqq t \leqq T$. Then x^* is the solution of (1) corresponding to $f^* \in C[0, T]$.

This overabundance of solutions is caused by the overabundance of forcing functions (f can be any continuous function). One would expect that as f and g become smoother the solution of (1) must become smoother. This is true to some extent but intuition should not be trusted too far. To see this consider the equation

$$x(t) = f(t) - \int_0^t (t - s)^{-1/2}x(s)\,ds. \tag{2}$$

If $f(t) \equiv 1$ (an entire function), then a Laplace transform argument may be used to see that

$$x(t) = \exp(\pi t)\operatorname{erfc}(\sqrt{\pi t}),$$

* Received by the editors June 1, 1970, and in revised form September 29, 1970. This research was supported in part by the National Aeronautics and Space Administration under Grant HGL 40-0020015 and in part by the Air Force Office of Scientific Research under Grant AF-AFOSR 67-0693A.

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where

$$\operatorname{erfc}(x) = (2/\sqrt{\pi})\int_x^\infty \exp(-r^2)\,dr$$

is the complementary error function. On the other hand, if $f(t) = 1 + 2\sqrt{t}$, then it is easy to verify that $x(t) \equiv 1$ (entire). In particular this shows that fortuitous choices of the function f yield smoothness properties at $t = 0$ which are the opposite of what one intuitively expects.

Since (2) is a linear equation of convolution type, then it is possible to analyze the behavior of solutions in some detail. Given any fixed continuous function f, let $x_0(t, f)$ be the corresponding solution of (2). Integration in (2) from zero to t and rearrangement of the double integral yields

$$\begin{aligned} x_1(t,f) &= \int_0^t x_0(\tau, f)\,d\tau \\ &= \int_0^t f(\tau)\,d\tau - \int_0^t (t-s)^{-1/2}\left\{\int_0^s x_0(\tau,f)\,d\tau\right\}ds \end{aligned}$$

or

$$x_1(t,f) = \int_0^t f(\tau_1)\,d\tau_1 - \int_0^t (t-s)^{-1/2} x_1(s,f)\,ds.$$

This integration process can be continued indefinitely:

$$\begin{aligned} x_{n+1}(t,f) &= \int_0^t x_n(\tau,f)\,d\tau \\ &= \int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_n} f(\tau_{n+1})\,d\tau_{n+1} \cdots d\tau_1 - \int_0^t (t-s)^{-1/2} x_{n+1}(s,f)\,ds \end{aligned}$$

or

$$x_{n+1}(t,f) = \int_0^t \{f(\tau)(t-\tau)^n/n!\}\,d\tau - \int_0^t (t-s)^{-1/2} x_{n+1}(s,f)\,ds.$$

Given a function $f \in C^{N+1}[0, T]$, write f in the form

$$f(t) = \sum_{j=0}^N f^{(j)}(0)t^j/j! + \int_0^t \{f^{(N+1)}(\tau)(t-\tau)^N/N!\}\,d\tau.$$

Let $u_j(t) = x_j(t, f)$ for the special choice $f(t) \equiv 1$. Then the solution of (2) may be written in the form

$$x_0(t,f) = \sum_{j=0}^N f^{(j)}(0)u_j(t)/j! + x_{N+1}(t, f^{(N+1)}).$$

The functions u_j can be computed explicitly. They are of class $C^j[0, T]$, indeed analytic in the complex plane cut by the negative real axis. The function $x_{N+1}(t, f^{(N+1)})$ is at least of class $C^{N+1}[0, T]$.

The foregoing analysis of (2) shows that as f becomes smoother, then $x(t, f)$ also becomes smoother but only for $t > 0$. In general there will be no increase in smoothness of the solution at $t = 0$. At the same time, very special choices of f

(for example, $f(0) = 0$ or $f(0) = f'(0) = 0$) may force smoother behavior at the origin. This general type of behavior seems to be typical of solutions of (1).

In (2) let $u_0(t)$ be the solution when $f(t) \equiv 1$. Then one can easily and explicitly compute $u_0'(t) = t^{-1/2} + \pi u_0(t)$. In the more general case where $f \in C^{N+1}$, $N \geqq 1$, then

$$x_0'(t, f) = \sum_{j=0}^{N} f^{(j)}(0) u_j'(t)/j! + x_{N+1}'(t, f^{(N+1)})$$
$$= f(0)\{t^{-1/2} + \pi u_0(t)\} + \text{continuous terms}.$$

More generally one could rewrite (1) in the form

$$x(t) = f(t) + \int_0^t a(s) g(t - s, x(t - s))\, ds \tag{1'}$$

and then formally differentiate to obtain

$$\begin{aligned} x'(t) = f'(t) + a(t) g(0, x(0)) + \int_0^t a(s)\{g_1(t - s, x(t - s)) \\ + g_2(t - s, x(t - s)) x'(t - s)\}\, ds, \end{aligned} \tag{3}$$

where $g_1 = \partial g/\partial t$ and $g_2 = \partial g/\partial x$. One might expect that if $g(0, x(0)) \neq 0$, then $x'(t) = O(a(t))$ as $t \to \infty$. We shall show that this is the case for a large class of kernels $a(t)$. In general, the nature of the singularity at $t = 0$ is hard to analyze in detail since the integral

$$\int_0^t a(s) g_2(t - s, x(t - s)) x'(t - s)\, ds$$

may also be singular at $t = 0$.

The remainder of this paper is organized in the following way. Section 2 contains preliminary definitions, lemmas and estimates. Section 3 contains the basic results on differentiability of the solution $x(t)$ of (1). The necessary hypotheses have been collected at the beginning of this section and have been labeled as assumptions (A1–A6). In Theorem 1 we show that if $a(t)$ is of class $C(0, T] \cap L^1(0, T)$ and if a, f and g satisfy certain other mild hypotheses, then $x(t)$ has a continuous first derivative which satisfies (3) on the half-open interval $0 < t \leqq T$. Theorem 3 covers the case where $a(t) \in C^{\nu-1}[0, T] \cap C(0, T]$ and $a^{(\nu)}(t) \in L^1(0, T)$. If $\nu \geqq 1$, then we show that $x(t) \in C^{\nu}[0, T] \cap C^{\nu+1}(0, T]$ with $x^{(\nu+1)}(t)$ of class L^1 near $t = 0$. Moreover, $x'(t)$ satisfies (3) while higher derivatives of $x(t)$ satisfy the appropriate equations obtained from (3) by formal differentiation.

The results in § 3 guarantee that for some integer $\nu \geqq 0$ the solution $x(t) \in C^{\nu}[0, T] \cap C^{\nu+1}(0, T]$. The function $x^{(\nu+1)}(t)$ is continuous when $t > 0$ and is L^1 near $t = 0$ but may become unbounded or may oscillate wildly as t approaches zero. The purpose of § 4 is to provide a partial answer to the question of whether or not $x^{(\nu+1)}(t)$ has any further derivatives when $t > 0$ and also to the question of the possible nature of the singularity in $x^{(\nu+1)}(t)$ at $t = 0$. We shall assume that the kernel $a(t)$ is a weakly singular function (Definition 1 below). Intuitively this means that for some integer $\nu \geqq 0$ one has $a(t) \in C^{\nu-1}[0, T] \cap C^{\nu+1}(0, T]$, $a^{(\nu)}(t) \in L^1$ in a neighborhood of $t = 0$ and $a^{(\nu+1)}(t)$ does not oscillate too wildly.

We seek sufficient conditions so that the solution $x(t)$ will be weakly singular of order $\nu + 1$. The case $\nu = 0$ is somewhat delicate. It is treated in Theorem 4 and Corollaries 2 and 3. When $\nu \geqq 1$ the analysis is straightforward. This case is discussed in Theorem 5.

Section 5 contains a detailed analysis of $x(t)$ in the special case where $a(t) = t^{\nu-p}$, $0 < p < 1$ and $\nu \geqq 0$ is an integer. The functions f and g are assumed to be real analytic functions of their arguments. Under these very special hypotheses the solution can be analyzed in greater detail. Earlier results in § 4 imply that $x^{(\nu+1)}(t) = O(t^{-p})$ as $t \to 0$. Here we show that there exists a computable constant K_1 such that $x^{(\nu+1)}(t) = f^{(\nu+1)}(0) + K_1 t^{-p} + O(t^{1-p})$. In Theorem 6 we show that $x(t)$ is an analytic function in a neighborhood of the set $0 < t \leqq T$. Moreover Corollary 4 asserts that if $p = r/q$ is a rational number, then the singularity in $x(t)$ at $t = 0$ is a branch point.

2. Preliminary lemmas. Consider a linear Volterra integral equation of the form

$$X(t) = F(t) + \int_0^t a(t-s)h(s)X(s)\,ds. \tag{4}$$

LEMMA 1. *For some $T > 0$ assume F and a are of class $L^1(0, T)$ and $h \in L^\infty(0, T)$. Then* (3) *has a unique solution $X \in L^1(0, T)$. If in addition h, F and a are scalars and are a.e. nonnegative, then $X(t) \geqq 0$ a.e.*

Proof. The usual contraction map and translation argument is applicable. Let h_0 be the essential supremum (ess sup) of $|h(t)|$ on $0 \leqq t \leqq T$. Pick an integer J such that if $S = T/J$, then

$$h_0 \int_0^S |a(t)|\,dt = \alpha < 1.$$

Then existence of $X(t)$ on $0 \leqq t \leqq S$ follows immediately by the principle of contraction mappings on $L^1(0, S)$.

Replacing t by $t + S$ in (4) one obtains

$$X_1(t) = F_1(t) + \int_0^t a(t-s)h(s+S)X_1(s)\,ds,$$

where $X_1(t) = X(t + S)$ and

$$F_1(t) = F(t+S) + \int_0^S a(t+S-s)h(s)X(s)\,ds.$$

Since $X \in L^1(0, S)$ is known and $F_1 \in L^1(0, S)$ is known, then the contraction mapping argument yields $X_1(t) \in L^1(0, S)$. Define $X(t + S) = X_1(t)$ a.e. on $0 < t < S$. Continue by induction on the intervals $jS < t < (j + 1)S$.

If h, F and a are nonnegative, then the argument is the same except that the contraction mappings are defined on the set $\{\varphi \in L^1(0, S) : \varphi(t) \geqq 0 \text{ a.e.}\}$. This completes the proof.

LEMMA 2. *Suppose F and $a \in L^1(0, T)$, $h \in L^\infty(0, T)$ and $h_0 \geqq \operatorname{ess\,sup} |h(t)|$. Suppose there exist $r > 0$ and a function $\beta \in L^1(0, r)$ such that*

$$|F(t)| + h_0 \int_0^t |a(t-s)|\beta(s)\,ds \leqq \beta(t), \qquad 0 \leqq t \leqq r.$$

Then there exists $r_0 \leqq \min\{r, T\}$ such that the unique L^1 solution of (4) *satisfies the estimate $|X(t)| \leqq \beta(t)$ a.e. on $0 < t < r_0$.*

Proof. Let S be the number given in the proof of Lemma 1 and let $r_0 = \min\{r, S\}$. Define

$$A = \{\varphi \in L^1(0, r_0) : |\varphi(t)| \leqq \beta(t) \text{ a.e.}\}.$$

Apply a contraction mapping on A.

COROLLARY 1. *Suppose the hypotheses of Lemma* 1 *are satisfied. Assume* $|F(t)|$ *and* $|a(t)| \leqq K\beta(t)$ *a.e. on* $0 \leqq t \leqq r$ *and that*:

(H1) *for each* $\varepsilon > 0$ *there exists* $\delta(\varepsilon) > 0$ *such that*

$$\int_0^t \beta(t-s)\beta(s)\,ds \leqq \varepsilon\beta(t) \qquad \text{a.e. on } 0 < t < \delta.$$

Then there exists $r_0 \leqq \min\{T, r\}$ *such that the unique* L^1 *solution of* (4) *satisfies the estimate* $|X(t)| \leqq (K+1)\beta(t)$ *a.e. on* $0 < t < r_0$.

Proof. Pick $\varepsilon > 0$ so small that $\varepsilon h_0 K(K+1) < 1$, where $h_0 \geqq \operatorname{ess\,sup}|h(t)|$. Then pick $\delta = \delta(\varepsilon)$ using (H1). Let $r_0 = \min\{\delta, r, T\}$. For almost all t in $0 < t < r_0$ one has

$$\begin{aligned}|F(t)| + h_0\int_0^t |a(t-s)|(K+1)\beta(s)\,ds &\leqq K\beta(t) + h_0K(K+1)\int_0^t \beta(t-s)\beta(s)\,ds\\ &\leqq (K+1)\beta(t).\end{aligned}$$

Now apply Lemma 2 to complete the proof.

It is easy to find examples of functions which satisfy hypotheses (H1). For example, if $0 < p < 1$, then

$$\int_0^t (t-s)^{-p}s^{-p}\,ds = Kt^{1-2p} = (Kt^{1-p})t^{-p} = o(t^{-p}),$$

where $K = \Gamma(1-p)^2/\Gamma(2-p)$ and $\Gamma(z)$ is the gamma function. If $\beta(t) = -\log t$ and $0 < t \leqq 1$, then

$$\begin{aligned}0 &\leqq \int_0^t \log(t-s)\log s\,ds = \int_0^{t/2} \log(t-s)\log s\,ds + \int_{t/2}^t \log(t-s)\log s\,ds\\ &\leqq \log\frac{t}{2}\int_0^{t/2}\log s\,ds + \log\frac{t}{2}\int_{t/2}^t \log(t-s)\,ds = 2\log\frac{t}{2}\int_0^{t/2}\log s\,ds\\ &= t\log(t/2)[\log(t/2) - 1] = o(\log t), \qquad t \to 0^+.\end{aligned}$$

If $\beta(t) = \sum_{n=1}^\infty e^{-n^2t}$, then

$$\begin{aligned}\int_0^t \beta(t-s)\beta(s)\,ds &= \sum_{\substack{n,m=1\\ n\neq m}}^\infty \frac{e^{-n^2t} - e^{-m^2t}}{m^2 - n^2} + t\sum_{n=1}^\infty e^{-n^2t}\\ &= 2\gamma(t) + t\beta(t),\end{aligned}$$

where $\gamma(t)$ is defined by

$$\begin{aligned}\gamma(t) &= \sum_{n=1}^\infty \sum_{m=n+1}^\infty \frac{e^{-n^2t} - e^{-m^2t}}{m^2 - n^2} = \sum_{n=1}^\infty\sum_{m=1}^\infty \frac{e^{-n^2t} - e^{-(m+n)^2t}}{m^2 + 2mn}\\ &= \sum_{n=1}^\infty e^{-n^2t}\sum_{m=1}^\infty \frac{1 - e^{-(m^2+2nm)t}}{m^2 + 2nm}.\end{aligned}$$

Clearly $t\beta(t) = o(\beta(t))$ as $t \to 0$. To see that $\gamma(t) = o(\beta(t))$, fix any $\varepsilon > 0$ and then pick M so large that for $m \geqq M$,

$$\sum_{m+1}^{\infty} 2m^{-2} < \varepsilon/2, \quad \sum_{m=1}^{\infty} (m^2 + 2nm)^{-1} < \varepsilon, \qquad n \geqq m.$$

Now pick $\delta > 0$ so small that if $0 < t < \delta$, then

$$\sum_{m=1}^{M} (1 - e^{-(m^2+2mn)t}) < \varepsilon/2, \qquad 1 \leqq m \leqq M.$$

Then one has

$$\begin{aligned} 0 \leqq \gamma(t) &\leqq \sum_{n=1}^{M} e^{-n^2t}\left\{\sum_{m=1}^{M} (1 - e^{-(m^2+2mn)t}) + \sum_{M+1}^{\infty} m^{-2}\right\} \\ &\quad + \sum_{n=M+1}^{\infty} e^{-n^2t} \sum_{m=1}^{\infty} (m^2 + 2mn)^{-1} \\ &\leqq \sum_{n=1}^{M} e^{-n^2t}\{\varepsilon/2 + \varepsilon/2\} + \sum_{n=M+1}^{\infty} e^{-n^2t}\varepsilon = \varepsilon\beta(t). \end{aligned}$$

Using this result it is easy to show that the function $\beta(t) = K\sum_{n=0}^{\infty} e^{-n^2t}$ also satisfies (H1).

The *resolvent* $R(t)$ associated with a given kernel function $a(t)$ is defined as the unique L^1 solution of the linear equation

$$R(t) = a(t) + \int_0^t a(t-s)R(s)\,ds. \tag{5}$$

It is well known (for example, see Tricomi [3, Chap. 1]) that the solution of a linear equation

$$X(t) = f(t) + \int_0^t a(t-s)X(s)\,ds \tag{6}$$

may be represented in terms of f and the resolvent:

$$X(t) = f(t) + \int_0^t R(t-s)f(s)\,ds. \tag{7}$$

Consider a pair of nonlinear equations

$$X_j(t) = F_j(t) + \int_0^t a(t-s)g_j(s, X_j(s))\,ds, \qquad j = 1, 2. \tag{8}$$

LEMMA 3. *Assume:*

(i) *a, F_1 and $F_2 \in L^1(0, T)$,*

(ii) *$g_1(t, x)$ and $g_2(t, x)$ are continuous in (t, x) for $0 \leqq t \leqq T$ and all x,*

(iii) *$g_1(t, x)$ is Lipschitz continuous in x with Lipschitz constant L (independent of t and x), and*

(iv) *X_1 and X_2 exist a.e. on $0 \leqq t \leqq T$ and are L^1.*

Let $r(t)$ be the resolvent of the kernel $L|a(t)|$ and define

$$Q(t) = F_1(t) - F_2(t) + \int_0^t a(t-s)\{g_1(s, X_2(s)) - g_2(s, X_2(s))\}\, ds.$$

Then a.e. on $0 < t < T$ one has

$$|X_1(t) - X_2(t)| \leqq |Q(t)| + \int_0^t r(t-s)|Q(s)|\, ds.$$

Proof. Define $z(t) = X_1(t) - X_2(t)$, $F(t) = F_1(t) - F_2(t)$ and

$$G(t) = \{g_1(t, X_1(t)) - g_1(t, X_2(t))\}/z(t)$$

when $z(t) \neq 0$ and $G(t) = 0$ when $z(t) = 0$. Clearly z and $F \in L^1(0, T)$, $G \in L^\infty(0, T)$ and $|G(t)| \leqq L$ a.e. Using (8) and the definitions above it follows that

$$\begin{aligned} z(t) &= F(t) + \int_0^t a(t-s)\{g_1(s, X_2(s)) - g_2(s, X_2(s))\}\, ds \\ &\quad + \int_0^t a(t-s)\{g_1(s, X_1(s)) - g_1(s, X_2(s))\}\, ds \\ &= Q(t) + \int_0^t a(t-s)G(s)z(s)\, ds, \end{aligned}$$

so that

$$|z(t)| \leqq |Q(t)| + L\int_0^t |a(t-s)|\,|z(s)|\, ds.$$

Let $p(t)$ be a nonnegative function such that

$$|z(t)| = \{Q(t) - p(t)\} + L\int_0^t |a(t-s)|\,|z(s)|\, ds.$$

Since $r(t)$ is the resolvent of $L|a(t)|$, then (7) implies that

$$|z(t)| = Q(t) - p(t) + \int_0^t r(t-s)\{Q(s) - p(s)\}\, ds.$$

Since r and p are nonnegative the lemma follows.

If $a(t) \equiv 1$ and both $F_1(t) \equiv F_1$ and $F_2(t) \equiv F_2$ are constants, then $r(t) = L \exp(Lt)$. In this case Lemma 3 reduces to a familiar estimate for ordinary differential equations.

In certain cases the resolvent associated with a kernel $a(t) \in C(0, T] \cap L^1(0, T)$ is not only $L^1(0, T)$ but also continuous for $t > 0$. This is trivial to see if $a \in L^2(0, T)$. Another instance is given by the following result.

LEMMA 4. *Suppose $a(t) \in C(0, T] \cap L^1(0, T)$. If $a(t)$ is nonnegative and non-increasing, then its resolvent is continuous on $0 < t \leqq T$.*

Proof. Let $r(t)$ be the resolvent of $a(t)$. By Lemma 1, $r(t) \in L^1(0, T)$ and $r(t) \geqq 0$ a.e. Therefore the function $a(\delta - s)r(s) \in L^1(0, \delta)$ for almost all $\delta \in (0, T)$.

Fix any such δ. Then

$$r(t+\delta) = \left\{ a(t+\delta) + \int_0^\delta a(t+\delta-s)r(s)\,ds \right\} + \int_0^t a(t-s)r(s+h)\,ds. \tag{9}$$

Note that the function $a(t+\delta)$ is continuous in $[0, T-\delta]$. The function E, defined by the relation

$$E(t) = \int_0^\delta a(t+\delta-s)r(s)\,ds,$$

is easily seen to be continuous on $0 < t \leqq T$. To see that $E(t)$ is continuous at $t = 0$ we must show that for any sequence t_n tending monotonically to zero one has

$$\int_0^\delta a(t_n+\delta-s)r(s)\,ds \to \int_0^\delta a(\delta-s)r(s)\,ds.$$

But $a(t)$ is nonincreasing so that $a(t_n+\delta-s)r(s) \to a(\delta-s)r(s)$ monotonically. Now apply the dominated convergence theorem.

We have shown that (9) has the form

$$x(t) = f(t) + \int_0^t a(t-s)x(s)\,ds, \quad x(t) = r(t+\delta),$$

where $f \in C[0, T-\delta]$ and $a(t) \in L^1(0, T-\delta)$. Using an argument similar to the proof of Lemma 1 it follows that $x(t) = r(t+\delta) \in C[0, T-\delta]$. Since $\delta > 0$ can be made arbitrarily small, then the proof of Lemma 4 is complete.

A similar proof establishes the following result.

LEMMA 5. *Suppose F, a and $\beta \in C(0, T] \cap L^1(0, T)$, $h \in L^\infty(0, T)$ and $|a(t)| \leqq \beta(t)$ on $0 < t \leqq T$. If β is nonincreasing, then the solution X of* (4) *is continuous on $0 < t \leqq T$.*

3. Differentiability of solutions. Consider the integral equation

$$x(t) = f(t) + \int_0^t a(s)g(t-s, x(t-s))\,ds \tag{1'}$$

and its formal derivative

$$x'(t) = f'(t) + a(t)g(0, f(0)) + \int_0^t a(t-s)\{g_1(s, x(s)) + g_2(s, x(s))x'(s)\}\,ds. \tag{3}$$

This last equation may be written in the form

$$X(t) = F(t) + \int_0^t a(t-s)g_2(s, x(s))X(s)\,ds, \tag{10}$$

where $X(t) = x'(t)$ and

$$F(t) = f'(t) + a(t)g(0, x(0)) + \int_0^t a(t-s)g_1(s, x(s))\,ds. \tag{11}$$

In the sequel we shall need some or all of the following hypotheses.

(A1) $f(t)$ and $g(t, x)$ are of class C^1 in t and respectively (t, x) for $0 \leqq t \leqq T$ and for all x.

(A2) The function $g_2(t, x) = \partial g(t, x)/\partial x$ is locally Lipschitz continuous in x.

(A3) $a(t) \in L^1(0, T) \cap C(0, T]$ and there exists a nonincreasing function $\alpha(t) \in L^1(0, T) \cap C(0, T]$ such that $|a(t)| \leqq \alpha(t)$ on $0 < t \leqq T$.

(A4) The unique continuous solution of (1) exists on the entire interval $0 \leqq t \leqq T$.

(A5) $f(t)$ and $g(t, x)$ are of class $C^{\nu+1}$ for some integer $\nu \geqq 1$.

(A6) $a(t) \in C^{\nu-1}[0, T] \cap C^{\nu}(0, T]$ and $|a^{(\nu)}(t)| \leqq \alpha(t)$, where α is nonincreasing and integrable on $0 < t < T$ and $\alpha \in C(0, T]$.

THEOREM 1. *Suppose* (A1–A4) *are true. Let* $X(t)$ *be the solution of* (10) *with* F *defined by* (11). *Then the solution* $x(t)$ *of* (1) *is of class* $C[0, T] \cap C^1(0, T]$ *and* $x'(t) = X(t)$ *on the interval* $0 < t \leqq T$.

Proof. Note that by Lemmas 1 and 5 it follows that $X \in C(0, T] \cap L^1(0, T)$. Let $M = \max |x(t)|$ on $0 \leqq t \leqq T$ and let $P(x)$ be a C^∞-function such that $P(x) = 1$ if $|x| \leqq M + 1$ and $P(x) \equiv 0$ if $|x| \geqq M + 2$. If the function $g(t, x)$ in (1) is replaced by $g(t, x)P(x)$, then nothing is changed in the range of interest. Therefore we shall assume that g has compact support. In particular then g, g_1 and g_2 are bounded and $g_2(t, x)$ is globally Lipschitz continuous in x.

Fix a number δ in the range $0 < \delta < T/2$. Define

$$Z(t, h) = \{x(t + h) - x(t)\}/h$$

for $0 < h \leqq \delta$ and $0 < t \leqq T - \delta$. Since $x(t)$ satisfies (1), then Z satisfies an equation of the form

$$Z(t, h) = R(t, h) + \int_0^t a(t - s)g_2(s, x^*(s))Z(s, h)\, ds,$$

where $x^*(s)$ is between $x(t)$ and $x(t + h)$, $0 < \theta(h) < h$ and

$$R(t, h) = (f(t + h) - f(t))/h + h^{-1} \int_t^{t+h} a(s)g(t + h - s, x(t + h - s))\, ds$$
$$+ \int_0^t a(s)g_1(t + \theta(h) - s, x(t - s))\, ds.$$

Let $r(t)$ be the resolvent of $L|a(t)|$. By Lemma 3 above,

$$|Z(t, h) - X(t)| \leqq Q(t, h) + \int_0^t r(t - s)Q(s, h)\, ds$$

on $0 < t \leqq T - \delta$, where

$$Q(t, h) = |R(t, h) - F(t)| + \int_0^t |a(t - s)|\,|g_2(s, x^*(s)) - g_2(s, x(s))|\, ds.$$

Let $K > 0$ be a bound for all of the functions $|f'(t)|$, $|g(t, x)|$, $|g_1(t, x)|$ and $|g_2(t, x)|$. Then the definitions of Q, R and F may be used to obtain the bound

$$\begin{aligned} |Q(s, h)| &\leqq K + (K/h) \int_s^{s+h} |a(u)|\, du + 3K \int_0^T |a(u)|\, du + |F(s)| \\ &\leqq 2K + (K/h) \int_s^{s+h} \alpha(u)\, du + 4K \int_0^T |a(u)|\, du + K\alpha(s) \\ &\leqq 2K \left\{1 + 2 \int_0^T |a(u)|\, du + \alpha(s)\right\}, \qquad 0 < s < t. \end{aligned}$$

Write this bound in the form $|Q(t, h)| \leqq K_0 + K_1\alpha(t)$.

Given $\varepsilon > 0$ let K_2 be a bound for $r(t)$ over $\delta \leqq t \leqq T - \delta$. Pick η in the range $0 < \eta \leqq \delta$ and so small that

$$\int_0^\eta K_2\{K_0 + K_1\alpha(t)\}\,dt < \varepsilon.$$

Now pick h_0 so small that whenever $0 < h \leqq h_0$, then

$$|Q(t,h)| \leqq \varepsilon\left\{\int_0^T r(s)\,ds + 1\right\}^{-1}$$

uniformly over $\eta \leqq t \leqq T - \eta$. Then for h and t in the range $0 < h \leqq h_0$, $\delta \leqq t \leqq T - \delta$ one has

$$\begin{aligned}|Z(t,h) - X(t)| &\leqq Q(t,h) + \int_0^\eta r(t-s)Q(s,h)\,ds + \int_\eta^t r(t-s)Q(s,h)\,ds\\ &\leqq \varepsilon + \int_0^\eta K_2\{K_0 + K_1\alpha(s)\}\,ds + \int_\eta^t r(t-s)\left\{\varepsilon\Big/\int_0^T r(u)\,du\right\}ds\\ &< 3\varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary this shows that $z(t,h) \to X(t)$ as $h \to 0^+$ uniformly in $\delta \leqq t \leqq T - \delta$. But $\delta > 0$ is also arbitrary so that $X(t)$ is the continuous right derivative of $x(t)$ on the interval $0 < t < T$.

By virtue of the uniform convergence to $X(t)$ it follows that on any interval $I = \{t : \delta \leqq t \leqq T - \delta\}$ the set $\{Z(\cdot, h) : 0 < h < \delta\}$ is equicontinuous. Therefore,

$$\lim_{h\to 0+} Z(t,h) = \lim_{h\to 0+} Z(t-h,h) = X(t)$$

uniformly on I. But $Z(t-h,h)$ is a left difference. For $t = T$ a separate but similar argument shows that $X(T)$ is the left derivative of $x(t)$ at $t = T$. This completes the proof of Theorem 1.

Exactly the same proof will establish the following theorem.

THEOREM 2. *Theorem* 1 *remains true if the assumptions on f are weakened to* $f \in C[0,T] \cap C^1(0,T]$ *and* $\int_0^\eta |(f(t+h) - f(t))/h - f'(t)|\,dt \to 0$ *as* $\eta \to 0$ *uniformly in h.*

In case (A4–A6) are true, then one can formally differentiate (1′) as follows:

$$\text{(12)}\qquad \begin{aligned}x^{(n)}(t) = f^{(n)}(t) &+ \sum_{k=0}^{n-1} a^{(k)}(t)\{D^{n-k-1}g(u, x(u))\}_{u=0}\\ &+ \int_0^t a(s)\{D^n g(u, x(u))\}_{u=t-s}\,ds,\end{aligned}$$

where $D^j = d^j/du^j$ denotes the jth derivative and $n = 0, 1, \cdots, v + 1$. We shall prove the following theorem.

THEOREM 3. *Suppose* (A4–A6) *are true with $v \geqq 1$. Then the solution of* (1) *satisfies the following*:

(i) $x \in C^v[0,T] \cap C^{v+1}(0,T]$,

(ii) $x^{(v+1)} \in L^1(0,T)$, *and*

(iii) $x(t)$ *satisfies* (12) *for* $1 \leqq n \leqq v + 1$ *and* $0 < t \leqq T$.

Proof. Since the hypotheses of Theorem 1 are trivially satisfied, then $x'(t) \in C(0, T] \cap L^1(0, T)$ and $x(t)$ satisfies (12) on $0 < t \leqq T$ for $n = 1$. Since $a(t)$ is continuous at $t = 0$, it is clear that

$$\{x(h) - x(0)\}h^{-1} = \{f(h) - f(0)\}h^{-1} + h^{-1}\int_0^h a(s)g(h - s, x(h - s))\,ds$$
$$\to f'(0) + a(0)g(0, x(0))$$

as $h \to 0^+$. Therefore, $x'(0)$ exists and satisfies (12).

Continuing by induction one can use Theorem 1 to establish (12) for $n = 1, 2, \cdots, \nu$. Applying Theorem 2 to (12) with $n = \nu$ one then obtains (12) for $n = \nu + 1$.

4. Weakly singular kernels.

DEFINITION 1. Suppose ν is a nonnegative integer and F is a function defined on $(0, T]$ or on $[0, T]$. Then F is called *weakly singular of order* ν if and only if:

(i) $F \in C(0, T]$ if $\nu = 0$ or $F \in C^{\nu-1}[0, T] \cap C^{\nu}(0, T]$ if $\nu > 0$;

(ii) for each $\varepsilon > 0$, $F^{(\nu)}(t)$ is absolutely continuous on $\varepsilon \leqq t \leqq T$; and finally

(iii) the function defined by

$$\alpha_\nu(t, F) = F(T) + \int_t^T |F^{(\nu+1)}(s)|\,ds, \qquad 0 < t \leqq T,$$

is of class $L^1(0, T)$.

For any integer $\nu \geqq 0$ let $WS(\nu)$ denote the set of all functions F which are weakly singular of order ν ($T > 0$ is fixed). The function $\alpha_\nu(t, F)$ is a measure of the singularity of $F^{(\nu)}$ at $t = 0$. Indeed, it is easy to see that α_ν is nonnegative, nonincreasing and that $|F^{(\nu)}(t)| \leqq \alpha_\nu(t, F)$ on $0 < t \leqq T$. The precise value of T is unimportant in the sense that if T is replaced by another value T', then α_ν must be adjusted only by an additive constant.

THEOREM 4. *Suppose* (A4) *is true,* (A5) *is true with* $\nu = 1$ *and* $a(t) \in WS(0)$. *Then the solution* $x(t)$ *of* (1) *is of class* $C^2(0, T]$ *and there exists a constant* K^* *such that*

$$\int_\tau^T |x''(t)|\,dt \leqq K^*\left(1 + \int_\tau^T |a'(t)|\,dt + \int_\tau^T |x'(t)|^2\,dt + \int_0^\tau |x'(s)|\left(\int_{\tau-s}^T |a'(t)|\,dt\right)ds\right) \tag{13}$$

on $0 < \tau \leqq T$.

Proof. We shall assume that $g(t, x) = g(x)$ is independent of t. The only additional complications in the general case are notational. By Theorem 1 above, $x \in C[0, T] \cap C^1(0, T]$ and $x'(t) \in L^1(0, T)$. For any $\tau \in (0, T)$ one has

$$x'(t + \tau) = \left\{f'(t + \tau) + a(t + \tau)g(x(0)) + \int_0^\tau a(t + \tau - s)g'(x(s))x'(s)\,ds\right\} + \int_0^t a(t - s)g'(x(s + \tau))x'(s + \tau)\,ds \tag{14}$$

on $0 < t \leqq T - \tau$. Note that $f'(t + \tau)$, $a(t + \tau)$ and $g'(x(s + \tau))$ are of class

$C^1[0, T-\tau]$. Also note that the function defined by

$$E(t) = \int_0^\tau a(t+\tau-s)g'(x(s))x'(s)\,ds$$

is of class $C[0, T-\tau] \cap C^1(0, T-\tau]$ and that $E' \in L^1(0, T-\tau)$. Indeed, one has

$$\begin{aligned}\int_0^{T-\tau} |E'(t)|\,dt &= \int_0^{T-\tau} \left| \int_0^\tau a'(t+\tau-s)g'(x(s))x'(s)\,ds \right| dt \\ &\leqq \int_0^\tau \left(\int_0^{T-\tau} |a'(t+\tau-s)|\,dt \right) |g'(x(s))x'(s)|\,ds \qquad (15) \\ &\leqq K \int_0^\tau |x'(s)| \left(\int_{\tau-s}^{T-s} |a'(t)|\,dt \right) ds < \infty,\end{aligned}$$

where K is an a priori constant which bounds $|g'(x(s))|$. Moreover, for all small h one has

$$\begin{aligned}&h^{-1} \int_0^\eta |E(t+h) - E(t)|\,dt \\ &\leqq (K/h) \int_0^\eta \int_0^\tau \left| \int_0^h a'(u+t+\tau-s)\,du \right| |x'(s)|\,ds\,dt \\ &= (K/h) \int_0^h \left\{ \int_0^\tau \left(\int_0^\eta |a'(u+t+\tau-s)|\,dt \right) |x'(s)|\,ds \right\} du \\ &= (K/h) \int_0^h \left\{ \int_0^\tau (\alpha_0(u+\eta+\tau-s) - \alpha_0(u+\tau-s))|x'(s)|\,ds \right\} du,\end{aligned}$$

where K is the bound on $|g'(x(s))|$ and $\alpha_0(t) = \alpha_0(t, a)$. The expression inside the brackets in the last integral will tend to zero as $\eta \to 0$ uniformly for $0 \leqq u \leqq 1$. Indeed, if this were not true, then there would exist sequences $\eta_n \to 0$ and $u_n \to u_0$ such that $0 \leqq u_0 \leqq 1$ and such that along this sequence the expression is larger than some preassigned $\varepsilon > 0$. But $\alpha_0(t)$ is continuous for $t > 0$ and

$$\begin{aligned}0 &\leqq \alpha_0(t+\eta+\tau-s) - \alpha_0(u+\tau-s) \\ &\leqq \alpha_0(u+\eta+\tau-s) \leqq \alpha_0(\tau-s)\end{aligned}$$

when $0 \leqq u \leqq 1$, $\eta > 0$ and $0 \leqq s \leqq \tau$. Therefore the dominated convergence theorem implies that

$$\varepsilon \leqq \lim_{h\to\infty} \int_0^\tau (\alpha_0(u_n+\eta_n+\tau-s) - \alpha_0(u_n+\tau-s))|x'(s)|\,ds = 0.$$

These remarks show that

$$\begin{aligned}&\lim_{\eta\to 0} \int_0^\eta |(E(t+h) - E(t))/h - E'(t)|\,dt \\ &\qquad \leqq \lim \left\{ \int_0^\eta |E(t+h) - E(t)|/h\,dt + \int_0^\eta |E'(t)|\,dt \right\} = 0\end{aligned}$$

uniformly for $0 < h \leqq 1$. In particular Theorem 2 applies to (14). Therefore

$x''(t + \tau)$ exists and is continuous on $0 < t \leqq T - \tau$. Since $\tau > 0$ can be made arbitrarily small, it follows that $x' \in C^1(0, T]$.

For any $\tau \in (0, T)$ the function $x''(t + \tau)$ satisfies an equation of the form

$$\begin{aligned} x''(t+\tau) &= f''(t+\tau) + a'(t+\tau)g(x(0)) + E'(t) + F_1(t) + F_2(t) \\ &\quad + \int_0^t a(t-s)g'(x(s+\tau))x''(s+\tau)\,ds, \end{aligned} \tag{16}$$

where E is the function defined above, $F_1(t) = a(t)g'(x(\tau))x'(\tau)$ and

$$F_2(t) = \int_0^t a(t-s)g''(x(\tau+s))x'(\tau+s)^2\,ds.$$

Let K be a bound on $0 \leqq t \leqq T$ for the functions $f''(t)$, $g(x(t))$, $g'(x(t))$ and $g''(x(t))$. There will be no loss of generality in assuming that T is small enough so that

$$\alpha = K\int_0^T |a(t)|\,dt < 1/2.$$

Take absolute values in (16) and integrate:

$$\begin{aligned} \int_\tau^T |x''(t)|\,dt &= \int_0^{T-\tau} |x''(t+\tau)|\,dt \\ &\leqq K(T-\tau) + K\int_\tau^T |a'(t)|\,dt + \int_0^{T-\tau} \{|E'(t)| + |F_1(t)| + |F_2(t)|\}\,dt \\ &\quad + \int_0^{T-\tau} \left| \int_0^t a(t-s)g'(x(\tau+s))x''(\tau+s)\,ds \right| dt. \end{aligned}$$

The last term in this inequality may be bounded as follows:

$$\begin{aligned} &K\int_0^{T-\tau}\int_s^{T-\tau} |a(t-s)|\,|x''(\tau+s)|\,dt\,ds \\ &\quad = K\int_\tau^T \left(\int_s^T |a(t-s)|\,dt\right)|x''(s)|\,ds \leqq \alpha\int_\tau^T |x''(s)|\,ds. \end{aligned}$$

Furthermore,

$$\begin{aligned} \int_0^{T-\tau} |F_1(t)|\,dt &\leqq K\int_\tau^T |a(t)|\,dt|x'(\tau)| \leqq \alpha|x'(\tau)| \\ &\leqq \alpha\left\{|x'(T)| + \int_\tau^T |x''(t)|\,dt\right\}, \end{aligned}$$

and

$$\begin{aligned} \int_0^{T-\tau} |F_2(t)|\,dt &\leqq K\int_0^{T-\tau}\left(\int_s^{T-\tau} |a(t-s)|\,dt\right)|x'(\tau+s)|^2\,ds \\ &\leqq K\int_\tau^T \left(\int_0^T |a(t)|\,dt\right)|x'(s)|^2\,ds = \alpha\int_\tau^T |x'(s)|^2\,ds. \end{aligned}$$

Combining these inequalities with (15) and rearranging we obtain

$$(1-2\alpha)\int_\tau^T |x''(s)|\,ds \leqq KT + K\int_\tau^T |a'(t)|\,dt + K\int_0^\tau |x'(s)| \int_{\tau-s}^{T-s} |a'(t)|\,dt\,ds$$
$$+\alpha|x'(T)| + \alpha\int_\tau^T |x'(s)|^2\,ds.$$

Since $1-2\alpha > 0$, then the proof is complete.

COROLLARY 2. *Assume the hypotheses of Theorem* 4. *If in addition the function* $\alpha_0(t, a) \in L^2(0, T)$, *then* $x \in WS(1)$.

Proof. First note that

$$\int_0^T \left(\int_\tau^T |x'(t)|^2\,dt\right) d\tau = \int_0^T \left(\int_0^t |x'(t)|^2\,d\tau\right) dt = \int_0^T t|x'(t)|^2\,dt < \infty$$

and

$$\int_\tau^T |a'(t)|\,dt \leqq \alpha_0(t, a).$$

Finally note that $x'(t)$ and $\alpha_0(t, a) \in L^1(0, T)$. Therefore,

$$\int_0^\tau |x'(s)|\left(\int_{\tau-s}^T |a'(t)|\,dt\right) ds \leqq \int_0^\tau |x'(s)|\alpha_0(\tau - s)\,ds$$

with the last function of class L^1 in τ, $0 \leqq \tau \leqq T$. Using these estimates in (13) it follows that

$$\alpha_1(t, x(\cdot)) = |x'(T)| + \int_t^T |x''(u)|\,du \in L^1(0, T).$$

This completes the proof of Corollary 2.

COROLLARY 3. *Assume the hypotheses of Theorem* 4. *If in addition* $\beta(t) = \alpha_0(t, a)$ *satisfies assumption* (H1), *then* $x(t) \in WS(1)$ *and* $\alpha_1(t, x(\cdot)) \leqq K\beta(t)$ *on* $0 < t \leqq T$ *for some a priori constant* K.

Proof. Theorem 1 and Corollary 1 imply that $|x'(t)| \leqq K_0\beta(t)$ on $0 < t \leqq T$ for some fixed constant $K_0 > 0$. Since β is nonincreasing, then

$$\int_\tau^T |x'(t)|^2\,dt \leqq \int_\tau^T K_0^2\beta(t)^2\,dt \leqq K_0^2\beta(\tau)\int_0^T \beta(t)\,dt.$$

Moreover (H1) may be used to see that

$$\int_0^\tau |x'(s)|\left(\int_{\tau-s}^T |a'(t)|\,dt\right) ds \leqq \int_0^\tau |x'(s)|\beta(\tau - s)\,ds$$
$$\leqq K_0\int_0^\tau \beta(s)\beta(\tau - s)\,ds \leqq K_1\beta(\tau).$$

Using these estimates in (13) one obtains

$$\int_\tau^T |x''(t)|\,dt \leqq K^*\left\{1 + \left(1 + K_0^2\int_0^T \beta(t)\,dt + K_1\right)\right\}\beta(\tau).$$

This proves Corollary 3.

THEOREM 5. *Suppose* (A4) *is true,* $a(t) \in WS(\nu)$ *where* $\nu \geqq 1$ *and both* f *and* g *are of class* $C^{\nu+2}$. *Then the solution of* (1) *is of class* $WS(\nu + 1)$ *and* $\alpha_{\nu+1}(t, x(\cdot)) \leqq K\alpha_\nu(t, a)$ *on* $0 < t \leqq T$ *for some fixed constant* $K > 0$.

Proof. Apply Theorem 3 to obtain (12). Replace t by $t + \tau$ in (12) and proceed as in Theorem 4.

5. An example of special kernels. Suppose $x(t)$ is the solution of

$$x(t) = f(t) + \int_0^t (t - s)^{\nu - p} g(s, x(s))\, ds \tag{17}$$

on $0 \leqq t \leqq T$, where $\nu \geqq 0$ is an integer and $0 < p < 1$. If f and g are sufficiently smooth, then the results in § 4 show that $x \in WS(\nu)$ and $x^{(\nu)}(t) = O(t^{-p})$ as $t \to 0$. Further information may be obtained for this special kernel. Change variables in the integral to $s = t \sin^2 \theta$:

$$x(t) = f(t) + 2\int_0^{\pi/2} t^{\nu+1-p} \cos^{\nu+1-p}\theta \sin\theta g(t\sin^2\theta, x(t\sin^2\theta))\, d\theta. \tag{17'}$$

If f and g are of class $C^{\nu+2}$, then by differentiating $\nu + 1$ times on both sides of (17′) one obtains

$$\begin{aligned} x^{(\nu+1)}(t) = &\Big\{2(\nu + 1 - p)(\nu - p) \cdots (1 - p) \\ &\cdot \int_0^{\pi/2} \cos^{\nu+1-p}\theta \sin\theta g(t\sin^2\theta, x(t\sin^2\theta))\, d\theta\Big\} t^{-p} \\ &+ \text{continuous terms of order } t^{1-p} (t^{1-2p} \text{ if } \nu = 0) \text{ or higher.} \end{aligned}$$

In particular then not only is $x^{(\nu+1)}(t) = O(t^{-p})$ but

$$x^{(\nu+1)}(t) = f^{(\nu+1)}(0) + K_1 t^{-p} + O(t^{1-p}), \qquad \nu \geqq 1,$$

or

$$x'(t) = K_1 t^{-p} + O(t^{1-2p}), \qquad \nu = 0,$$

where

$$\begin{aligned} K_1 &= 2(\nu + 1 - p)(\nu - p) \cdots (1 - p) \int_0^{\pi/2} \cos^{\nu+1-p} \sin\theta g(0, x(0))\, d\theta \\ &= 2g(0, f(0))(\nu + 1 - p) \cdots (1 - p)/(\nu + 2 - p). \end{aligned}$$

Even more information is available when f and g are analytic.

THEOREM 6. *Assume* ν *is a nonnegative integer,* $0 < p < 1$ *and that* $f(t)$ *is real analytic in a neighborhood of* $0 \leqq t \leqq T$. *Suppose* $g(t, x)$ *is real analytic on an open set which contains all real ordered pairs* (t, x), $0 \leqq t \leqq T$, *and* $|x| < \infty$. *Then* $x(t)$ *is real analytic in a neighborhood of the set* $0 < t \leqq T$.

Proof. Let $\|x\| = \max |x(t)|$ on $0 \leqq t \leqq T$. Given $\varepsilon > 0$ define

$$D(\varepsilon) = \{z : 0 \leqq \operatorname{Re} z \leqq T + \varepsilon \text{ and } |\operatorname{Im} z| \leqq \varepsilon\}$$

and

$$E(\varepsilon) = \{(z, w) : z \in D(\varepsilon) \text{ and } |w| \leqq \|x\| + 1\}.$$

Define

$$M = \max\left\{|g(z,w)|, \left|\frac{\partial g}{\partial w}(z,w)\right| : (z,w) \in E(\varepsilon)\right\},$$

$$K = \max\{|f(z)| + TM|(z+s)^{\nu+1-p}| : z \in D(\varepsilon) \text{ and } 0 \leqq s \leqq T\},$$

and

$$S(\varepsilon) = \{z \text{ complex}: 0 \leqq \operatorname{Re} z \leqq \varepsilon, |\operatorname{Im} z| \leqq \varepsilon/2\}.$$

Pick ε_0 so small that whenever $0 < \varepsilon \leqq \varepsilon_0$, then f is analytic in $D(\varepsilon)$ and g is analytic in $E(\varepsilon)$.

Let $F(\varepsilon)$ denote the set of all functions φ, real analytic in the interior of $S(\varepsilon)$, continuous on $S(\varepsilon)$ and satisfying the bound $|\varphi(z)| \leqq K + 1$ for all $z \in S(\varepsilon)$. Given φ in $F(\varepsilon)$ define

$$(R\varphi)(z) = f(z) + \int_0^{\pi/2} 2z^{\nu+1-p} \cos^{\nu+1-p}\theta \sin\theta g(z\sin^2\theta, \varphi(z\sin^2\theta))\, d\theta.$$

If ε is chosen so that

$$\beta = (\sqrt{5}\varepsilon/2)^{\nu+1-p}\pi M < 1,$$

then for any $z \in S(\varepsilon)$,

$$|(R\varphi)(z)| \leqq |f(z)| + \int_0^{\pi/2} 2(\sqrt{5}\varepsilon/2)^{\nu+1-p}|g(z\sin^2\theta, \varphi(z\sin^2\theta))|\, d\theta$$

$$\leqq K + \int_0^{\pi/2} 2(\sqrt{5}\varepsilon/2)^{\nu+1-p} M\, d\theta \leqq K + \beta < K + 1.$$

Therefore $S\varphi \in F(\varepsilon)$ when $\varphi \in F(\varepsilon)$. Moreover, if φ_1 and $\varphi_2 \in F(\varepsilon)$, then

$$|R\varphi_1(z) - R\varphi_2(z)| \leqq \int_0^{\pi/2} 2(\sqrt{5}\varepsilon/2)^{\nu+1-p} M |\varphi_1(z\sin^2\theta) - \varphi_2(z\sin^2\theta)|\, d\theta$$

$$\leqq \beta \max\{|\varphi_1(z) - \varphi_2(z)| : z \in S(\varepsilon)\}.$$

Therefore R is a contraction mapping on $F(\varepsilon)$.

Let $x(z)$ be the unique fixed point of R. Then $x(z)$ is real analytic in the interior of $S(\varepsilon)$, continuous on all of $S(\varepsilon)$ and $x(t)$ solves (17) if $0 \leqq t \leqq \varepsilon$. This means that the solution of (17) is analytic in a neighborhood of $0 < t < \varepsilon$.

Suppose we know that $x(z)$ is analytic in a set $\{z : 0 < \operatorname{Re} z \leqq \tau, |\operatorname{Im} z| \leqq \varepsilon/2\}$, where $\tau < T$.

Translation in (17) shows that

$$x(t+\tau) = f_\tau(t) + \int_0^t (t-s)^{\nu-p} g(s+\tau, x(s+\tau))\, ds,$$

where

$$f_\tau(t) = f(t+\tau) + \int_0^\tau (t+\tau-s)^{\nu-p} g(s, x(s))\, ds.$$

Since f_τ is real analytic in t and $|f_\tau(z)| \leqq K$ if $z \in S(\varepsilon)$, then the first part of the proof applies. This means that $x(z + \tau)$ is in the class $F(\varepsilon)$. Since the number ε has been fixed beforehand, one may step across the interval $0 \leqq t \leqq T$ in a finite number of steps. This completes the proof of Theorem 6.

COROLLARY 4. *Assume the hypotheses of Theorem 6. If p is a rational number, $p = r/q$ in lowest terms, then $x(t^q)$ is analytic in a neighborhood of $t = 0$.*

Proof. First replace t by z^q in (17′):

$$(17'') \quad x(z^q) = f(z^q) + \int_0^{\pi/2} 2z^{(\nu+1-p)q} \cos^{\nu+1-p}\theta \sin tg(z^q \sin^2\theta, x(z^q \sin^2\theta))\, d\theta.$$

Let $F(\varepsilon) = \{\varphi : \varphi$ is real analytic in $|z| < \varepsilon$ and continuous on $|z| \leqq \varepsilon\}$. Define

$$R\varphi(z) = f(z^q) + \int_0^{\pi/2} 2z^{(\nu+1-p)q} \cos^{\nu+1-p}\theta \sin\theta g(z^q \sin^2\theta, \varphi(z \sin^{2/q}\theta))\, d\theta$$

for $\varphi \in F(\varepsilon)$ and $|z| \leqq \varepsilon$. As in the proof of Theorem 6 one can show that if ε is sufficiently small, then R is a contraction mapping on $F(\varepsilon)$.

It would be interesting to know whether or not Theorem 6 can be generalized to a large class of kernels $a(t)$ which are analytic for $\operatorname{Re} t > 0$. The proof of Theorem 6 cannot be generalized too much since it depends on the monotonicity and homogeneity of $a(t) = t^{\nu-p}$. Corollary 4, which establishes the exact nature of the singularity of $x(z)$ at $z = 0$, is even more firmly wedded to the particular properties of $t^{\nu-p}$.

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SIAM J. MATH. ANAL.
Vol. 2, No. 2, May 1971

ON THE ASYMPTOTIC SOLUTION OF INITIAL VALUE PROBLEMS FOR DIFFERENTIAL EQUATIONS WITH SMALL DELAY*

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Abstract. This paper obtains asymptotic expansions for initial value problems of the form

$$\dot{x}(t) = f(t, x(t), x(t-\mu), \dot{x}(t-\mu)), \qquad t \geqq 0, \tag{1}$$

$$x(t) = \phi(t), \qquad -\mu \leqq t \leqq 0, \tag{2}$$

as the positive delay parameter μ tends to zero. The critical hypotheses are: (i) that the reduced problem ((1)–(2) with $\mu = 0$) has a unique solution $X_0(t)$, and (ii) that $|f_u(t, x, y, u)| < a < 1$ everywhere. Then $x(t)$ will converge to $X_0(t)$ as $\mu \to 0$ and boundary layer behavior will occur in higher order approximations near $t = 0$.

1. Introduction and summary. Consider the initial value problem consisting of the nonlinear differential-difference equation

$$\dot{x}(t) = f(t, x(t), x(t-\mu), \dot{x}(t-\mu)) \quad \text{for } t \geqq 0 \tag{1.1}$$

and the initial condition

$$x(t) = \phi(t) \quad \text{for } -\mu \leqq t \leqq 0. \tag{1.2}$$

We wish to determine the behavior of a continuous solution $x(t)$ as the positive delay parameter μ tends to zero. Let us assume that:

(i) the nonlinear reduced problem (1.1)–(1.2) with $\mu = 0$)

$$\begin{aligned} \dot{X}_0(t) &= f(t, X_0(t), X_0(t), \dot{X}_0(t)), \\ X_0(0) &= \phi(0) \end{aligned} \tag{1.3}$$

has a unique continuously differentiable solution $X_0(t)$ for $0 \leqq t \leqq T$;

(ii) $f(t, x, y, u)$ and $\phi(t)$ are infinitely differentiable (in all arguments); and

(iii) for some $\kappa > 0$,

$$|f_u(t, x, y, u)| < e^{-\kappa} < 1 \quad \text{everywhere}. \tag{1.4}$$

Under these conditions, we shall show that the solution $x(t)$ converges to $X_0(t)$ as $\mu \to 0$ on the closed interval $0 \leqq t \leqq T$. Higher order approximations, as in the familiar boundary layer theory, will converge nonuniformly at $t = 0$.

Before proceeding, we note that for moderate values of μ the problem could be solved by a stepwise integration scheme on $0 \leqq t \leqq T$. We would define

$$x(t) = x_0(t) = \phi(t) \quad \text{for } -\mu \leqq t \leqq 0$$

and we would define $x(t)$ to be the solution $x_j(t)$ of the initial value problem

$$\begin{aligned} &\dot{x}_j(t) = f(t, x_j(t), x_{j-1}(t-\mu), \dot{x}_{j-1}(t-\mu)), \\ &x_j((j-1)\mu) = x_{j-1}((j-1)\mu) \quad \text{for } (j-1)\mu \leqq t \leqq j\mu \end{aligned}$$

for each j with $1 \leqq j \leqq 1 + [T/\mu]$ (here, $[\alpha]$ represents the greatest integer $\leqq \alpha$).

* Received by the editors June 16, 1970, and in revised form September 24, 1970.

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Note that $\dot{x}(t)$ will generally be discontinuous at the values $t = j\mu$, $j \geqq 0$. For our problem, this method is unsatisfactory because the stepsize μ is too small. Instead, asymptotic methods are appropriate. Asymptotic approximation procedures for solving such problems were previously given by Vasil'eva and others (see [4]). The presentation here is simpler and is closely related to previous work on the asymptotic solution of initial value problems and boundary value problems (see Vasil'eva [3], O'Malley [1] and [2], and Wasow [6]).

We shall construct a formal solution of (1.1)–(1.2) on the interval $0 \leqq t \leqq T$ of the form

$$x(t) = X(t;\mu) + \mu v(\theta;\mu), \tag{1.5}$$

where the "outer expansion"

$$X(t;\mu) = \sum_{j=0}^{\infty} X_j(t)\mu^j \tag{1.6}$$

formally solves (1.1) and the "boundary layer correction" $v(\theta;\mu)$ has the form

$$v(\theta;\mu) = \sum_{j=0}^{\infty} v_j(\theta)\mu^j \tag{1.7}$$

in the stretched variable

$$\theta = t/\mu. \tag{1.8}$$

Taking $\phi(t)$ independent of μ, we have

$$\begin{aligned} X_0(0) &= \phi(0), \\ X_j(0) &= -v_{j-1}(0) \quad \text{for each } j \geqq 1. \end{aligned} \tag{1.9}$$

Further, the values $v_j(0)$ will be chosen so that

$$v_j(\theta) \to 0 \quad \text{as} \quad \theta \to \infty. \tag{1.10}$$

Away from $t = 0$, then, the asymptotic solution will be determined by the outer expansion alone, i.e., for each integer $N \geqq 0$,

$$x(t) = \sum_{j=0}^{N} X_j(t)\mu^j + O(\mu^{N+1}) \tag{1.11}$$

uniformly on each interval $\delta \leqq t \leqq T$ for any $\delta > 0$. As (1.9) shows, however, the initial values for the boundary layer correction terms must be calculated in order to determine the outer expansion.

2. Construction of the formal expansions. The outer expansion is obtained by substituting the sum (1.6) into the differential equation (1.1) and equating coefficients of like powers of μ^j. We have equality at $\mu = 0$ since X_0 solves the reduced problem. Equating first order coefficients, we see that X_1 must satisfy the linear equation

$$\begin{aligned} \dot{X}_1(t) = {} & f_x(t, X_0(t), X_0(t), \dot{X}_0(t))X_1(t) \\ & + f_y(t, X_0(t), X_0(t), \dot{X}_0(t))(X_1(t) - \dot{X}_0(t)) \\ & + f_u(t, X_0(t), X_0(t), \dot{X}_0(t))(\dot{X}_1(t) - \ddot{X}_0(t)), \end{aligned}$$

which we shall rewrite as

$$\begin{aligned}\dot{X}_1(t) &= (1 - f_{u0}(t))^{-1}(f_{x0}(t) + f_{y0}(t))X_1(t) + B_0(t)\\ &\equiv A(t)X_1(t) + B_0(t).\end{aligned}$$

In general, we find the X_j's successively as solutions of linear equations of the form

$$\dot{X}_j(t) = A(t)X_j(t) + B_{j-1}(t), \tag{2.1}$$

where the B_{j-1}'s are determined by lower order terms. Thus, the terms X_j of the outer expansion (1.6) can be generated recursively on $0 \leqq t \leqq T$ with only the initial values $X_j(0)$ for $j \geqq 1$ to be specified. These terms are infinitely differentiable.

Similarly, the terms of the boundary layer correction $v(\theta;\mu)$ are determined by stepwise integration on the intervals $p \leqq \theta \leqq p + 1$, $p \geqq 0$. By (1.5), we have

$$v_\theta(\theta;\mu) = \begin{cases} f(\mu\theta, X(\mu\theta;\mu) + \mu v(\theta;\mu), \phi(\mu(\theta-1)), \dot{\phi}(\mu(\theta-1))) \\ \quad - f(\mu\theta, X(\mu\theta;\mu), X(\mu(\theta-1);\mu), \dot{X}(\mu(\theta-1);\mu)) & \text{for } 0 \leqq \theta \leqq 1, \\ f(\mu\theta, X(\mu\theta;\mu) + \mu v(\theta;\mu), X(\mu(\theta-1);\mu) \\ \quad + \mu v(\theta-1;\mu), \dot{X}(\mu(\theta-1);\mu) + v_\theta(\theta-1;\mu)) \\ \quad - f(\mu\theta, X(\mu\theta;\mu), X(\mu(\theta-1);\mu), \dot{X}(\mu(\theta-1);\mu)) & \text{for } \theta \geqq 1. \end{cases} \tag{2.2}$$

For $\mu = 0$, then, we ask that $v_0(\theta)$ be a continuous function such that

$$v_{0\theta}(\theta) = \begin{cases} f(0, X_0(0), \phi(0), \dot{\phi}(0)) - f(0, X_0(0), X_0(0), \dot{X}_0(0)) & \text{for } 0 \leqq \theta \leqq 1, \\ f(0, X_0(0), X_0(0), \dot{X}_0(0) + v_{0\theta}(\theta-1)) \\ \quad - f(0, X_0(0), X_0(0), \dot{X}_0(0)) & \text{for } \theta \geqq 1. \end{cases} \tag{2.3}$$

Thus $v_{0\theta}$ is stepwise constant. Setting

$$v_{0\theta}(\theta) = G_p^0 \quad \text{for } p \leqq \theta \leqq p + 1$$

and integrating, we have

$$v_0(\theta) = v_0(0) + \sum_{l=0}^{p-1} G_l^0 + (\theta - p)G_p^0 \quad \text{for } p \leqq \theta \leqq p + 1. \tag{2.4}$$

Since $v_0(\theta) \to 0$ as $\theta \to \infty$, we must select

$$X_1(0) = -v_0(0) = \sum_{l=0}^{\infty} G_l^0 \tag{2.5}$$

assuming that this limit exists. To clarify this issue, we introduce the mapping F such that

$$Fu = f(0, \phi(0), \phi(0), u)$$

and

$$F^j u = f(0, \phi(0), \phi(0), F^{j-1}u) \quad \text{for integers } j > 1.$$

Note that (1.3) and (1.4) imply that F is a contraction having $\dot{X}_0(0)$ as its unique fixed point. Thus, $\dot{X}_0(0)$ can be obtained by successive approximations, i.e.,

$$\dot{X}_0(0) = \lim_{p \to \infty} F^p \dot{\phi}(0).$$

Further, since

$$G_0^p = F^p \dot{\phi}(0) - \dot{X}_0(0) \quad \text{for each } p \geqq 0,$$

(2.5) becomes

$$X_1(0) = \lim_{p \to \infty} \left\{ \sum_{l=0}^{p-1} (F^l \dot{\phi}(0)) - p\dot{X}_0(0) \right\}$$

and the limit is finite. This result was obtained by Vasil'eva [4].

In order to calculate higher order terms in $v(\theta; \mu)$, we need the following lemma.

LEMMA 1.

$$v_0(\theta) = O(e^{-\kappa\theta}) = v_{0\theta}(\theta) \quad \textit{as} \quad \theta \to \infty. \tag{2.6}$$

The proof of the lemma will be postponed to § 5.

Equating coefficients of μ^j in (2.2) successively to zero, we ask that

$$v_{j\theta}(\theta) = M_j(\theta) + v_{j\theta}(\theta - 1)N_j(\theta), \tag{2.7}$$

where $M_j(\theta)$ is a successively known linear combination of the $v_l(\theta)$, $v_l(\theta - 1)$ and $v_{l\theta}(\theta - 1)$ for $l < j$ and

$$N_j(\theta) = \begin{cases} 0 & \text{for } 0 \leqq \theta \leqq 1, \\ f_u(0, \phi(0), \phi(0), \dot{X}_0(0) + v_{0\theta}(\theta - 1)) & \text{for } \theta \geqq 1. \end{cases}$$

Setting

$$v_{j\theta}(\theta) = G_j^p(\theta) \quad \text{for } p \leqq \theta \leqq p + 1, \tag{2.8}$$

we determine the G_j^p's in turn, for $p = 0, 1, 2, \cdots$. Integrating stepwise, we find

$$v_j(\theta) = v_j(0) + \sum_{l=0}^{p-1} \left(\int_l^{l+1} G_j^l(s)\, ds \right) + \int_p^\theta G_j^p(s)\, ds \quad \text{for } p \leqq \theta \leqq p + 1. \tag{2.9}$$

(Note that these boundary layer correction terms will generally have discontinuous derivatives at positive integer values of θ.) Since $|N_j(\theta)| < e^{-\kappa}$, we can easily show that the following lemma holds.

LEMMA 2.

$$v_j(\theta) = O(e^{-\kappa(1-\delta)\theta}) = v_{j\theta}(\theta) \quad \textit{as} \quad \theta \to \infty \tag{2.10}$$

for any $\delta > 0$.

The proof of this lemma will be postponed to § 5.

We then obtain

$$X_{j+1}(0) = -v_j(0) = \int_0^\infty v_{j\theta}(s)\,ds = \sum_{l=0}^{\infty}\left(\int_l^{l+1} G_j^l(s)\,ds\right), \tag{2.11}$$

where the limit is finite. Thus, the complete expansion can be formally obtained termwise.

3. Main result. In order to show the asymptotic validity of the preceding formal results, we shall prove the following theorem.

THEOREM. *Under the assumptions* (i)–(iii), *the initial value problem* (1.1)–(1.2) *has a unique solution for* μ *sufficiently small. It is of the form*

$$x(t) = X^N(t) + \mu v^N(t/\mu) + \mu^{N+1}R_N(t;\mu)$$

for each integer $N \geqq 0$, *where*

$$X^N(t) = \sum_{j=0}^{N} X_j(t)\mu^j,$$

$$v^N(t/\mu) = \sum_{j=0}^{N-1} v_j(t/\mu)\mu^j,$$

and $R_N(t;\mu)$ *is uniformly bounded throughout* $0 \leqq t \leqq T$ *for* μ *sufficiently small.*

Note. Here assumption (iii) is crucial. The limiting case where $|f_u| = 1$ is discussed in Vasil'eva [5].

4. Examples.

Example 1. *Equations with retarded argument.* Consider the nonlinear problem

$$\begin{aligned} \dot{x}(t) &= g(t, x(t), x(t-\mu)) && \text{for } t \geqq 0,\\ x(t) &= \phi(t) && \text{for } -\mu \leqq t \leqq 0. \end{aligned}$$

Since the differential-difference equation is independent of $\dot{x}(t-\mu)$, the construction of the boundary layer correction terms becomes considerably simplified. In particular, the determination of the initial values $X_j(0)$ by the infinite sums (2.11) is avoided.

Here (2.2) becomes

$$v_\theta(\theta;\mu) = \begin{cases} g(\mu\theta, X(\mu\theta;\mu) + \mu v(\theta;\mu), \phi(\mu(\theta-1))) \\ \quad -g(\mu\theta, X(\mu\theta;\mu), X(\mu(\theta-1);\mu)) \quad \text{for } 0 \leqq \theta \leqq 1, \\ g(\mu\theta, X(\mu\theta;\mu) + \mu v(\theta;\mu), X(\mu(\theta-1);\mu) + \mu v(\theta-1;\mu)) \\ \quad -g(\mu\theta, X(\mu\theta;\mu), X(\mu(\theta-1);\mu)) \quad \text{for } \theta \geqq 1. \end{cases}$$

Setting $\mu = 0$, we obtain

$$v_{0\theta}(\theta) = 0 \quad \text{for } \theta \geqq 0$$

since $X_0(0) = \phi(0)$. Because $v_0 \to 0$ as $\theta \to \infty$, we select

$$v_0(\theta) \equiv v_0(0) = 0 \quad \text{for } \theta \geqq 0.$$

Thus, $X_1(0) = 0$ also. Likewise, from the coefficient of μ, we have

$$v_{1\theta}(\theta) = \begin{cases} g_y(0, \phi(0), \phi(0))(\dot{\phi}(0) - \dot{X}_0(0))(\theta - 1) & \text{for } 0 \leqq \theta \leqq 1, \\ 0 & \text{for } \theta \geqq 1. \end{cases}$$

Because $v_1(\theta) \to 0$ as $\theta \to \infty$, we must set $v_1(1) = 0$ so that

$$v_1(\theta) = \begin{cases} \frac{1}{2} g_y(0, \phi(0), \phi(0))(\dot{\phi}(0) - \dot{X}_0(0))(\theta - 1)^2 & \text{for } 0 \leqq \theta \leqq 1, \\ 0 & \text{for } \theta \geqq 1. \end{cases}$$

As an induction hypothesis, we suppose that the $v_l(\theta)$'s are known for $l < j$ and satisfy

$$v_l(\theta) \equiv 0 \quad \text{for } \theta \geqq l.$$

Then,

$$v_{j\theta}(\theta) = M_j(\theta),$$

where $M_j(\theta)$ is known for $\theta \geqq 0$ and satisfies $M_j(\theta) \equiv 0$ for $\theta \geqq j$. Since $v_j(\theta) \to 0$ as $\theta \to \infty$, we then have

$$v_j(\theta) = \begin{cases} \displaystyle\int_\theta^j M_j(s)\, ds & \text{for } \theta \leqq j, \\ 0 & \text{for } \theta \geqq j \end{cases}$$

for all integers j and

$$X_{j+1}(0) = -\int_0^j M_j(s)\, ds.$$

Further, our theorem implies that for any integer $N \geqq 0$,

$$x(t) = \sum_{j=0}^{N} \mu^j X_j(t) + O(\mu^{N+1}) \quad \text{for } \mu N \leqq t \leqq T.$$

Example 2. *Linear difference equations for* $\dot{x}(t)$. Consider the linear problem

$$\begin{aligned} \dot{x}(t) &= a\dot{x}(t - \mu) && \text{for } t \geqq 0, \\ x(t) &= \phi(t) && \text{for } -\mu \leqq t \leqq 0 \end{aligned}$$

for $|a| < 1$. The outer expansion here is such that

$$X_0(t) = \phi(0),$$

and for each $j \geqq 1$,

$$X_j(t) = p_{j-1}(t) e^{at},$$

where $p_j(t)$ is a polynomial of degree j and $X_j(0)$ must be determined.

Equation (2.2) implies that the boundary layer correction $v(\theta; \mu)$ must satisfy

$$v_\theta(\theta; \mu) = a(\dot{\phi}(\mu(\theta - 1)) - \dot{X}(\mu(\theta - 1); \mu)) \equiv \sum_{j=0}^{\infty} \mu^j G_j^0(\theta) \quad \text{for } 0 \leqq \theta \leqq 1$$

and

$$v_\theta(\theta;\mu) = av_\theta(\theta - 1;\mu) \quad \text{for } \theta \geqq 1.$$

Equating coefficients, then, we have

$$v_{j\theta}(\theta) = G_j^p(\theta) = a^p G_j^0(\theta - p) \quad \text{for } p \leqq \theta \leqq p + 1.$$

Here, $G_j^0(\theta)$ is known successively in terms of the $v_l(\theta)$ for $l < j$. For example,

$$G_0^0(\theta) = a\dot{\phi}(0)$$

and

$$G_1^0(\theta) = a(\dot{\phi}(0)(\theta - 1) + v_0(0)).$$

Integrating then, we have

$$v_j(\theta) = a^p \left\{ \left(\frac{1}{1 - a} \right) \left(\int_0^1 G_j^0(s)\, ds \right) + \int_p^\theta G_j^0(s - p)\, ds \right\} \quad \text{for } p \leqq \theta \leqq p + 1$$

and

$$X_{j+1}(0) = \left(\frac{1}{1 - a} \right) \left(-\int_0^1 G_j^0(s)\, ds \right).$$

The expansions so obtained will be uniformly valid on any finite t interval.

5. Proofs.

Proof of Lemma 1. Rewriting (2.3) as

$$v_{0\theta}(\theta) = v_{0\theta}(\theta - 1)\tilde{F}_p(\theta) \quad \text{for } p \leqq \theta \leqq p + 1, \quad p \geqq 1,$$

we see that each $\tilde{F}_p$ satisfies $|\tilde{F}_p(\theta)| < e^{-\kappa} < 1$. Thus

$$|v_{0\theta}(\theta)| \leqq w_0(\theta) \quad \text{for all } \theta \geqq 0,$$

where

$$w_0(\theta) = \begin{cases} |v_{0\theta}(\theta)| & \text{for } 0 \leqq \theta \leqq 1, \\ e^{-\kappa} w_0(\theta - 1) & \text{for } \theta \geqq 1. \end{cases}$$

Noting that

$$w_0(\theta) = e^{-\kappa p} w_0(\theta - p) \quad \text{for } p \leqq \theta \leqq p + 1,$$

we have

$$|v_{0\theta}(\theta)| \leqq w_0(\theta) \leqq Be^{-\kappa\theta} \quad \text{for all } \theta,$$

where $B \geqq \max_{0 \leqq \alpha \leqq 1} |w_0(\alpha)/e^{-\kappa\alpha}|$. Since $v_0(\theta) \to 0$ as $\theta \to \infty$, we also have

$$|v_0(\theta)| \leqq \int_0^\infty |v_{0\tau}(s)|\, ds \leqq \frac{B}{\kappa} e^{-\kappa\theta} \quad \text{for all } \theta.$$

Proof of Lemma 2. As an induction hypothesis, suppose that the $v_l(\theta)$'s are known for $l < j$ and satisfy

$$v_l(\theta) = O(e^{-\kappa(1-\delta)\theta}) = v_{l\theta}(\theta) \quad \text{as} \quad \theta \to \infty$$

for any $\delta > 0$. This will imply that

$$|M_j(\theta)| \leqq Be^{-\kappa(1-\delta)\theta}$$

for some $B > 0$. Since $|N_j(\theta)| < e^{-\kappa}$, (2.7) implies that

$$|v_{j\theta}(\theta)| \leqq w_j(\theta) \quad \text{for all } \theta \geqq 0,$$

where

$$w_j(\theta) \equiv \begin{cases} |v_{j\theta}(\theta)| & \text{for } 0 \leqq \theta \leqq 1, \\ Be^{-\kappa(1-\delta)\theta} + e^{-\kappa}w_j(\theta - 1) & \text{for } \theta \geqq 1. \end{cases}$$

Noting that

$$w_j(\theta) = e^{-\kappa p}w_j(\theta - p) + Be^{-\kappa(1-\delta)\theta}\left(\frac{1 - e^{-p\kappa\delta}}{1 - e^{-\kappa\delta}}\right) \quad \text{for } p \leqq \theta \leqq p + 1,$$

we have

$$0 \leqq |v_{j\theta}(\theta)| \leqq w_j(\theta) \leqq ce^{-\kappa(1-\delta)\theta} \quad \text{for all } \theta,$$

where $c > B + \max_{0 \leqq \alpha \leqq 1} |w_j(\alpha)/e^{-\kappa\alpha}|$. Since $v_j(\theta) \to 0$ as $\theta \to \infty$, this implies that

$$|v_0(\theta)| \leqq \frac{c}{\kappa(1 - \delta)}e^{-\kappa(1-\delta)\theta} \quad \text{for all } \theta.$$

Proof of the theorem. The initial value problem (1.1)–(1.2) implies that R_N must satisfy the equation

$$\mu^{N+1}\dot{R}_N(t;\mu) = f(t, x(t), x(t - \mu), \dot{x}(t - \mu)) - \dot{X}^N(t) - v_\theta^N(\theta) \quad \text{for } 0 \leqq t \leqq T$$

and the initial condition

$$R_N(0;\mu) = 0.$$

Moreover, since the outer expansion $X(t;\mu)$ satisfies (1.1), we have

$$\dot{X}^N(t) = f(t, X^N(t), X^N(t - \mu), \dot{X}^N(t - \mu)) + \mu^{N+1}B_1(t;\mu) \quad \text{for } 0 \leqq t \leqq T,$$

where B_1 is bounded. Likewise, since the boundary layer correction $v(\theta;\mu)$ satisfies (2.2), we have

$$\begin{aligned} v_\theta^N(\theta) = {} & f(\mu\theta, X^N(\mu\theta) + \mu v^N(\theta), \phi(\mu(\theta - 1)), \dot{\phi}(\mu(\theta - 1))) \\ & - f(\mu\theta, X^N(\mu\theta), X^N(\mu(\theta - 1)), \dot{X}^N(\mu(\theta - 1))) \\ & + \mu^N B_2(\theta;\mu) \quad \text{for } 0 \leqq \theta \leqq 1 \end{aligned}$$

with B_2 bounded, and

$$\begin{aligned} v_\theta^N(\theta) = {} & f(\mu\theta, X^N(\mu\theta) + \mu v^N(\theta), X^N(\mu(\theta - 1)) \\ & + \mu v^N(\theta - 1), \dot{X}^N(\mu(\theta - 1)) + v_\theta^N(\theta - 1)) \\ & - f(\mu\theta, X^N(\mu\theta), X^N(\mu(\theta - 1)), \dot{X}^N(\mu(\theta - 1))) \\ & + \mu^N B_3(\theta;\mu)\, e^{-\kappa(1-\delta)\theta} \quad \text{for } \theta \geqq 1 \end{aligned}$$

with B_3 bounded.

Thus for $0 \leqq t \leqq \mu$,

$$\begin{aligned}\mu^{N+1}\dot{R}_N(t;\mu) = f(t, X^N(t) &+ \mu v^N(\theta) \\ &+ \mu^{N+1}R_N(t;\mu), \phi(t-\mu), \dot{\phi}(t-\mu)) \\ &- f(t, X^N(t) + \mu v^N(\theta), \phi(t-\mu), \dot{\phi}(t-\mu)) \\ &- \mu^{N+1}B_1(t;\mu) - \mu^N B_2(\theta;\mu)\end{aligned}$$

or

$$\dot{R}_N(t;\mu) = R_N(t;\mu)F_1(t) + \frac{1}{\mu}C_1(\theta;\mu),$$

where F_1 is a function of t and $\mu^{N+1}R_N(t;\mu)$ and $C_1(\theta;\mu) = -B_2(\theta;\mu) - \mu B_1(\mu\theta;\mu)$. Integrating, we have

$$R_N(t;\mu) = \mu\int_0^{t/\mu} R_N(\mu s;\mu)F_1(\mu s)\,ds + R_N^0(t;\mu) \quad \text{for } 0 \leqq t \leqq \mu,$$

where $R_N^0(t;\mu) = \int_0^{t/\mu} C_1(s;\mu)\,ds$ is bounded. This Volterra integral equation can be simply solved by successive approximations. One sets

$$R_N^{j+1}(t;\mu) = R_N^0(t;\mu) + \mu\int_0^{t/\mu} R_N^j(\mu s;\mu)F_1(\mu s)\,ds$$

for each $j \geqq 0$ and defines

$$R_N(t;\mu) = \lim_{j\to\infty} R_N^j(t;\mu).$$

Convergence to a unique continuous solution on $0 \leqq t \leqq \mu$ is assured since F_1 is bounded (cf. Willett [7]).

Likewise, for $t \geqq \mu$, we have

$$\begin{aligned}\mu^{N+1}\dot{R}_N(t;\mu) = f(t, X^N(t) &+ \mu v^N(\theta) + \mu^{N+1}R_N(t;\mu), X^N(t-\mu) \\ &+ \mu v^N(\theta-1) + \mu^{N+1}R_N(t-\mu;\mu), \dot{X}^N(t-\mu) \\ &+ v_\theta^N(\theta-1) + \mu^{N+1}\dot{R}_N(t-\mu;\mu)) \\ &- f(t, X^N(t) + \mu v^N(\theta), X^N(t-\mu) \\ &+ \mu v^N(\theta-1), \dot{X}_N(t-\mu) + v_\theta^N(\theta-1)) \\ &+ \mu^{N+1}C(t;\mu),\end{aligned}$$

where $C(t;\mu) = -B_1(t;\mu) - (1/\mu)B_3(t/\mu;\mu)\,e^{-\kappa(1-\delta)t/\mu}$ is integrable. Thus,

$$\dot{R}_N(t;\mu) = R_N(t;\mu)F_2(t) + R_N(t-\mu;\mu)F_3(t) + \dot{R}_N(t-\mu;\mu)F_4(t) + C(t;\mu),$$

where F_2, F_3 and F_4 are smooth functions of $t, \mu^{N+1}R_N(t;\mu), \mu^{N+1}R_N(t-\mu;\mu)$ and $\mu^{N+1}\dot{R}_N(t-\mu;\mu)$. Integrating, then, we see that $R_N(t;\mu)$ must solve an integral equation of the form

$$\begin{aligned}R_N(t;\mu) = R_N(t-\mu;\mu)F_4(t) &+ R_N^0(t;\mu) \\ &+ \int_\mu^t L(s, R_N(s;\mu), R_N(s-\mu;\mu))\,ds \quad \text{for } \mu \leqq t \leqq T.\end{aligned}$$

Here,

$$L(t, R_N(t;\mu), R_N(t-\mu;\mu)) = R_N(t;\mu)F_2(t) + R_N(t-\mu;\mu)(F_3(t) - \dot{F}_4(t))$$

and

$$R_N^0(t;\mu) = R_N(\mu;\mu) - R_N(0;\mu)F_4(\mu) + \int_\mu^t C(s;\mu)\,ds$$

is bounded since $R_N(\mu;\mu)$ is bounded. This equation, too, can be solved by successive approximations for μ sufficiently small. For each $j \geqq 0$, we define

$$R_N^{j+1}(t;\mu) = R_N(t;\mu) \quad \text{for } 0 \leqq t \leqq \mu$$

and

$$R_N^{j+1}(t;\mu) = R_N^j(t-\mu;\mu)F_4(t) + R_N^0(t;\mu) + \int_\mu^t L(s, R_N^j(s;\mu), R_N^j(s-\mu;\mu))\,ds \quad \text{for } \mu \leqq t \leqq T$$

and set

$$R_N(t;\mu) = \lim_{j\to\infty} R_N^j(t;\mu).$$

All iterates will be bounded. Moreover, they will converge to a unique continuous solution on $\mu \leqq t \leqq T$ by the contraction mapping principle because $|F_4(t)| < e^{-\kappa} < 1$ and the kernel $L(t, R, R)$ is Lipschitz continuous in its second and third arguments for μ small. Further, majorants can be simply obtained to yield explicit bounds for R_N throughout $0 \leqq t \leqq T$.

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SIAM J. MATH. ANAL.
Vol. 2, No. 2, May 1971

THE MAXIMUM LIKELIHOOD ESTIMATE OF THE NONCENTRALITY PARAMETER OF A NONCENTRAL F VARIATE*

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Abstract. The maximum likelihood estimate of the noncentrality parameter of a noncentral F distribution with p and q degrees of freedom is obtained. This estimate is expressed as a solution of an equation involving confluent hypergeometric functions for which tables are readily available [8]. Our derivation is based upon a crucial lemma that the quotient of the derivative of a confluent hypergeometric function by itself is a strictly monotonic function for different positive parameters and positive argument. The maximum likelihood estimate of the noncentrality parameter of noncentral χ^2 and noncentral t^2 distributions are also derived as very special cases of our result.

1. Introduction. In 1949, P. B. Patnaik dealt with various applications of noncentral χ^2 and noncentral F distributions in testing hypotheses and calculating the power of such tests [2]. In 1967 the maximum likelihood estimate of the noncentrality parameter of a noncentral χ^2 distribution with 2 degrees of freedom was determined by P. L. Meyer [1]. Recently the maximum likelihood estimate of the noncentrality parameter of a noncentral χ^2 variate with p degrees of freedom was also determined [9]. The object of this paper is to find the maximum likelihood estimate of the noncentrality parameter of a noncentral F distribution. It is understood that the noncentral F variate under consideration is the quotient of a noncentral χ^2 variate by a central χ^2 variate. Following the notations of F. A. Graybill [3, p. 78] we assume that the noncentral F variate is denoted by F'. It is a fact that if $u = wq/pz$, where w is a random variable distributed as $\chi^2(p, \lambda)$, that is, as the noncentral chi-square with p degrees of freedom and noncentrality parameter λ, and if another random variable z is distributed as $\chi^2(q, 0)$, that is, as the central chi-square with q degrees of freedom, and if w and z are independent, then u is distributed as the noncentral F distribution with p, q degrees of freedom and with noncentrality parameter λ, and the frequency function of u is

$$f(u) = \sum_{i=0}^{\infty} \frac{\Gamma((2i + p + q)/2)(p/q)^{(2i+p)/2}}{\Gamma(q/2)\Gamma((2i + p)/2)i!} \lambda^i e^{-\lambda} \frac{u^{(2i+p-2)/2}}{(1 + pu/q)^{(2i+p+q)/2}}, \tag{1}$$

$0 \leqq u < \infty$. (See [3, p. 78].)

In a situation like this we shall say that U is distributed as $F'(p, q, \lambda)$. It is easily seen that

$$f(u) = \frac{\Gamma((p + q)/2)}{\Gamma(p/2)\Gamma(q/2)} \frac{(pu/q)^{p/2}}{u(1 + pu/q)^{(p+q)/2}} e^{-\lambda} M\left(\frac{p + q}{2}, \frac{p}{2}; \frac{\lambda pu}{q + pu}\right), \tag{2}$$

where $M(a, b; x)$ stands for the confluent hypergeometric function with para-

* Received by the editors April 7, 1970, and in revised form September 14, 1970.

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meters a, b, and argument x, that is,

$$M(a, b; x) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{x^n}{n!}, \tag{3}$$

where $(a)_n$ is Pochhammer's symbol:

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1), \quad (a)_0 = 1.$$

2. The likelihood function. Let us assume that $u_1, u_2, \cdots, u_n$ represent a sample of size n from a random variable U which is distributed as $F'(p, q, \lambda)$ whose probability density function is given by (2). Hence the likelihood function L will be given by

$$L \equiv L(u_1, u_2, \cdots, u_n; \lambda)$$

$$= \frac{e^{-n\lambda}}{B^n(p/2, q/2)} \prod_{i=1}^{n} \frac{(pu_i/q)^{p/2}}{u_i(1 + pu_i/q)^{(p+q)/2}} M\left(\frac{p+q}{2}, \frac{p}{2}; \frac{\lambda pu_i}{q + pu_i}\right).$$

Therefore,

$$\log L = -n \log B\left(\frac{p}{2}, \frac{q}{2}\right) - n\lambda + \frac{p}{2} \sum_{i=1}^{n} \log\left(\frac{pu_i}{q}\right)$$

$$- \sum_{i=1}^{n} \log u_i - \frac{p+q}{2} \sum_{i=1}^{n} \log\left(1 + \frac{pu_i}{q}\right)$$

$$+ \sum_{i=1}^{n} \log M\left(\frac{p+q}{2}, \frac{p}{2}; \frac{\lambda pu_i}{q + pu_i}\right).$$

Hence, our maximum likelihood equation is

$$n = \sum_{i=1}^{n} \frac{pu_i}{q + pu_i} F(\lambda; u_i), \tag{4}$$

where

$$F(\lambda; u) = \frac{M'((p+q)/2, p/2; \lambda pu/(q + pu))}{M((p+q)/2, p/2; \lambda pu/(q + pu))}. \tag{5}$$

If the maximum likelihood estimate exists, it will be obtained by solving for λ from (4). In the next section we will deal with existence as well as uniqueness of such an estimate. To discuss these results we shall need some results which we prove below.

3.

LEMMA 1. *For the confluent hypergeometric function $M(a, b; x)$ with parameters a, b and argument $x(\geqq 0)$, $M'(a, b; x)/M(a, b; x)$ is:*

(i) *a strictly decreasing function of x if $a > b > 0$,*

(ii) *a strictly increasing function of x if $0 < a < b$.*

Proof. We have

$$\frac{d}{dx}\left[\frac{M'(a,b;x)}{M(a,b;x)}\right] = \frac{M''(a,b;x)M(a,b;x) - M'^2(a,b;x)}{M^2(a,b;x)}.$$

Therefore, our result will be proved by showing that $\{M''(a,b;x)M(a,b;x) - M'^2(a,b;x)\}$ is strictly negative or strictly positive according as $a > b$ or $a < b$. Now it can be readily seen that

$$M'(a,b;x) = \frac{a}{b}\sum_{n=0}^{\infty}\frac{(a+1)_n}{(b+1)_n}\frac{x^n}{n!}, \tag{6}$$

and

$$M''(a,b;x) = \frac{a(a+1)}{b(b+1)}\sum_{n=0}^{\infty}\frac{(a+2)_n}{(b+2)_n}\frac{x^n}{n!}. \tag{7}$$

By using the formula for the Cauchy product of two series one can readily see that

$$M''(a,b;x)M(a,b;x) - M'^2(a,b;x) = \sum_{n=0}^{\infty} x^n\gamma_n, \tag{8}$$

where

$$\gamma_n = \frac{a}{b}\frac{\Gamma(b)\Gamma(b+1)}{\Gamma(a)\Gamma(a+1)}(b-a)\sum_{r=0}^{n}\frac{1}{n!}\binom{n}{r}\frac{\Gamma(a+r)\Gamma(a+n+1-r)}{\Gamma(b+r+1)\Gamma(b+n+2-r)}(n+1-2r). \tag{9}$$

Our lemma will be proved if we can show that for all n, γ_n is strictly negative or strictly positive according as $a > b$ or $a < b$.

In the finite sum in (9) there are $n + 1$ terms, and it is evident that if n is odd the first $(n + 1)/2$ terms are positive, the $((n + 3)/2)$th term is zero and the last $(n - 1)/2$ terms are negative. On the other hand, if n is even, the first $(n + 2)/2$ terms are positive and the last $n/2$ terms are negative. This makes it possible to pair t_1 with t_n, t_2 with $t_{n-1}, \cdots, t_r$ with t_{n-r+1} for $r \leqq n/2$ when n is even, where

$$t_r = \frac{1}{n!}\binom{n}{r}\frac{\Gamma(a+r)\Gamma(a+n+1-r)}{\Gamma(b+1+r)\Gamma(b+n+2-r)}(n+1-2r). \tag{10}$$

It can be easily seen that

$$t_r + t_{n-r+1} = \frac{\Gamma(a+r)\Gamma(a+n+1-r)(n+1-2r)^2}{\Gamma(b+1+r)\Gamma(b+n+2-r)(n-r+1)!r!}. \tag{11}$$

Clearly, $t_r + t_{n-r+1} > 0$ whether $r \leqq (n - 1)/2$ or $r \leqq n/2$ as they are in two different cases.

Now

$$\gamma_n = \frac{a\Gamma(b)\Gamma(b+1)(b-a)}{b\Gamma(a)\Gamma(a+1)}\left[\sum_{r=1}^{k(n)}(t_r + t_{n-r+1}) + t_0\right],$$

where $k(n) = (n - 1)/2$ when n is odd, and $k(n) = n/2$ when n is even. Note that t_0 is obtained by taking $r = 0$ in (10) which is clearly positive, and $t_r + t_{n-r+1}$

is evaluated by (11). Thus it is quite evident that $\gamma_n < 0$ if $a > b$ and $\gamma_n > 0$ if $a < b$. So the proof of our lemma is complete.

Now we obtain the following result quite trivially.

COROLLARY. *If $a > b > 0$, we have for $x > 0$,*

$$\frac{a}{b} = \frac{M'(a,b;0)}{M(a,b;0)} > \frac{M'(a,b;x)}{M(a,b;x)}.$$

It is obvious, therefore, that for $\lambda > 0$ and $u > 0$,

$$\frac{M'((p+q)/2, p/2; \lambda pu/(q+pu))}{M((p+q)/2, p/2; \lambda pu/(q+pu))} < \frac{p+q}{p}, \tag{12}$$

where p and q are positive integers.

THEOREM 1. *The solution to the likelihood equation*

$$n = \sum_{i=1}^{n} \frac{pu_i}{q+pu_i} F(\lambda; u_i),$$

if it exists at all, is unique.

Proof. If the solution to the maximum likelihood equation is not unique, assume that there are two solutions λ and λ'. For the sake of definiteness let us say that $\lambda' > \lambda$.

Therefore,

$$\sum_{i=1}^{n} \frac{pu_i}{q+pu_i}[F(\lambda; u_i) - F(\lambda'; u_i)] = 0, \tag{13}$$

which is a contradiction, as in view of Lemma 1, we have,

$$F(\lambda; u_i) - F(\lambda'; u_i) > 0.$$

THEOREM 2. *If*

$$\frac{p+q}{p} \sum_{i=1}^{n} \frac{pu_i}{q+pu_i} \leqq n,$$

the maximum value of $\log L$ *is obtained at* $\lambda = 0$.

Proof. From (4) we have

$$\begin{aligned}\frac{\partial(\log L)}{\partial \lambda} &= -n + \sum_{i=1}^{n} \frac{pu_i}{q+pu_i} F(\lambda; u_i) \\ &\leqq -\frac{p+q}{p} \sum_{i=1}^{n} \frac{pu_i}{q+pu_i} + \sum_{i=1}^{n} \frac{pu_i}{q+pu_i} F(\lambda; u_i) \\ &= -\sum_{i=1}^{n} \left[\frac{p+q}{p} - F(\lambda; u_i)\right] \frac{pu_i}{q+pu_i} < 0,\end{aligned}$$

assuming that all u_i's are not zero. Therefore, $\log L$ is a decreasing function of λ. Consequently, $\hat{\lambda}$, the maximum likelihood (ML) estimate of the parameter λ, based on a sample $u_1, u_2, \cdots, u_n$, is zero if

$$\frac{p+q}{p} \sum_{i=1}^{n} \frac{pu_i}{q+pu_i} \leqq n. \tag{14}$$

However, it will be shown later that the situation (14) is unlikely to occur when $n \to \infty$.

THEOREM 3. *If the random variable U has probability density function given by* (2), *then λ, the maximum likelihood estimate of the parameter λ, based on a sample $u_1, u_2, \cdots, u_n$, when*

$$\frac{p+q}{p}\sum_{i=1}^{n}\frac{pu_i}{q+pu_i} > n,$$

is obtained as the unique solution of

$$-n + \sum_{i=1}^{n}\frac{pu_i}{q+pu_i}F(\lambda; u_i) = 0. \tag{15}$$

Proof. We have already observed in the corollary to Lemma 1 that $F(\lambda; u) \to (p+q)/p$ as $\lambda \to 0+$. Therefore, if $\phi(\lambda)$ denotes $\partial(\log L)/\partial\lambda$,

$$\lim_{\lambda\to 0+}\phi(\lambda) = \frac{p+q}{p}\sum_{i=1}^{n}\frac{pu_i}{q+pu_i} - n > 0.$$

But $\phi(\lambda)$ is continuous for $\lambda > 0$; therefore, $\phi(\lambda) > 0$ when λ is sufficiently small and positive. Further it can be readily seen that

$$M'(a, b; x) = \frac{a}{b}M(a+1, b+1; x), \qquad a, b > 0.$$

Therefore, in view of the fact that as $|z| \to \infty$,

$$M(a, b; z) = \frac{\Gamma(b)}{\Gamma(a)}e^z z^{a-b}[1 + O(|z|^{-1})], \qquad \operatorname{Re} z > 0$$

(see [8, p. 504]), we have

$$\lim_{x\to\infty}\frac{M'(a, b; x)}{M(a, b; x)} = 1.$$

Thus $\lim_{\lambda\to\infty} F(\lambda; u) = 1$ when $u > 0$, and

$$\lim_{\lambda\to\infty} F(\lambda; u) = \frac{p+q}{p} \quad \text{when} \quad u = 0. \tag{16}$$

Now

$$\begin{aligned}\lim_{\lambda\to\infty}\phi(\lambda) &= -n + \sum_{i=1}^{n}\frac{pu_i}{q+pu_i} \qquad \text{(all } u_i\text{'s} > 0)\\ &= -\sum_{i=1}^{n}\frac{q}{q+pu_i} < 0.\end{aligned}$$

If however some of the u_i's happen to be zero, the limit of $\phi(\lambda)$ as $\lambda \to \infty$ will be smaller. Thus we see that $\phi(\lambda)$ is positive when λ is sufficiently large. Therefore, there exists a positive zero of $\phi(\lambda)$. Uniqueness has already been proved in Theorem 1.

Since $\phi(\lambda)$ changes from positive to negative values if λ passes the zero of $\phi(\lambda)$, it follows that the zero of $\phi(\lambda)$ obtained must correspond to the global maximum of $\log L$.

THEOREM 4.

$$\lim_{n\to\infty} P\left[\frac{p+q}{p}\sum_{i=1}^{n}\frac{pu_i}{q+pu_i} > n\right] = 1. \tag{17}$$

Proof. Let U be a noncentral F distribution whose frequency function is given by (2). Let $E^2 = pu/(q + pu)$. It is well known that the frequency function of E^2 is

$$g(E^2; p, q; \lambda) = \sum_{i=0}^{\infty}\frac{\Gamma((2i+p+q)/2)}{\Gamma(q/2)\Gamma((2i+p)/2)i!}\lambda^i e^{-\lambda}(E^2)^{(2i+p-2)/2}(1-E^2)^{(q-2)/2}, \tag{18}$$

$0 \leqq E^2 \leqq 1$. (See [3, p. 79].)

It can be readily shown that the expectation of E^2 is given by

$$\mathscr{E}(E^2) = \sum_{i=0}^{\infty}\frac{e^{-\lambda}\lambda^i}{i!}\frac{2i+p}{2i+p+q} > \sum_{i=0}^{\infty}\frac{e^{-\lambda}\lambda^i}{i!}\frac{p}{p+q} = \frac{p}{p+q}.$$

Thus,

$$\mathscr{E}(E^2) > \frac{p}{p+q}.$$

Therefore,

$$\mathscr{E}(E^2) = \frac{p}{p+q} + \eta, \tag{19}$$

where η is a positive quantity depending upon p, q and λ. Let ε be an arbitrarily small positive quantity less than η.

Now if $u_1, u_2, \cdots, u_n$ are the sample elements from the random variable U, invoking the weak law of large numbers [5] we obtain

$$\lim_{n\to\infty} P\left[\frac{p}{p+q} + \eta - \varepsilon < \left|\sum_{i=1}^{n}\frac{pu_i}{q+pu_i}\Big/n\right| < \frac{p}{p+q} + \eta + \varepsilon\right] = 1.$$

Thus,

$$\lim_{n\to\infty} P\left[\sum_{i=1}^{n}\frac{pu_i}{q+pu_i}\Big/n > \frac{p}{p+q}\right] = 1$$

or

$$\lim_{n\to\infty} P\left[\frac{p+q}{p}\sum_{i=1}^{n}\frac{pu_i}{q+pu_i} > n\right] = 1.$$

Remark. Thus with probability approaching 1, as $n \to \infty$, a solution of the maximum likelihood equation exists which depends on sample values.

4. Particular cases. In this section we shall deal with a few important special cases of our main result.

(a) $p = 1$. It is well known that if $p = 1$, the distribution of the noncentral F variate whose frequency function is given by (1) reduces to that of noncentral

t^2 variate [2, p. 220]; therefore, the ML estimate of the noncentrality parameter of the noncentral t^2 distribution will be obtained by solving for λ from

$$n = \sum_{i=1}^{n} \frac{M'((q+1)/2, 1/2; \lambda u_i/(q+u_i))}{M((q+1)/2, 1/2; \lambda u_i/(q+u_i))} \frac{u_i}{q+u_i} \tag{20}$$

or

$$n = \sum_{i=1}^{n} \frac{(q+1)u_i}{q+u_i} \frac{M((q+3)/2, 3/2; \lambda u_i/(q+u_i))}{M((q+1)/2, 1/2; \lambda u_i/(q+u_i))}. \tag{21}$$

(b) We now deal with the case when $q \to \infty$, and p is fixed. It can be seen that when $q \to \infty$, the distribution of U whose probability density function is given by (2) reduces to that of $\chi'^2(p, \lambda)/p$ (see [2, p. 221]). Now since

$$M'\left(\frac{p+q}{2}, \frac{p}{2}; \frac{\lambda pu}{q+pu}\right) = \frac{p+q}{p} M\left(\frac{p+q+2}{2}, \frac{p+2}{2}; \frac{\lambda pu}{q+pu}\right),$$

we have

$$F(\lambda; u) = \frac{p+q}{p} \frac{M((p+q+2)/2, (p+2)/2; \lambda pu/(q+pu))}{M((p+q)/2, p/2; \lambda pu/(q+pu))}.$$

Hence,

$$\lim_{q \to \infty} \frac{F(\lambda; u)}{q+pu} = \frac{1}{p} \frac{\Gamma((p+2)/2)(\lambda pu/2)^{1/2-(p+2)/4} I_{p/2}(\sqrt{2\lambda pu})}{\Gamma(p/2)(\lambda pu/2)^{1/2-(p+2)/4} I_{p/2-1}(2\lambda pu)}$$

[8, p. 506]

$$= \frac{1}{\sqrt{2\lambda pu}} \cdot \frac{I_{p/2}(\sqrt{2\lambda pu})}{I_{p/2-1}(\sqrt{2\lambda pu})}.$$

Thus, when $q \to \infty$, the ML equation (15) reduces to

$$n = \sum_{i=1}^{n} \frac{I_{p/2}(\sqrt{2\lambda pu_i})}{I_{p/2-1}(\sqrt{2\lambda pu_i})} \frac{pu_i}{\sqrt{2\lambda pu_i}}. \tag{22}$$

Using a suitable transformation one can easily show that the ML equation for $\chi'^2(p, \lambda)$ is

$$n = \sum_{i=1}^{n} F(aR_i)R_i^2, \tag{23}$$

where

$$F(x) = \frac{1}{x} \frac{I_{p/2}(x)}{I_{p/2-1}(x)}$$

and $R_1, R_2, \cdots, R_n$ are sample values from $\chi'^2(p, \lambda)$.

The ML equation (23) is in complete agreement with that obtained in [9, p. 8].

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SIAM J. MATH. ANAL.
Vol. 2, No. 2, May 1971

AN EXISTENCE AND UNIQUENESS THEOREM FOR BOUNDARY VALUE PROBLEMS OF NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS*

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Abstract. A condition is given for the existence and uniqueness of solutions for n-point boundary value problems which generalizes the condition of Pólya for linear differential equations. A unique solution exists if the nth order differential equation can be integrated in n steps, reducing the order by one at each step.

1. Introduction. In 1922 G. Pólya [3] gave a set of conditions which assure uniqueness and existence for boundary value problems of linear differential equations. The purpose of this paper is to generalize his ideas to include nonlinear differential equations. Some of Pólya's results can be transferred almost verbatim to the general case, but additional considerations are required, owing to the fact that uniqueness does not automatically assure existence in the nonlinear case.

2. Statement of the theorem. Consider the nth order differential equation (D.E.)

$$y^{(n)} = f_n(x, y, y', \cdots, y^{(n-1)}), \tag{1}$$

where f_n is continuous in $(a, b) \times R^n$ and fulfills a Lipschitz condition for the $y^{(i)}$'s uniform in every compact subset of $(a, b) \times R^n$.

This D.E. is called *reducible* if it is equivalent to a family of $(n - 1)$st order D.E. of the same type, i.e., if there exists a function f_{n-1}, continuous in $(a, b) \times R^n$ with Lipschitz condition for the $y^{(i)}$'s, such that any solution of (1) is also a solution of

$$y^{(n-1)} = f_{n-1}(x, y, y', \cdots, y^{(n-2)}; c) \tag{2}$$

for some value of the parameter c, and, conversely, if a solution of (2) for any value of c, $-\infty < c < +\infty$, is a solution of (1). Moreover, different members of (2) will have no solution in common. This establishes a one-to-one relation between $y^{(n-1)}$ and c, and as a consequence, there exists a function F_n continuous in $(a, b) \times R^n$ (see Lemma 4) such that

$$c = F_n(x, y, y', \cdots, y^{(n-1)}). \tag{2a}$$

The D.E. (1) is called decomposable if (2) is again reducible for every value of the parameter c and so forth; that is, if there are equivalent families of D.E.'s: D.E.'s:

$$\begin{aligned} y^{(n)} &= f_n(x, y, y', \cdots, y^{(n-1)}), \\ y^{(n-1)} &= f_{n-1}(x, y, y', \cdots, y^{(n-2)}; c_n), \\ &\cdots \\ y' &= f_1(x, y; c_2, c_3, \cdots, c_n), \\ y &= f_0(x; c_1, c_2, \cdots, c_n), \end{aligned} \tag{3}$$

* Received by the editors July 28, 1970.

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and the parameters c_i are determined uniquely for every point $(x, y, y', \cdots, y^{(n-1)})$ by the continuous functions F_i:

$$
\begin{aligned}
c_n &= F_n(x, y, y', \cdots, y^{(n-1)}),\\
c_{n-1} &= F_{n-1}(x, y, y', \cdots, y^{(n-2)}; c_n),\\
&\cdots\\
c_1 &= F_1(x, y; c_2, \cdots, c_n).
\end{aligned}
\tag{4}
$$

In particular, the last equation of (3) establishes a principal set of solutions of (1).

A k-point nth order boundary value problem (k-n-B.V.P.) of the D.E. (1) is defined as follows: Given k points ξ_i, $i = 1, \cdots, k, a < \xi_1 < \xi_2 < \cdots < \xi_k < b$, k positive integers n_i, $\sum_{i=1}^k n_i = n$, and a set of n values η_i^j, $i = 1, \cdots, k, j = 0, \cdots, n_i - 1$; we require a solution $y(x)$ of (1) defined in $\xi_1 \leqq x \leqq \xi_k$, which satisfies $y^{(j)}(\xi_i) = \eta_i^j$. In particular, a 1-$n$-B.V.P. is an initial value problem. This definition excludes some mixed boundary value problems.

With these definitions, the following is true.

THEOREM 1. *If the n-th order D.E.* (1) *is decomposable, then every k-n-B.V.P. has one and only one solution.*

In Pólya's paper the linear D.E.

$$y^{(n)} = a_1(x)y^{(n-1)} + \cdots + a_n(x)y \tag{5}$$

is brought into the form

$$u_n(x)\frac{d}{dx}(u_{n-1}(x)\frac{d}{dx}(\cdots \frac{d}{dx}(u_0(x)y(x)) \cdots)) = 0, \tag{5a}$$

from which the decomposition (3), (4) can be obtained by simple integration; for example,

$$F_n = u_{n-1}(x)\frac{d}{dx}(\cdots \frac{d}{dx}(u_0(x)y(x)) \cdots) = c_n$$

and f_{n-1} has the form

$$f_{n-1} = b_1(x)y^{n-2} + \cdots + b_{n-1}(x)y + c_n\phi_n(x).$$

We observe that for any solution $y(x)$ of (1),

$$\frac{d}{dx}(F_n(x, y(x), y'(x), \cdots, y^{(n-1)}(x))) \equiv 0. \tag{6}$$

If F_n is differentiable, then it is the solution of the first order partial differential equation

$$F_{n,x} + y'F_{n,y} + y''F_{n,y'} + \cdots + f_n(x, y, y', \cdots, y^{(n-1)})F_{n,y^{(n-1)}} = 0 \tag{7}$$

($F_{n,x} = \partial F_n/\partial x$, etc.), which is obtained from (6) by replacing $y^{(n)}$ by f_n. Conversely, the D.E. (1) is uniquely determined by (6) using (7) if

$$F_{n,y^{(n-1)}} \neq 0 \quad \text{in } (a, b) \times R^n. \tag{8}$$

Equivalent conditions hold for the other functions in (3), (4).

Equations (3), (4) establish for fixed x a one-to-one mapping $R^n \leftrightarrow R^n$ from the parameters c_i to the derivatives $y^{(j)}$. Conditions are especially simple if this mapping is differentiable in both directions. The following theorem holds.

THEOREM 2. *Let*

$$y = f(x; c_1, c_2, \cdots, c_n) \tag{9}$$

be a set of principal solutions of (1) *with the following properties*:

(i) *Equation* (9) *satisfies* (1) *for every set of values* $(c_i) \in R^n$; *any solution of* (1) *is of the form* (9) *for one and only one set of parameters* c_i.

(ii) f *and the first* $n - 1$ *derivatives with respect to* x *are differentiable with respect to* (c_i).

(iii)

$$\begin{aligned} &\frac{\partial y}{\partial c_1} \neq 0, \\ &\frac{\partial(y, y')}{\partial(c_1, c_2)} \neq 0, \\ &\vdots \\ &\frac{\partial(y, y', \cdots, y^{(n-1)})}{\partial(c_1, c_2, \cdots, c_n)} \neq 0, \end{aligned} \tag{10}$$

i.e., these Jacobians do not vanish for any point in $(a, b) \times R_n$.

Then (1) *is decomposable.*

It is easy to see that under the stated conditions systems of the form (3), (4) can be obtained from (9) and its derivatives. An easy, but tedious, computation shows also that (10) is equivalent to (8) for the functions (4) if the differentiability conditions are fulfilled. For a linear D.E. the Jacobians (10) are the Wronskians considered by Pólya.

The proof of Theorem 1 is divided into two parts. The first part is concerned with the uniqueness, the second part with the existence of the boundary value problems.

3. Proof of uniqueness. As in Pólya's paper the proof is based on generalizations of Rolle's theorem.

LEMMA 1. *Let* (1) *be reducible and* $\phi(x)$ *a* C^n*-function in* (a, b). *Define*

$$\hat{c}(\phi; x) = F_n(x, \phi(x), \cdots, \phi^{(n-1)}(x)) \tag{11}$$

with F_n *as in* (2a). *Let* $\hat{c}(\phi; x_1) = \hat{c}(\phi; x_2)$ *for some pair of points* $a < x_1 < x_2 < b$. *Then there exists a point* ξ, $x_1 < \xi < x_2$, *such that*

$$\phi^{(n)}(\xi) = f_n(\xi, \phi(\xi), \phi'(\xi), \cdots, \phi^{(n-1)}(\xi)). \tag{12}$$

Proof. If F_n is differentiable, then (12) is a simple application of Rolle's theorem. If $\hat{c}(\phi; x)$ is only assumed to be continuous, then it is either constant in $x_1 \leqq x \leqq x_2$ or has a maximum or minimum different from $\hat{c}(\phi; x_{1,2})$. In the first case $\phi(x)$ is a solution of (1) in (x_1, x_2) and (12) must be true. In the second case let $\hat{c}(\phi\,;\,\xi) = c_M$ be a maximum and

$$\phi^{(n)}(\xi) \neq f_n(\xi, \phi(\xi), \phi'(\xi), \cdots, \phi^{(n-1)}(\xi)),$$

say, $\phi^{(n)}(\xi) > f_n(\xi, \cdots)$. Let $u(x)$ be a solution of (1) with initial conditions at ξ, $u^{(j)}(\xi) = \phi^{(j)}(\xi), j = 0, \cdots, n-1$. Then $u^{(n)}(x) < \phi^{(n)}(x)$ in some neighborhood of ξ, and there are constants d, D and δ such that $0 < d < \phi^{(n)}(t_1) - u^{(n)}(t_2) < D$ for all t_1, t_2 in $\langle \xi - \delta, \xi + \delta \rangle$. Thus,

$$\phi^{(n-1)}(x) \lesseqgtr u^{(n-1)}(x) + d(x - \xi) \tag{13}$$

depending on whether $\xi - \delta \leqq x < \xi$ or $\xi < x \leqq \xi - \delta$. Moreover,

$$|\phi^{(j)}(x) - u^j(x)| < \frac{D}{2}(x - \xi)^2 \tag{14}$$

for all x in $\langle \xi - \delta, \xi + \delta \rangle, j = 0, \cdots, n-2$.

We have

$$u^{(n-1)}(x) = f_{n-1}(x, u(x), u'(x), \cdots, u^{(n-2)}(x); c_M) \tag{15}$$

and

$$\begin{aligned} &|f_{n-1}(x, u(x), \cdots, u^{(n-2)}(x); c_M) - f_{n-1}(x, \phi(x), \cdots, \phi^{(n-2)}(x); c_M)| \\ &\qquad < \frac{LD}{2}(x - \xi)^2, \end{aligned} \tag{16}$$

where L is the Lipschitz constant for f_{n-1} in a sufficiently large cube around $(\xi, u(\xi), \cdots, u^{(n-2)}(\xi))$. From (13)–(16) it follows that

$$\phi^{(n-1)}(x) \lesseqgtr f_{n-1}(x, \phi(x), \cdots, \phi^{(n-2)}(x); c_M) \tag{17}$$

for $\xi - \delta' < x < \xi$ or $\xi < x < \xi + \delta'$, respectively, and a sufficiently small $\delta' > 0$. According to (2) and (2a) there is a one-to-one continuous relation between $y^{(n-1)}$ and c; that is, F_n and f_n are monotone functions in $y^{(n-1)}$ and c respectively for fixed values of $x, y, y', \cdots, y^{(n-2)}$. Moreover, these functions are either all decreasing or all increasing, independent of $x, y, \cdots, y^{(n-2)}$. Thus from (17) it follows that

$$\hat{c}(\phi, x) \lesseqgtr c_M$$

depending on whether $x \lesseqgtr \xi$; that is, c_M cannot be a maximum of $\hat{c}(\phi, x)$. Thus $\phi^{(n)}(\xi) = u^{(n)}(\xi)$ or (12) is true.

In this proof the Lipschitz condition can be replaced by a Nagumo condition around ξ (see [1, p. 31]). The lemma becomes false if the initial value problem at ξ of the D.E. (2) has more than one solution. It is an open question whether Lemma 1 remains true, if f_{n-1} is continuous and every initial value problem of (2) has only one solution.

Repeated application of Lemma 1 leads to a second lemma.

LEMMA 2. *Let* (1) *be decomposable and* $u(x) = f_0(x; c_1, c_2, \cdots, c_n)$, *a solution of* (1) *for some set* (c_i). *Let* $\phi(x)$ *be* C^{k-1} *in* (a, b) *and* $u^{(j)}(x_i) = \phi^{(j)}(x_i), i = 1, \cdots, l, u < x_1 < x_2 < \cdots < x_l < b, j = 0, \cdots, n_i - 1, \sum_{i=1}^{l} n_i = k$. *Then*

$$F_k(\xi, \phi(\xi), \cdots, \phi^{(k-1)}(\xi); c_{k+1}, \cdots, c_n) = c_k$$

for some $\xi, x_1 \leqq \xi \leqq x_l$.

An immediate consequence of Lemma 2 is the following.

LEMMA 3. *Let* (1) *be decomposable and*

$$u(x) = f_0(x; c_1, \cdots, c_k, c_{k+1}, \cdots, c_n),$$
$$\hat{u}(x) = f_0(x; \hat{c}_1, \cdots, \hat{c}_k, c_{k+1}, \cdots, c_n)$$

be two solutions of (1). *Further, let* $u^{(j)}(x_i) = \hat{u}^{(j)}(x_i)$, $i = 1, \cdots, l, u < x_1 < \cdots < x_l < b, j = 1, \cdots, n_i - 1, \sum_{i=1}^{l} n_i = k$. *Then* $\hat{c}_i = c_i, i = 1, \cdots, k$, *i.e.*, $\hat{u}(x) \equiv u(x)$.

Proof. According to Lemma 2 there exists a ξ, $x_1 \leqq \xi \leqq x_l$, such that

$$F_k(\xi, \hat{u}(\xi), \hat{u}(\xi), \cdots, \hat{u}^{(k-1)}(\xi); c_{k+1}, \cdots, c_k) = c_k;$$

but, since $\hat{u}(x)$ is a solution with parameters $(\hat{c}_1, \cdots, \hat{c}_k, c_{k+1}, \cdots, c_n)$, $F_k(x, \hat{u}(x), \cdots, \hat{u}^{(k-1)}(x); c_{k+1}, \cdots, c_n) = \hat{c}_k$ for all x. Thus $\hat{c}_k = c_k$. Repetition of the process leads to $\hat{c}_{k-1} = c_{k-1}, \cdots$ down to $\hat{c}_1 = c_1$.

Setting $k = n$ we obtain the following corollary.

COROLLARY. *If* (1) *is decomposable, then every k-n-B.V.P. has at most one solution.*

4. Proof of existence. As a first step in the proof we have to establish that the solutions of a B.V.P. depend continuously on the parameters of the problem. The basis for this is a little known topological theorem.

THEOREM 3. *Let D be a domain (nonempty, open, connected subset) in R^n and T a continuous one-to-one mapping from D into R^n. Then the image $T(D)$ is also a domain in R^n and the inverse mapping is continuous.*

For a proof, see Rado and Reichelderfer [4, p. 135]. An immediate consequence is the following lemma.

LEMMA 4. *Let the functions*

$$y_i = f_i(x_1, \cdots, x_n; c_1, \cdots, c_k) \tag{18}$$

be continuous in $R^n \times D^k$, where D^k is a domain in R^k, and let (18) *represent a one-to-one mapping from R^n onto R^n for any set $(c_1, \cdots, c_k) \in D^k$. Then the inverse functions*

$$x_i = g_i(y_1, \cdots, y_n; c_1, \cdots, c_k), \qquad i = 1, \cdots, n, \tag{19}$$

are continuous in $R^n \times D^k$.

Proof. By adding the equations

$$c_j = c_j, \quad j = 1, \cdots, k, \tag{20}$$

we obtain a one-to-one continuous mapping of $R^n \times D^k$ onto itself whose inverse is (19) and (20) and is thus continuous in $R^n \times D^k$.

The following lemma lists conditions under which solutions of B.V.P.'s depend continuously on the parameters of the problem.

LEMMA 5. *Let $y(x; \eta; c)$ be the solution of the k-n-B.V.P.*

$$y^{(n)} = f(x, y, \cdots, y^{(n-1)}; c_1, \cdots, c_l) \tag{21}$$

with $y^{(j)}(\xi_i; c; \eta) = \eta_i^j, i = 1, \cdots, k; j = 0, \cdots, n_i, \sum_{i=1}^{k} n_i = n, (\eta_i^j) \in R^n, (\xi_i) \in D^k \subset R^k$, *where D^k is a domain such that $a < \xi_1 < \xi_2 < \xi_k < b$, for all sets $(\xi_i) \in D^k$.*

Consider $y^{(j)}(\xi;\eta;c)$, $j = 0, \cdots, n-1$, $a < \xi < b$. *These are continuous functions in* $\xi \in (a, b)$, $(\xi_i) \in D^k$, $(\eta_i^j) \in R^n$, $(c_m) \in R^l$ *if the following conditions are satisfied*:

(i) $f(x, y, \cdots, y^{n-1}; c_1, \cdots, c_l)$ *is continuous in* $(a, b) \times R^n \times R^l$.

(ii) *All solutions of* (21) *can be extended through the interval* (a, b).

(iii) *The k-n-B.V.P. has a unique solution for any set* $(\eta_i^j) \in R^n$, $(\xi_i) \in D^k$, $(c_m) \in R^l$.

(iv) *All initial value problems have a unique solution.*

Proof. Under the stated conditions a solution of a B.V.P. is a solution of an initial value problem and there is a one-to-one continuous mapping from a set of initial values to a set η_i^j for fixed ξ_i, ξ and c_m, these being considered the parameters of Lemma 4. Thus the initial values are continuously dependent on η_i^j, ξ_i, ξ and c_m.

LEMMA 6. *Let* $y(x;\eta;c)$ *be defined as in Lemma 5 and let* $\eta_t^{n_t-1}$ *vary from* $-\infty$ *to* $+\infty$, *for some* t, $1 \leqq t \leqq k$, *all other* η*'s*, ξ_i*'s and* c_m*'s being fixed. Then* $y(\xi;\eta;c)$ *for* $\xi \neq \xi_i$, $i = 1, \cdots, k$, *and* $y^{(n_j)}(\xi_j, \eta; c)$ *vary also from* $-\infty$ *to* $+\infty$ *and are monotone continuous functions of* $\eta_t^{n_t-1}$ *under conditions* (i)–(iii) *of Lemma* 5 *and the additional condition*:

(iv)* *The following modified B.V.P. has a unique solution for all values of* η_{k+1}^0 *or* $\eta_j^{n_j}$ *respectively*: $y^{(j)}(\xi_i) = \eta_i^j$, $i = 1, \cdots, k$; $j = 0, \cdots, n_i - 1$ *for* $i \neq t$, $j = 0, \cdots, n_t - 2$ *for* $i = t$; $y(\xi) = \eta_{k+1}^0$ *or* $y^{(n_j)}(\xi_j) = \eta_j^{n_j}$, *respectively.*

The proof of this lemma is straightforward.

LEMMA 7. *Let the D.E.* (1) *be reducible and* (2) *be its reduced form. There exists a solution for any k-n-B.V.P. of* (1) *if the following conditions are fulfilled*:

(i) *All solutions of* (1) *can be extended through the whole interval* (a, b).

(ii) *A solution of a k-n-B.V.P. of* (1) *if it exists is unique.*

(iii) *There is a unique solution of any k-*$(n-1)$*-B.V.P. of* (2) *for any value of* c.

Proof. The basic idea of this proof is to eliminate one condition of the k-n-B.V.P. and solve the ensuing k'-$(n-1)$-B.V.P. of (2) for all values of c. By varying c one hopes to fulfill the missing condition. The technique is similar to that used by Lasota and Opial [2].

We shall discuss in detail only the n-n-B.V.P. The general case can be handled in the same manner. Let $a < \xi_1 < \xi_2 < \cdots < \xi_{n-1} < \xi_n < b$, and let $y(x;\eta;c)$ be the solution of the $(n-1)$-$(n-1)$-B.V.P. of (2) with

$$y(\xi_i;\eta;c) = \eta_i, \qquad i = 1, \cdots, n-1.$$

Let $\hat{\eta}(c) = y(\xi_n;\eta;c)$. According to Lemma 5, $\hat{\eta}(c)$ is continuous; it is also monotone because different values of c lead to different solutions of (1) and no two solutions can have the same boundary values. Lemma 7 is established if one can show that the range of $\hat{\eta}(c)$ is R.

Let us assume that $\hat{\eta}(c)$ is monotone increasing and that for some sequence $c_0 < c_1 < \cdots < c_\nu \to +\infty$, $\lim_{\nu\to\infty} \hat{\eta}(c_\nu) = \eta^+ < +\infty$. Let $\hat{\eta}(c_0) < \eta^+ - \varepsilon$ for some $\varepsilon > 0$.

We first establish that $y^{(j)}(x;\eta;c_u)$ is uniformly bounded for $j = 0, \cdots, n-2$ in $\xi_1 + \delta \leqq x \leqq \xi_n - \delta$ for all $\delta > 0$. This is done by constructing certain "limiting" solutions of (2) for a fixed $c = c_0$; let us call them $y_j^+(x)$ and $y_j^-(x)$, $j = 0, \cdots, n-2$. These are solutions of k-$(n-2)$-B.V.P.'s to be specified later with the additional conditions

$$\begin{aligned} &y_j^-(\xi_n) \leqq \eta^+ - \varepsilon < \eta^+ + \varepsilon \leqq y_j^+(\xi_n) \quad \text{and} \\ (22) \qquad &y_j^-(\xi_1) \leqq \eta_1 - \varepsilon < \eta_1 + \varepsilon < y_j^+(\xi_1) \quad \text{or} \\ &y_j^+(\xi_1) \leqq \eta_1 - \varepsilon < \eta_1 + \varepsilon \leqq y_j^-(\xi_1). \end{aligned}$$

These combined with any $n - 2$ additional boundary conditions can always be satisfied according to Lemma 6 and part (iii) of Lemma 7. More specifically, let

$$y_0^+(\xi_i) = y_0^-(\xi_i) = \eta_i, \qquad i = 2, \cdots, n-1.$$

It is easy to see that $y(x;\eta;c_\nu)$ is always between $y_0^+(x)$ and $y_0^-(x)$ for $\xi_1 \leqq x \leqq \xi_n$, $x \neq \xi_i$, $i = 2, \cdots, n-1$, and $y'(\xi_i;\eta;c_\nu)$ is between $y_0^{+\prime}(\xi_i)$ and $y^{-\prime}(\xi_i)$, $i = 2, \cdots, n = 1$.[1] This is true for $x = \xi_1, \xi_n$ because of (22). If it is not true at some other point, then $y(x;\eta;c_\nu)$ has to cross either y_0^+ or y_0^- and has to cross the same solution again in order to return. If this happens at one of the points ξ_i it must be $y'(\xi_i;\eta;c_\nu) = y_0^+(\xi_i)$ or $y_0^{-\prime}(\xi_i)$. Under either condition $y(x;\eta;c_\nu)$ would have n points in common with either y^+ or y^- counting multiplicity, and a k-n-B.V.P. of (1) would have more than one solution. Thus $y(x;\eta;c_\nu)$ is uniformly bounded.

For y_1^+, y_1^-, we require that the following conditions be satisfied:

$$y_1^+(\xi) = y_1^-(\xi) = y(\xi;\eta;c_\nu)$$

for some ξ, $\xi_1 + \delta \leqq \xi \leqq \xi_n - \delta$, $\xi \neq \xi_i$, and

$$y_1^+(\xi_i) = y_1^-(\xi_i) = \eta_i, \qquad 1 \leqq i \leqq n-1, \quad i \neq i_0,$$

for some i_0. By the same argument as above, $y'(\xi;\eta;c_\nu)$ is between $y_1^{+\prime}(\xi)$ and $y_1^{-\prime}(\xi)$. Of course, the limiting functions depend on ξ and $y(\xi;\eta;c_\nu)$; but these values can be restricted to compact sets. This is true for $y(\xi;\eta;c_\nu)$ because it is uniformly bounded. The value of ξ can be restricted to one of the overlapping intervals $\xi_{i_0-1} + \delta \leqq \xi \leqq \xi_{i_0+1} - \delta$, $2 \leqq i_0 \leqq n-1$; $y_1^{+\prime}(\xi)$ and $y_1^{-\prime}(\xi)$ are continuous functions of ξ in these intervals provided one of the inequalities (22) is replaced by equality. This is possible by Lemma 6. Thus $y'(\xi;\eta;c_\nu)$ is uniformly bounded.

The same method is used for the higher derivatives. For instance, the conditions for y_{n-1}^+, y_{n-1}^- read

$$y_{n-2}^{+(j)}(\xi) = y_{n-2}^{-(j)}(\xi) = y^{(j)}(\xi;\eta;c_\nu), \qquad j = 0, \cdots, n-3,$$

from which it follows that $y^{(n-2)}(\xi;\eta;c_\nu)$ lies between $y_{n-1}^{+(n-2)}(\xi)$ and $y_{n-1}^{-(n-2)}(\xi)$.

Since $y^{(n-2)}(\xi;\eta;c_\nu)$ is uniformly bounded, the mean value theorem shows that there exists for every ν a point $x = t_\nu$ such that $|y^{(n-1)}(t_\nu;\eta;c_\nu)| < M$ for sufficiently large M independent of ν. We can therefore select a subsequence c_{ν_j} such that $t_{\nu_j} \to t_0$, $y^{(j)}(t_{\nu_j};\eta;c_{\nu_j}) \to u_j \neq \infty$, $j = 0, \cdots, n-1$. But (2a) gives

$$F_n(t_{\nu_j}, y(t_{\nu_j};\eta;c_{\nu_j}) \cdots) = c_{\nu_j}$$

[1] Note that y_0^+ and y_0^- have no points in common outside ξ_i and the derivatives are different at these points.

and the left side converges to $F_n(t_0, u_0, u_1, \cdots, u_{n-1}) \neq \infty$ contrary to the assumption that $c_{\nu_j} \to \infty$. Thus $\eta^+ = \infty$.

It remains to show that the set of D.E.'s (3) satisfies the conditions of Lemma 7, starting from the bottom. We observe first that any solution of (3) exists in the whole interval (a, b). Because of the relation $y = f_0(x; c_1, \cdots, c_n)$, y remains finite in (a, b) for any fixed set (c_i). So does y' since

$$y' = f_1(x, f_0(x; c_1, \cdots, c_n); c_2, \cdots, c_n)$$

and so on up to $y^{(n)}$. Furthermore, the D.E.

$$y' = f_1(x, y; c_2, c_3, \cdots, c_n) \tag{3a}$$

has a unique solution for any 1-1-B.V.P. because of the Lipschitz condition on f_1 with respect to y.

Also a k-2-B.V.P. of the D.E.

$$y' = f_2(x, y, y'; c_3, \cdots, c_n), \tag{3b}$$

whose reduced form is (3a), has at most one solution (Lemma 3). Thus every k-2-B.V.P. of (3b) has a unique solution. Repeated application of this process establishes the existence of all k-n-B.V.P. of the D.E. (1).

5. Example and conclusion. As an example for a nonlinear decomposable D.E. let the function $\phi(a, b)$ be defined as the solution of

$$\phi^3 + a\phi = b. \tag{23}$$

This equation has a unique real solution for all $0 < a < +\infty$, $-\infty < b < +\infty$. Its partial derivatives are

$$\phi_b = -\phi\phi_a = (\beta\phi^2 + a)^{-1}. \tag{24}$$

Consider the D.E.

$$y'' = y' \tanh x + (2\cosh x)^{-1}\phi((\cosh x)^{-1}, y - y' \tanh x). \tag{25}$$

This has the following decomposition:

$$y' = -2\cosh x + e^x\phi(e^x, y - 2c_2 \sinh x), \tag{26a}$$

$$y = c_1 e^x + c_2 e^{-x} + (c_1 + c_2)^3 \tag{26b}$$

and

$$c_2 = (2\cosh x)^{-1}\left(-y' + \frac{e^x}{2}\phi((\cosh x)^{-1}, y - y' \tanh x\right), \tag{27a}$$

$$c_1 = -c_2 + \phi(e^x; y - 2\sinh x). \tag{27b}$$

Equations (27a), (27b) are obtained solving (26a), (26b) for c_2 and c_1, respectively. Equivalence of (26a), (26b) with (25) is established by differentiation of (27a), (27b); using (23), (24), one obtains

$$\frac{dc_2}{dx} = (2\cosh x)^{-1}\left(1 + \phi_b \frac{e^x}{2} \tanh x\right)(y^n - y' \tanh x - (2\cosh x)^{-1}\phi) = 0,$$

$$\frac{dc_1}{dx} = \phi_b(y' - 2c_2 \cosh x - e^x\phi) = 0,$$

i.e., (25) and (26a), except for the nonvanishing factor (8). The Jacobians (10) are

$$\frac{\partial y}{\partial c_1} = e^x + 3(c_1 + c_2)^2 > 0,$$

$$\frac{\partial(y, y')}{\partial(c_1, c_2)} = -2 - 6(c_1 + c_2)^2 \cosh x < 0,$$

i.e., the conditions of Theorem 2 are satisfied. Thus every 2-2-B.V.P. of (25) has a solution which can be verified directly. A similar D.E. is discussed by Lasota and Opial in [2].

The above example is somewhat artificial due to the fact that not many D.E.'s can be decomposed in the prescribed manner. However, most of the proofs remain valid if the decomposition can be achieved in a suitable subset of $(a, b) \times R^n$. Consider, for instance, the D.E.

$$y'' = y'^2/y, \tag{28}$$

which satisfies all conditions in D: $-\infty < x < +\infty$, $0 < y < +\infty$, $-\infty < y' < +\infty$. A decomposition of (28) is

$$y' = c_2 y, \tag{29a}$$

$$y = e^{c_1 + c_2 x} \tag{29b}$$

and

$$c_2 = y'/y, \tag{30a}$$

$$c_1 = -c_2 x + \log y. \tag{30b}$$

All 2-2-B.V.P.'s of (28) in D (i.e., $\eta_1^0, \eta_2^0 > 0$) have a solution. Further work is necessary to determine conditions for D and possible modifications of the theorem.

Another interesting question is whether the converse of Theorem 1 is true, i.e., is a D.E. decomposable if every B.V.P. has a unique solution? If this condition is satisfied for some interval $a < x < b$, the decomposition may be obtained in a smaller interval, say, (a', b), $a < a'$, provided that the parameters c_n are chosen to satisfy the initial conditions at some point ξ, $a < \xi < a'$, $c_i = y^{(n-i)}(\xi)$. Existence and uniqueness for every 2-n-B.V.P. assures the existence of continuous functions in (3), (4). However, the D.E.'s (3) may have more than one solution of an initial value problem, since there is no guarantee that the functions so obtained are of Lipschitz type. This may introduce solutions of (3) which are not solutions of the original D.E. (1). Whether this can be avoided is an open question.

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SIAM J. MATH. ANAL.
Vol. 2, No. 2, May 1971

A REPRESENTATION THEOREM FOR A CLASS OF CONVOLUTION TRANSFORMABLE GENERALIZED FUNCTIONS*

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Abstract. Let $K_{c,d}(t)$ be an infinitely differentiable function defined over $(-\infty, \infty)$ such that

$$K_{c,d}(t) = \begin{cases} e^{ct} & \text{for } t \geqq 1, \\ e^{dt} & \text{for } t \leqq -1, \end{cases}$$

and $K_{c,d}(t) \neq 0$ in $(-1, 1)$, the quantities c and d being real. We say that an infinitely differentiable function $\phi(x)$ defined over $(-\infty, \infty)$ belongs to Zemanian's $L_{c,d}$-space if

$$\gamma_k(\phi) = \sup_{-\infty < t < \infty} |K_{c,d}(t)\phi^{(k)}(t)| < \infty$$

for $k = 0, 1, 2, \cdots$. The topology on $L_{c,d}$-space is generated by the sequence of seminorms $\{\gamma_k\}_{k=0}^{\infty}$. Zemanian extended the real inversion theory (for the convolution transform of functions) of Hirschman and Widder to $L'_{c,d}$-space, but did not give a structure formula.

In this paper an extension of $L_{c,d}$-space and its dual space to n dimensions is given and a structure formula obtained which shows that, globally, every element of the dual space of $L(c, d)$ is the linear combination of the finite order distributional derivative of continuous functions. The space $L(c, d)$ is the strict inductive limit of L_{c_ν, d_ν}, $c_\nu \to c+$, $d_\nu \to d-$. Some special cases are also derived.

The Hirschman-Widder convolution transformation [1] has recently been extended to certain classes of generalized functions [3], and their real inversion formula [1, pp. 127–132] has been shown to be still valid when the limiting operation in that formula is understood as weak convergence in the space D' of Schwartz distributions [8]. The complex inversion formula [1, Theorem 7.1b, p. 231] has been extended in a similar way to the convolution transform of generalized functions. In 1967 Zemanian introduced the space $L'_{c,d}$ of generalized functions [3], where c and d are fixed real (arbitrary) constants. The real as well as complex inversion formulas of Hirschman and Widder are shown to be valid for the space of generalized functions $L'_{c,d}$, where the constants c and d are restricted in some way [3], [5]. The testing function space $L_{c,d}$ which was dealt with in [3] and [5] was defined over R^1. In this note we shall deal with an $L_{c,d}$-space defined over R^n, and the constants c and d will be fixed arbitrary elements of R^n. Our object is to find a representation formula for a certain subspace of $L'_{c,d}$.

The notation and terminology will follow that of [7]. Unless otherwise stated t and x will be understood to be variables in R^n and the letters c, d and a will signify constants in R^n. If a and b are in R^n, by $a > b$ we mean that $a_i > b_i$ for $i = 1, 2, \cdots, n$, where a_i and b_i are the components of a and b respectively. When c and x both belong to R^n, the expression cx is understood to be the scalar product of c and x. The differentiation operator D^k is understood to be the operator

$$\frac{\partial^{k_n}}{\partial x_n^{k_n}} \cdots \frac{\partial^{k_2}}{\partial x_2^{k_2}} \frac{\partial^{k_1}}{\partial x_1^{k_1}},$$

where $k = (k_1, k_2, \cdots, k_n)$ and the k_i are nonnegative integers. The order of the differentiation operator D^k will be defined as the number $|k| = k_1 + k_2 + \cdots + k_n$.

* Received by the editors March 26, 1970, and in revised form August 17, 1970.

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The testing function space $L_{c,d}$. Choose a real-valued infinitely differentiable positive function $K_{c,d}(t)$ over R^n such that

$$K_{c,d}(t) = \begin{cases} e^{ct} & \text{for } t \geqq 1, \\ e^{dt} & \text{for } t \leqq -1. \end{cases}$$

One way of choosing such a function $K_{c,d}(t)$ over R^n is as follows: Define

$$K_{c,d}(t) = \prod_{i=1}^{n} K_{c_i,d_i}(t_i),$$

where

$$K_{c_i,d_i} = \begin{cases} e^{c_i t_i} & \text{for } t_i \geqq 1, \\ e^{d_i t_i} & \text{for } t_i \leqq -1, \end{cases}$$

and $K_{c_i,d_i}(t_i)$ is infinitely differentiable and positive on $(-\infty, \infty)$. A complex-valued and infinitely differentiable function $\phi(t)$ defined over R^n is said to belong to the space $L_{c,d}$ if

$$\gamma_m(\phi) = \max_{0 \leqq |k| \leqq m} \sup_{t \in R^n} |K_{c,d}(t) D^k \phi| < \infty$$

for $m = 0, 1, 2, \cdots$. Clearly $L_{c,d}$ is a vector space closed with respect to differentiation.

The convergence in $L_{c,d}$. A sequence $\{\phi_\nu(t)\}_{\nu=1}^{\infty}$, where $\phi_\nu(t)$ is in $L_{c,d}$ for each ν, is said to converge to $\phi(t)$ in $L_{c,d}$ if $\gamma_k(\phi_\nu - \phi) \to 0$ as $\nu \to \infty$ for $k = 0, 1, 2, \cdots$. We further add that a sequence $\{\phi_\nu(t)\}_{\nu=1}^{\infty}$, where each $\phi_\nu(t) \in L_{c,d}$, is a Cauchy sequence in $L_{c,d}$ if $\gamma_k(\phi_\nu - \phi_\mu) \to 0$ as μ and ν both go to ∞, independently of each other, for $k = 0, 1, 2, \cdots$. It has been proved by Zemanian [3] that for $n = 1$, $L_{c,d}$ is a sequentially complete, Hausdorff, locally convex topological vector space. This result is also true for $L_{c,d}$ defined over R^n, $n > 1$, and the proof is quite similar to that given by Zemanian for $n = 1$.

The dual space $L'_{c,d}$ contains all distributions of compact support in R^n. Also, the regular distribution f corresponding to any locally integrable function $f(x)$ defined over R^n such that

$$\int_{R^n} \left| \frac{f(x)}{K_{c,d}(x)} \right| dx < \infty,$$

is a member of $L'_{c,d}$.

DEFINITION. For fixed real values of c and d, a smooth and complex-valued function $\phi(t)$ defined over R^n is said to be in the space $\bar{L}_{c,d}$ if there exists an $\eta > 0$ such that

$$\phi^{(k)}(t) = \begin{cases} O[e^{-(c+\eta)t}] & \text{for } t > 0 \text{ and } |t| \to \infty, \\ O[e^{-(d-\eta)t}] & \text{for } t < 0 \text{ and } |t| \to \infty \cdot \end{cases} \tag{1}$$

where $|t| = \sqrt{t_1^2 + t_2^2 + \cdots + t_n^2}$; $\eta = (\eta_1, \eta_2, \cdots, \eta_n)$, and $|k|$ assumes the values $0, 1, 2, \cdots$. It can be readily seen that $\bar{L}_{c,d}$ is a linear submanifold of $L_{c,d}$.

One may observe that the elements of the space $\bar{L}_{c,d}$ are actually the same as those of the space $L(c, d)$, where $L(c, d)$ is the countable union of L_{c_ν,d_ν}, $c_\nu \to c+$,

$d_\nu \to d-$ [2]. Also $\bar{L}_{c,d}$ is a proper subspace of $L_{c,d}$ [2]. When we say that $c_\nu \to c+$ we mean that every component of c_ν approaches the corresponding component of c from the right. Similarly $d_\nu \to d-$.

LEMMA. *Let $\bar{L}_{c,d}$ be the space of infinitely differentiable complex-valued functions defined over R^n satisfying the asymptotic orders* (1). *Then for $\phi(t) \in \bar{L}_{c,d}$ and $f \in L'_{c,d}$ there exist a constant C and an integer $r \geqq 0$ such that*

$$|\langle f, \phi \rangle| \leqq C\beta_{r+n}(\phi), \tag{2}$$

where

$$\beta_m(\phi) = \max_{0 \leqq |k| \leqq m} \int_{R^n} |K_{c,d}(t)\phi^{(k)}(t)| \, dt. \tag{3}$$

Proof. For every $\phi(x) \in \bar{L}_{c,d}$ and $f \in L'_{c,d}$ we have, in view of the boundedness property of generalized functions,

$$|\langle f, \phi \rangle| \leqq P\gamma_r(\phi) \tag{4}$$

for appropriate constants P and r. Further,

$$K_{c,d}(t)\phi^{(k)}(t) = \int_{-\infty}^{t_n} \cdots \int_{-\infty}^{t_1} \Delta[K_{c,d}(x)\phi^{(k)}(x)] \, dx_1 \, dx_2 \cdots dx_n, \tag{5}$$

where Δ is the differentiation monomial $\partial/\partial x_1 \, \partial/\partial x_2 \cdots \partial/\partial x_n$. It can be seen quite readily that for an appropriate constant $q > 0$ we have for a fixed x in R^n,

$$|D^m[K_{c,d}(x)\phi^{(k)}(x)]| \leqq q \max_{0 \leqq |p| \leqq |m| + |k|} |K_{c,d}(x)\phi^{(p)}(x)|. \tag{6}$$

Therefore, (2) is obtained quite easily in view of (4), (5) and (6).

THEOREM. *Let $f \in L'_{c,d}$ and $\phi \in \bar{L}_{c,d}$. Let N be the number of n-tuples i satisfying the condition $|i| \leqq r + n$. Then there exist N bounded measurable functions $g_i(x)$ defined over R^n such that*

$$\langle f, \phi \rangle = \sum_{|i| \leqq r+n} \langle g_i(x), K_{c,d}(x)D^i\phi(x) \rangle. \tag{7}$$

Proof. The result follows quite readily in view of inequality (2), the representation theorem of Riesz and the Hahn-Banach theorem. The proof is very similar to that given in [7, pp. 273–274], and therefore the details are omitted.

Remark. At first glance it would seem that the result expressed in (7) is an easy consequence of a general result of Gel'fand and Shilov [9, pp. 110–113]. But in fact it could not be obtained from Gel'fand and Shilov's result as the spaces $\bar{L}_{c,d}$ and $L_{c,d}$ do not satisfy either of the conditions (N) and (P) (see [9, pp. 110–113]). Formula (7) gives only the structure of the space $L'_{c,d}$, the dual space of the countable union space $L(c, d)$, and it is invalid for the whole space $L'_{c,d}$ [2, p. 50, Ex. 3. 2–1].

COROLLARY. *Let $f \in L'_{c,d}$ and $\phi \in D(R^n)$, the space of infinitely differentiable functions with compact support in R^n. Then, there exist N bounded measurable* *functions $g_i(x)$ defined over R^n such that*

$$\langle f, \phi \rangle =$$

$$\left\langle \sum_{|i| \leqq n+r} (-1)^{|i|} D^i \Delta \int_{a_n}^{t_n} \cdots \int_{a_1}^{t_1} K_{c,d}(x) g_i(x_1, x_2, \cdots, x_n)\, dx_1 \cdots dx_n, \phi(t_1, \cdots, t_n) \right\rangle. \tag{8}$$

Here N is the number of n-tuples i satisfying $|i| \leqq n + r$, r is the same nonnegative integer that appeared in (2), *and Δ is the differentiation monomial $\partial/\partial t_1\ \partial/\partial t_2 \cdots \partial/\partial t_n$.*

Proof. Let $\phi(t) \in D(R^n)$. Then in view of the previous theorems, there exist bounded measurable functions $g_i(x)$ defined over R^n satisfying the relation

$$\langle f, \phi \rangle = \sum_{|i| \leqq n+r} \langle g_i(x), K_{c,d}(x) D^i \phi(x) \rangle. \tag{9}$$

Again, since $D^i \phi(t) \in D(R^n)$ and the regular distribution corresponding to the integral appearing in (8) belongs to $D'(R^n)$, the relation (8) follows immediately using the rules of distributional differentiation.

If $n = 1$, the structure formula (8) for elements of Zemanian's space $L'_{c,d}$ takes the form

$$\langle f, \phi \rangle = \left\langle \sum_{i=0}^{r+1} (-1)^i D^{i+1} \int_a^t g_i(x) K_{c,d}(x)\, dx, \phi(t) \right\rangle. \tag{10}$$

Here the $g_i(x)$ are bounded measurable functions over R^1 and the operator D stands for the operator d/dt. The function $K_{c,d}(x)$ is the function defined over R^1 and the integer r is determined appropriately.

Acknowledgment. The author expresses his gratitude to a referee and Professor A. H. Zemanian for their constructive criticism.

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SIAM J. MATH. ANAL.
Vol. 2, No. 2, May 1971

UNIQUENESS OF POSITIVE SOLUTIONS TO NONLINEAR ELLIPTIC PROBLEMS*

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Abstract. The uniqueness of positive solutions to self-adjoint elliptic partial differential equations with nonlinear forcing terms subject to mixed Dirichlet and nonlinear Neumann boundary conditions on bounded domains is proved under relatively mild conditions on the nonlinear terms. The result generalizes known results.

1. Introduction. Recently, positive solutions to certain nonlinear elliptic partial differential equations have been of interest (cf. [1], [2], [4], [5]). The uniqueness of such solutions is known for nonlinear elliptic problems with linear boundary conditions under the assumption that the nonlinear terms are of a restrictive form and satisfy a concavity condition (see [1], [2], [4], [5]). In Theorem 1 below we generalize these known results in several directions: first, we consider a nonlinear differential equation of a general type (see (2.1) below); secondly we consider nonlinear boundary conditions (see (2.2) below); and, finally, we weaken the assumptions on the nonlinear terms (H1–H3 below).

2. Results. Consider the following general boundary value problem which will be referred to as Problem I:

$$Lu = F(x, u) \quad \text{on } D, \tag{2.1}$$

$$u(x) = \alpha(x) \quad \text{on } S^1, \qquad \partial u/\partial v = G(x, u) \quad \text{on } S^2, \quad S^1 + S^2 = S, \tag{2.2}$$

where $x = (x_1, \cdots, x_m)$ and

$$Lu \equiv \sum_{i,j=1}^{m} D_i(a_{ij}(x)D_j u) + a_0(x)u, \qquad D_i = \partial/\partial x_i,$$

$$\sum_{i,j=1}^{m} a_{ij}(x)\xi_i\xi_j > 0, \qquad \xi_1^2 + \cdots + \xi_m^2 \neq 0, \quad x \in D,$$

$$a_{ij}(x) = a_{ji}(x), \qquad x \in D,$$

$$\frac{\partial u}{\partial v} \equiv \sum_{i,j=1}^{m} a_{ij}(x)n_i(x)D_j u, \qquad x \in S.$$

Here D is a bounded region in m-dimensional space with boundary S whose outwardly directed normal at x is denoted by $(n_1(x), \cdots, n_m(x))$; S^1 and S^2 are disjoint measurable sets whose union is S. (Actually our proof and hence our result are valid when S^1 and S^2 are disjoint, measurable sets whose union equals S up to a set of measure zero, but we will not push this point.) The divergence theorem is assumed to hold on D and the coefficients $a_{ij}(x)$, $a_0(x)$ are assumed once continuously differentiable on $\bar{D}$, the closure of D. The functions α, F, G are presumed given in advance. By a solution to Problem I we mean a function

* Received by the editors March 19, 1970, and in revised form July 25, 1970.

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$u(x) \in C^1(\bar{D})$ for which the derivatives appearing in (2.1) exist and are continuous on $\bar{D}$, and the boundary conditions (2.2) hold on the appropriate regions.

We impose the following conditions on the given functions α, F, G:

H1. $\alpha(x)$, $F(x, z)$, $G(x, z)$ are all defined and continuous on S^1, $D \times [0, \infty)$, $S^2 \times [0, \infty)$, respectively, and $\alpha(x) > 0$ on S^1.

H2. $z'F(x, z) \geqq zF(x, z')$ for $z \geqq z' \geqq 0$ and $x \in D$.

H3. $z'G(x, z) \leqq zG(x, z')$ for $z \geqq z' \geqq 0$ and $x \in S^2$.

Our main result, which is proved in the next section, is contained in the following theorem.

THEOREM 1. *If* H1, H2, H3 *hold and if u, v are two solutions to Problem* I *satisfying $u > 0$, $v > 0$ on $\bar{D}$, then $u = kv$, $k =$ positive constant. Consequently, if S^1 is nonempty, there exists at most one positive solution to Problem* I. *In any case if a strict inequality holds in either* H2 *or* H3, *then at most one positive solution exists.*

Hypothesis H2 (H3) means geometrically that the slope of the line in the z, F-plane (z, G-plane) passing through the origin and the "point" $[z, F(x, z)]$ ($[z, G(x, z)]$) is a nondecreasing (nonincreasing) function of $z \geqq 0$ for each fixed value of x. If F, G are once differentiable in z for all values of x in the appropriate regions, then H2, H3 are equivalent to the requirements $zF_z - F \geqq 0$, $zG_z - G \leqq 0$ for all $z \geqq 0$ and appropriate x (see [2]). It is not difficult to see that any functions F, G which are concave up and concave down in z, respectively, and which satisfy $F(x, 0) \leqq 0$, $G(x, 0) \geqq 0$ for all appropriate x necessarily satisfy H2, H3, respectively. Moreover, H2 and H3 are certainly satisfied for functions F, G linear in z and, thus, these hypotheses (which do not necessarily restrict the concavity or monotonicity of F, G in the variable z) are weaker than the concavity assumption of Keller [4], [5] and Cohen [1], [2] for Problem I.

3. Proof of Theorem 1. The proof utilizes a generalization of Green's integral identity (due originally to M. H. Martin) which has been used by many authors to study uniqueness questions for nonlinear boundary problems (cf. Cushing [3] for bibliography). A straightforward application of the divergence theorem together with $a_{ij} = a_{ji}$ yields the identity

$$\int_S (\lambda - 1)\left(v\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu}\right) dx = \int_D [Q + (\lambda - 1)(vLu - uLv)]\, dx, \tag{3.1}$$

where $\lambda = u/v$ and $Q = v^2 \sum_{i,j=1}^m a_{ij} D_i\lambda D_j\lambda$. Supposing that u, v are two solutions to Problem I satisfying $u > 0$, $v > 0$ in $\bar{D}$, we see that this identity becomes

$$I_1 \equiv \int_{S^2} (\lambda - 1)[vG(x, u) - uG(x, v)]\, dx = I_2 + I_3, \tag{3.2}$$

where

$$I_2 \equiv \int_D Q\, dx, \qquad I_3 \equiv \int_D (\lambda - 1)[vF(x, u) - uF(x, v)]\, dx.$$

As $v > 0$ on $\bar{D}$, we have $\lambda \in C^1(\bar{D})$ and consequently the identity is valid. Now $I_2 \geqq 0$ by the definiteness of a_{ij}; moreover, the integrand of I_3 is nonnegative (H2) while the integrand of I_1 is nonpositive (H3), and hence $I_1 \leqq 0$, $I_3 \geqq 0$. We conclude from identity (3.2) that $I_i = 0$, $i = 1, 2, 3$. But $I_2 = 0$ together with the definiteness of Q implies $D_i\lambda = 0$, $i = 1, \cdots, m$, or $u = kv$, $k =$ const. If S^1 is nonempty, then clearly $k = 1$. In any case, if strict inequality holds in H2 or H3, then $I_3 = 0$ or $I_1 = 0$ implies $k = 1$ and the theorem follows.

Finally we note that for eigenvalue problems of the general type $F \equiv F(\lambda, x, u)$ and/or $G \equiv G(\mu, x, u)$, λ, $\mu =$ constants (to be determined as part of the solution), Theorem 1 remains valid for all eigenvalues λ and/or μ for which a positive solution exists provided H1–H3 hold for the given values of λ, μ.

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SIAM J. MATH. ANAL.
Vol. 2, No. 3, August 1971

ON THE SOLUTION OF THE INTEGRAL EQUATION FOR THE POTENTIAL OF TWO STRIPS*

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Abstract. An integral equation associated with the potential of two strips is solved in closed form and in series form. The closed form solution generalizes the well-known formula of Carleman for the single strip problem. The series solution concludes the work initiated by Shinbrot. Also, an erroneous solution found by Tricomi for a related problem is corrected.

1. Introduction and summary. We consider here the integral equation

$$(Au)(x) \equiv \int_{-1}^{-k} + \int_{k}^{1} \log|x - t|u(t)\,dt = f(x), \qquad k < |x| < 1, \quad 0 < k < 1, \tag{1.1}$$

associated with the potential of two strips. We shall show that this equation has the solution

$$\begin{aligned} \pi^2[R(x)]^{1/2} \operatorname{sgn} x\; u(x) = & \int_{-1}^{-k} + \int_{k}^{1} \frac{[R(t)]^{1/2} \operatorname{sgn} t\, f'(t)}{t - x}\,dt \\ & + \int_{-1}^{-k} + \int_{k}^{1} \left[1 - t^2 - \frac{E(k)}{K(k)} - \frac{xt}{\log(2/k')}\right] \\ & \cdot \operatorname{sgn} t\, f(t)[R(t)]^{-1/2}\,dt, \qquad k < |x| < 1, \end{aligned} \tag{1.2}$$

where

$$k' = \sqrt{1 - k^2}, \qquad R(x) = (1 - x^2)(x^2 - k^2), \tag{1.3}$$

and $K(k)$, $E(k)$ are the complete elliptic integrals

$$\begin{aligned} K(k) &= \int_0^1 (1 - k^2t^2)^{-1/2}(1 - t^2)^{-1/2}\,dt, \\ E(k) &= \int_0^1 (1 - k^2t^2)^{1/2}(1 - t^2)^{-1/2}\,dt. \end{aligned} \tag{1.4}$$

In (1.2), and throughout this paper, we interpret improper integrals in the principal value sense whenever necessary.

This result can be regarded as a generalization of a well-known formula due to Carleman [1] for the single strip. It is easily verified that (1.2) reduces to his solution in the limit as k vanishes.

In an alternative approach we will show that a solution to (1.1) can be found in the form of an infinite series

$$u(x) = \sum_{n=0}^{\infty} \left[\frac{1}{M_n}\int_{-1}^{-k} + \int_{k}^{1} \phi_n(t) f(t)\,dt\right] \phi_n(x), \qquad k < |x| < 1, \tag{1.5}$$

* Received by the editors July 30, 1970, and in revised form November 12, 1970.

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where the $\{\phi_n(x)\}$ are defined by

$$[R(x)]^{1/2}\phi_0(x) = |x|, \qquad k < |x| < 1,$$

$$[R(x)]^{1/2}\phi_1(x) = \operatorname{sgn} x, \qquad k < |x| < 1,$$

$$(1.6)\qquad \begin{aligned} [R(x)]^{1/2}\phi_{2n}(x) &= \begin{cases} \cos\left\{\dfrac{n\pi}{K(k')}F\left[\sin^{-1}\left(\dfrac{(x^2-k^2)^{1/2}}{k'|x|}\right), k'\right]\right\}, & k < x < 1, \\ 0, & -1 < x < -k, \end{cases} \\ [R(x)]^{1/2}\phi_{2n+1}(x) &= \begin{cases} 0, & k < x < 1, \\ \cos\left\{\dfrac{n\pi}{K(k')}F\left[\sin^{-1}\left(\dfrac{(x^2-k^2)^{1/2}}{k'|x|}\right), k'\right]\right\}, & -1 < x < -k, \end{cases} \\ & \qquad n = 1, 2, \cdots, \end{aligned}$$

with $F(\zeta, \gamma)$ denoting the elliptic integral of the first kind

$$(1.7)\qquad F(\zeta,\gamma) = \int_0^{\sin\zeta} (1-\gamma^2t^2)^{-1/2}(1-t^2)^{-1/2}\,dt.$$

The normalizing constants are

$$(1.8)\qquad M_n = \int_{-1}^{-k} + \int_k^1 \phi_n(s)\int_{-1}^{-k} + \int_k^1 \phi_n(t)\log|s-t|\,dt\,ds, \qquad n = 0, 1, 2, \cdots,$$

$M_0 = -\pi^2\log(2/k')$, $M_1 = -2\pi K(k)K(k')$, $M_{2n} = M_{2n+1} = -K^2(k')/2n$, $n = 1, 2, \cdots$.

The series solution is obtained by following the method of Shinbrot [3], who has shown that the integral equation (1.1) and similar equations can be related to an eigenvalue problem for an ordinary differential equation.

Another result which is an easy consequence of (1.1) and (1.2) concerns the integral equation

$$(1.9)\qquad \int_{-1}^{-k} + \int_k^1 \frac{v(t)}{t-x}\,dt = g(x), \quad k < |x| < 1, \quad 0 < k < 1.$$

It will be shown that this equation has the solution

$$(1.10)\qquad \pi^2[R(x)]^{1/2}\operatorname{sgn} x\; v(x) = \int_{-1}^{-k} + \int_k^1 \frac{[R(t)]^{1/2}\operatorname{sgn} t\, g(t)}{x-t}\,dt + C'x + C'', \qquad k < |x| < 1,$$

where C' and C'' are arbitrary constants.

This problem was treated earlier by Tricomi [4]. We shall demonstrate that his formula corresponding to (1.10) is incorrect.

2. Closed form solution. To begin the investigation of (1.1), we extend the interval of consideration to $-1 < x < 1$ by defining

$$U(x) = \begin{cases} u(x), & k < |x| < 1, \\ 0, & -k < x < k, \end{cases} \tag{2.1}$$

$$F(x) = \begin{cases} f(x), & k < |x| < 1, \\ h(x) = \displaystyle\int_{-1}^{-k} + \int_{k}^{1} \log|x - t| u(t)\, dt, & -k < x < k. \end{cases} \tag{2.2}$$

Although $h(x)$is not known a priori, these definitions allow us to express (1.1) as

$$\int_{-1}^{1} \log|x - t| U(t)\, dt = F(x), \qquad -1 < x < 1. \tag{2.3}$$

This equation can be solved for $U(x)$ in terms of $F(x)$ by the well-known formula of Carleman [1],

$$\pi^2(1 - x^2)^{1/2} U(x) = \int_{-1}^{1} \frac{(1 - t^2)^{1/2} F'(t)}{t - x} dt - \frac{1}{\log 2} \int_{-1}^{1} F(t)(1 - t^2)^{-1/2}\, dt, \qquad -1 < x < 1. \tag{2.4}$$

From the definition of $U(x)$, we see that (2.4) is equivalent to the two equations

$$\pi^2(1 - x^2)^{1/2} u(x) = \int_{-1}^{-k} + \int_{k}^{1} \frac{(1 - t^2)^{1/2} f'(t)}{t - x} dt + \int_{-k}^{k} \frac{(1 - t^2)^{1/2} h'(t)}{t - x} dt - D, \qquad k < |x| < 1, \tag{2.5}$$

and

$$\int_{-k}^{k} \frac{(1 - t^2)^{1/2} h'(t)}{t - x} dt = \int_{-1}^{-k} + \int_{k}^{1} \frac{(1 - t^2)^{1/2} f'(t)}{x - t} dt + D, \qquad -k < x < k, \tag{2.6}$$

where

$$D = \frac{1}{\log 2} \left[\int_{-k}^{k} h(t)(1 - t^2)^{-1/2}\, dt + \int_{-1}^{-k} + \int_{k}^{1} f(t)(1 - t^2)^{-1/2}\, dt \right]. \tag{2.7}$$

Once $h'(t)$ and D have been determined, (2.5) becomes the required solution of (1.1). In order to determine $h'(t)$ and D, we focus our attention on (2.6). We see that it can be interpreted as an integral equation for $h'(t)$. To solve it, we employ another well-known result from the theory of singular integral equations. It readily follows from the Carleman problem (2.3), (2.4) that the integral equation

$$\int_{-k}^{k} \frac{V(t)}{t - x} dt = G(x), \qquad -k < x < k, \tag{2.8}$$

has the solution

$$\pi^2(k^2 - x^2)^{1/2}V(x) = -\int_{-k}^{k} \frac{(k - t^2)^{1/2}G(t)}{t - x}\,dt + C, \qquad -k < x < k, \tag{2.9}$$

where C is an arbitrary constant.

Using this formula to solve (2.6) we find that

$$\begin{aligned} \pi^2[-R(x)]^{1/2}h'(x) &= \int_{-k}^{k} \frac{(k^2 - t^2)^{1/2}}{t - x}\left[\int_{-k}^{k} + \int_{k}^{1} \frac{(1 - s^2)^{1/2}f'(s)}{s - t}\,ds\right] dt \\ &\quad - D\int_{-k}^{k} \frac{(k^2 - t^2)^{1/2}}{t - x}\,dt + C, \qquad -k < x < k. \end{aligned} \tag{2.10}$$

Upon interchanging the order of integration we find that

$$\begin{aligned} \pi^2[-R(x)]^{1/2}h'(x) &= \int_{-1}^{-k} + \int_{k}^{1} \frac{(1 - s^2)^{1/2}f'(s)}{s - x}\left[\int_{-k}^{k} \frac{(k^2 - t^2)^{1/2}\,dt}{t - x}\right. \\ &\quad \left. - \int_{-k}^{k} \frac{(k^2 - t^2)^{1/2}}{t - s}\,dt\right] ds - D\int_{-k}^{k} \frac{(k^2 - t^2)^{1/2}}{t - x}\,dt + C, \\ &\qquad -k < x < k. \end{aligned} \tag{2.11}$$

Then the identity

$$\int_{-k}^{k} \frac{(k^2 - t^2)^{1/2}}{t - y}\,dt = \begin{cases} -\pi y, & -k < y < k, \\ -\pi y + \pi(y^2 - k^2)^{1/2}\,\mathrm{sgn}\,y, & k < |y| < 1, \end{cases} \tag{2.12}$$

reduces (2.11) to

$$\begin{aligned} \pi[-R(x)]^{1/2}h'(x) &= \int_{-1}^{-k} + \int_{k}^{1} \frac{[R(t)]^{1/2}\,\mathrm{sgn}\,t\,f'(t)}{x - t}\,dt \\ &\quad + \int_{-1}^{k} + \int_{k}^{1} (1 - t^2)^{1/2}f'(t)\,dt + Dx + \frac{C}{\pi}, \\ &\qquad -k < x < k. \end{aligned} \tag{2.13}$$

To find C, we divide (2.12) by $[-R(x)]^{1/2}$ and integrate over the interval $-k < x < k$. This yields

$$\pi[h(k) - h(-k)] = -N + \left[\frac{C}{\pi} + \int_{-1}^{-k} + \int_{k}^{1} (1 - t^2)^{1/2}f'(t)\,dt\right]\int_{-k}^{k} [-R(s)]^{-1/2}\,ds, \tag{2.14}$$

where the constant N is defined as

$$N = \int_{-k}^{k} [-R(s)]^{-1/2}\left[\int_{-1}^{-k} + \int_{k}^{1} \frac{[R(t)]^{1/2}\,\mathrm{sgn}\,t\,f'(t)}{t - s}\,dt\right] ds. \tag{2.15}$$

For sufficiently well-behaved solutions of the integral equation, we see that $F(x)$ will be continuous on the interval $-1 \leqq x \leqq 1$. Consequently $h(k) = f(k)$ and $h(-k) = f(-k)$. This argument and the integral identity

$$\int_{-k}^{k} [-R(s)]^{-1/2}\, ds = 2\int_{0}^{1} (1 - t^2)^{-1/2}(1 - k^2t^2)^{-1/2}\, dt = 2K(k) \tag{2.16}$$

allow us to determine C from (2.14) as

$$C = -\pi\left[\int_{-1}^{-k} + \int_{k}^{1}\right] (1 - t^2)^{1/2} f'(t)\, dt + \frac{\pi\{N + \pi[f(k) - f(-k)]\}}{2K(k)}. \tag{2.17}$$

As for the constant D, we postpone its determination for the moment.

The next step is to use (2.13) to find an expression for the integral

$$\int_{-k}^{k} (t - x)^{-1}(1 - t^2)^{1/2} h'(t)\, dt, \qquad k < |x| < 1,$$

which appears in (2.5). We hasten to point out that this integral is not the same as in (2.6), where the domain is $-k < x < k$. We find that

$$\begin{aligned}\pi\int_{-k}^{k} \frac{(1 - t^2)^{1/2}h'(t)}{t - x}\, dt = &\left[\int_{-1}^{-k} + \int_{k}^{1}\right] [R(s)]^{1/2} \operatorname{sgn} s\, f'(s) \int_{-k}^{k} \frac{(k^2 - t^2)^{-1/2}\, dt}{(t - x)(t - s)}\, ds \\ &+ D\int_{-k}^{k} \frac{(k^2 - t^2)^{-1/2} t\, dt}{t - x} \\ &+ \frac{N + \pi[f(k) - f(-k)]}{2K(k)} \int_{-k}^{k} \frac{(k^2 - t^2)^{-1/2}\, dt}{t - x}, \\ &\qquad k < |x| < 1.\end{aligned} \tag{2.18}$$

This expression can be simplified by using the identities

$$\int_{-k}^{k} \frac{(k^2 - t^2)^{-1/2}\, dt}{t - x} = -\pi(x^2 - k^2)^{-1/2} \operatorname{sgn} x, \qquad k < |x| < 1, \tag{2.19}$$

$$\int_{-k}^{k} \frac{(k^2 - t^2)^{-1/2} t\, dt}{t - x} = \pi - \pi|x|(x^2 - k^2)^{-1/2}, \qquad k < |x| < 1, \tag{2.20}$$

$$\int_{-k}^{k} \frac{(k^2 - t^2)^{-1/2}\, dt}{(t - x)(t - s)} = \frac{\pi}{x - s}\left[\frac{\operatorname{sgn} s}{(s^2 - k^2)^{1/2}} - \frac{\operatorname{sgn} x}{(x^2 - k^2)^{1/2}}\right], \qquad \begin{matrix} k < |x| < 1, \\ k < |s| < 1. \end{matrix} \tag{2.21}$$

We then have

$$\begin{aligned}\int_{-k}^{k} \frac{(1 - t^2)^{1/2}h'(t)}{t - x}\, dt = &\left[\int_{-1}^{-k} + \int_{k}^{1}\right] \frac{(1 - s^2)^{1/2} f'(s)}{x - s}\, ds \\ &- \frac{\operatorname{sgn} x}{(s^2 - k^2)^{1/2}} \left[\int_{-1}^{-k} + \int_{k}^{1}\right] \frac{[R(s)]^{1/2} \operatorname{sgn} s\, f'(s)}{x - s}\, ds \\ &+ D\left(1 - \frac{|x|}{(x^2 - k^2)^{1/2}}\right) \\ &- \left\{\frac{N + \pi[f(k) - f(-k)]}{2K(k)}\right\} \frac{\operatorname{sgn} x}{(x^2 - k^2)^{1/2}}, \\ &\qquad k < |x| < 1.\end{aligned} \tag{2.22}$$

This reduces the prospective solution (2.5) to

$$\pi^2[R(x)]^{1/2} \operatorname{sgn} x\, u(x) = \int_{-1}^{-k} + \int_k^1 \frac{[R(t)]^{1/2}}{t-x} \operatorname{sgn} t\, f'(t)\, dt \tag{2.23}$$
$$- \frac{N + \pi[f(k) - f(-k)]}{2K(k)} - Dx, \qquad k < |x| < 1.$$

In order to bring (2.23) into the desired form for the solution, we need the identities

$$\int_{-1}^{-k} + \int_k^1 \frac{\operatorname{sgn} t\,[R(t)]^{-1/2}}{x-t}\, dt = 0, \qquad k < |x| < 1, \tag{2.24}$$

$$\int_{-1}^{-k} + \int_k^1 \frac{|t|\,[R(t)]^{-1/2}}{x-t}\, dt = 0, \qquad k < |x| < 1, \tag{2.25}$$

$$\int_{-1}^{-k} + \int_k^1 |t| \log|x-t|\,[R(t)]^{-1/2}\, dt = -\pi \log \frac{2}{k'}, \qquad k < |x| < 1, \tag{2.26}$$

$$\int_{-1}^{-k} + \int_k^1 \left\{\operatorname{sgn} t\,[R(t)]^{-1/2}\left[1 - t^2 - \frac{E(k)}{K(k)}\right]\right\} \log|x-t|\, dt = \pi x, \tag{2.27}$$
$$k < |x| < 1.$$

The first two of these are found in [4], while the last two will be derived in § 5.

From (1.1), we can produce the equation

$$\int_{-1}^{-k} + \int_k^1 [R(t)]^{-1/2}|t| \left[\int_{-1}^{-k} + \int_k^1 \log|t-x| u(x)\, dx\right] dt \tag{2.28}$$
$$= \int_{-1}^{-k} + \int_k^1 [R(t)]^{-1/2}|t| f(t)\, dt.$$

Then by interchanging the order of integration and utilizing (2.26) we find

$$\pi^2 \int_{-1}^{-k} + \int_k^1 u(x)\, dx = -\frac{\pi}{\log(2/k')} \int_{-1}^{-k} + \int_k^1 |t| f(t)[R(t)]^{-1/2}\, dt. \tag{2.29}$$

Now (2.23) can be used to evaluate the left side of (2.29). Integrating (2.23) with the aid of (2.24) we find that

$$\pi^2 \int_{-1}^{-k} + \int_k^1 u(x)\, dx = -D \int_{-1}^{-k} + \int_k^1 \frac{|x|}{\sqrt{(1-x^2)(x^2-k^2)}}\, dx = -\pi D. \tag{2.30}$$

Comparison of these last two equations yields

$$D = \frac{1}{\log(2/k')} \int_{-1}^{-k} + \int_k^1 |t| f(t)[R(t)]^{-1/2}\, dt. \tag{2.31}$$

Hence (2.23) becomes

$$\begin{aligned}\pi^2[R(x)]^{1/2} \operatorname{sgn} x\, u(x) &= \int_{-1}^{-k} + \int_{k}^{1} \frac{[R(t)]^{1/2} \operatorname{sgn} t\, f'(t)}{t - x}\, dt - \frac{N + \pi[f(k) - f(-k)]}{2K(k)} \\ &\quad - \frac{x}{\log(2/k')} \int_{-1}^{-k} + \int_{k}^{1} |t| f(t)[R(t)]^{-1/2}\, dt, \quad k < |x| < 1. \end{aligned} \tag{2.32}$$

This gives a formula for the solution of (1.1). All quantities on the right side are known in terms of the given function $f(x)$; however, the constant term is somewhat cumbersome. This motivates the finding of an alternative expression which will finally bring the solution into the form of (1.2).

By integrating (2.32) and utilizing (2.25), we obtain the relationship

$$\pi \int_{-1}^{-k} + \int_{k}^{1} x u(x)\, dx = -\frac{N + \pi[f(k) - f(-k)]}{2K(k)}. \tag{2.33}$$

To obtain another expression for the left side of (2.33), we consider the following integral of (1.1):

$$\begin{aligned}&\int_{-1}^{-k} + \int_{k}^{1} \left\{ \operatorname{sgn} t\, [R(t)]^{-1/2} \left[1 - t^2 - \frac{E(k)}{K(k)} \right] \right\} \int_{-1}^{-k} + \int_{k}^{1} \log|t - x| u(x)\, dx\, dt \\ &\qquad = \int_{-1}^{-k} + \int_{k}^{1} \left\{ \operatorname{sgn} t\, [R(t)]^{-1/2} \left[1 - t^2 - \frac{E(k)}{K(k)} \right] \right\} f(t)\, dt. \end{aligned} \tag{2.34}$$

The interchange of the order of integration and utilization of the identity (2.27) gives

$$\int_{-1}^{-k} + \int_{k}^{1} x u(x)\, dx = \int_{-1}^{-k} + \int_{k}^{1} \left\{ \operatorname{sgn} t\, [R(t)]^{-1/2} \left[1 - t^2 - \frac{E(k)}{K(k)} \right] \right\} f(t)\, dt. \tag{2.35}$$

Thus the constant on the right side of (2.32) can be replaced by the constant in (2.34) to yield the final formula

$$\begin{aligned}\pi^2[R(x)]^{1/2} \operatorname{sgn} x\, u(x) &= \int_{-1}^{-k} + \int_{k}^{1} \frac{[R(t)]^{1/2} \operatorname{sgn} t\, f'(t)}{t - x}\, dt \\ &\quad + \int_{-1}^{-k} + \int_{k}^{1} \left[1 - t^2 - \frac{E(k)}{K(k)} - \frac{xt}{\log(2/k')} \right] \\ &\quad \cdot \operatorname{sgn} t\, f(t) [R(t)]^{-1/2}\, dt, \qquad k < |x| < 1, \end{aligned} \tag{2.36}$$

as in the solution of (1.1)

3. Series solution. Our task here is to show that the series

$$S(x) = \sum_{n=0}^{\infty} \left[\frac{1}{M_n} \int_{-1}^{-k} + \int_{k}^{1} \phi_n(t) f(t)\, dt \right] \phi_n(x), \qquad k < |x| < 1, \tag{3.1}$$

with the set of functions $\{\phi_n(x)\}$ defined by (1.6) satisfies (1.1). That is, $u(x) = S(x)$ whenever a solution exists. The existence question has been resolved essentially in the previous section.

The ease with which we can demonstrate that this series is a solution arises from our fortune in knowing the explicit form of the $\{\phi_n(x)\}$. We were able to construct these functions (with one exception) by following the work of Shinbrot [3]. He shows that the problem of solving the integral equation (1.1) is related to an eigenvalue problem for an ordinary differential equation. We have solved his eigenvalue problem, and use the solutions in forming the $\{\phi_n(x)\}$. Our method of forming the series (3.1) is somewhat different, and curiously we find it necessary to add a term $\phi_0(x)$ which is not a solution of his differential equation.

One of the essential properties of the $\{\phi_n(x)\}$ is orthogonality with respect to the integral operator A; that is, the integrals

$$I_{mn} = \int_{-1}^{-k} + \int_k^1 \phi_m(x) \int_{-1}^{-k} + \int_k^1 \log|x - t|\phi_n(t)\,dt\,dx, \tag{3.2}$$

$$m = 0, 1, \cdots, \quad n = 0, 1, \cdots,$$

satisfy the condition

$$I_{mn} = \begin{cases} 0, & m \neq n, \\ M_n, & m = n. \end{cases} \tag{3.3}$$

At this point, we enter into the details of constructing the $\{\phi_n(x)\}$ and show that these functions satisfy (3.3). The background for much of this is given in [3].

To study (3.2) we introduce the Fourier transform pair,

$$\hat{\phi}_n(\xi) = \int_{-\infty}^{\infty} e^{i\xi x}\phi_n(x)\,dx, \qquad \phi_n(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi x}\hat{\phi}_n(\xi)\,d\xi. \tag{3.4}$$

Then the Parseval formula gives

$$I_{mn} = -2\pi^2 \int_{-\infty}^{\infty} \frac{\hat{\phi}_m^*(\xi)\hat{\phi}_n(\xi)}{|\xi|}\,d\xi \tag{3.5}$$

for sufficiently well-behaved functions satisfying $\hat{\phi}_n(0) = 0$.

To find suitable candidates for the $\{\phi_n(x)\}$, we follow [3] in considering the eigenfunctions $\{\chi_n(x)\}$ which satisfy

$$\frac{d^2}{dx^2}[(1 - x^2)(x^2 - k^2)\chi_n(x)] + \frac{d}{dx}[x(2x^2 - 1 - k)\chi_n(x)] + \lambda_n\chi_n(x) = 0,$$

$$k < |x| < 1, \tag{3.6}$$

$$\lim_{x \to \pm 1, \pm k} \left[(1 - x^2)(x^2 - k^2)\frac{d\chi_n(x)}{dx} - x(2x^2 - 1 - k^2)\chi_n(x)\right] = 0.$$

Also, we recall that every solution of (3.6) also satisfies

$$\lim_{x \to \pm 1, \pm k} [(1 - x^2)(x^2 - k^2)\chi_n(x)] = 0. \tag{3.7}$$

Taking the Fourier transform of (3.6), we obtain

$$\left[\xi^2\frac{d^4}{d\xi^4} + 2\xi\frac{d^3}{d\xi^3} + (1 + k^2)\xi^2\frac{d^2}{d\xi^2} + \xi(1 + k^2)\frac{d}{d\xi} + \xi^2 k^2\right]\hat{\chi}_n(\xi) + \lambda_n\hat{\chi}_n(\xi) = 0,$$

$$(3.8) \qquad -\infty < \xi < \infty,$$

$$\lim_{\xi \to \pm\infty} \hat{\chi}_n(\xi) = 0.$$

By considering a similar equation for $\hat{\chi}_m^*(\xi)$, we can perform the usual manipulations for self-adjoint operators to find that, if $\hat{\chi}_m^*(0) = \hat{\chi}_n(0) = 0$, then

$$(3.9) \qquad (\lambda_n - \lambda_m)\int_{-\infty}^{\infty} \frac{\hat{\chi}_m^*(\xi)\hat{\chi}_n(\xi)}{|\xi|}\, d\xi = 0.$$

In view of (3.5) the $\{\chi_n(x)\}$ should be closely related to the $\{\phi_n(x)\}$. We proceed to investigate this connection.

To solve (3.6) it is convenient to change variables. Let

$$(3.10) \qquad \begin{aligned} \theta(x) &= \int_0^x \frac{ds}{[|s^2 - k^2|(1 - s^2)]^{1/2}} \\ &= \left\{K(k) + F\left[\sin^{-1}\left(\frac{(x^2 - k^2)^{1/2}}{k'|x|}\right), k'\right]\right\} \operatorname{sgn} x, \end{aligned}$$

and

$$(3.11) \qquad Z_n(\theta) = [R(x)]^{1/2}\chi_n(x),$$

whereupon (3.6) can be expressed as

$$(3.12) \qquad \begin{aligned} &\frac{d^2 Z_n(\theta)}{d\theta^2} + \lambda_n Z_n(\theta) = 0, \qquad k(k) < |\theta| < K(k) + K(k'), \\ &\left[\frac{dZ_n}{d\theta}\right]_{\theta = \pm K(k)} = \left[\frac{dZ_n}{d\theta}\right]_{\theta = \pm[K(k) + K(k')]} = 0. \end{aligned}$$

It is easily verified that this problem is satisfied by a function of the form

$$(3.13) \qquad \begin{aligned} &Z(\theta) = (a + b\operatorname{sgn}\theta)\cos\{\lambda_n^{1/2}[|\theta| - K(k)]\}, \\ &\qquad\qquad K(k) < |\theta| < K(k) + K(k'), \\ &\lambda_n = \left[\frac{n\pi}{K(k')}\right]^2, \qquad n = 0, 1, \cdots, \end{aligned}$$

where a and b are arbitrary constants. It follows that the solution of (3.6) is of the form

$$(3.14) \qquad \chi(x) = (a + b\operatorname{sgn} x)\cos\{\lambda_n^{1/2}[|\theta(x)| - K(k)]\}, \qquad k < |x| < 1.$$

Nevertheless, care must be exercised in the arrangement of these possible solutions into an acceptable set $\{\chi_n(x)\}$. An important criterion is that their Fourier transforms $\{\hat{\chi}_n(\xi)\}$ satisfy (3.9), thereby establishing the desired orthogonality.

For the appropriate $\{\chi_n(x)\}$ we choose, for $\lambda > 0$,

$$
\begin{aligned}
&[R(x)]^{1/2}\chi_{2n}(x) = \begin{cases} \cos\left\{\dfrac{n\pi}{K(k')}[\theta(x) - K(k)]\right\}, & k < x < 1, \\ 0, & -1 < x < -k, \end{cases} \\
(3.15)\quad & \\
&[R(x)]^{1/2}\chi_{2n+1}(x) = \begin{cases} 0, & k < x < 1, \\ \cos\left\{\dfrac{n\pi}{K(k')}[\theta(x) + K(k)]\right\}, & -1 < x < -k, \quad n = 1, 2, \cdots. \end{cases}
\end{aligned}
$$

Unfortunately, this scheme cannot be extended to $\lambda = 0$ because the resulting solutions do not possess the desired orthogonality. Even though they satisfy (3.6), their Fourier transforms are inappropriate for (3.9). This anomaly is partly remedied by taking a linear combination of these solutions corresponding to $\lambda = 0$. Hence we define

$$
(3.16) \qquad [R(x)]^{1/2}\chi_1(x) = \operatorname{sgn} x, \qquad k < |x| < 1,
$$

which satisfies (3.6) for $\lambda = 0$, and its Fourier transform has the desired behavior.

Now the $\{\chi_n(x)\}$ defined by (3.15), (3.16) do give Fourier transforms $\{\hat{\chi}_n(\xi)\}$ for which (3.9) applies, and therefore they have the desired orthogonality with respect to the operator A. Consequently they are acceptable as members of the set $\{\phi_n(x)\}$, but they fail to provide all of the elements of that set. Ultimately, it will be seen that the required $\{\phi_n(x)\}$ are given by

$$
(3.17) \qquad \phi_0(x) = |x|[R(x)]^{-1/2}, \qquad \phi_n(x) = \chi_n(x), \qquad n > 0, \quad k < |x| < 1.
$$

We note that $\phi_0(x)$ is not a solution of the eigenvalue problem (3.6).

Let us confirm that the $\{\phi_n(x)\}$, defined by (1.6) and equivalently by (3.17), satisfy the orthogonality relation (3.3). By virtue of (3.5), (3.9), and (3.17) we immediately have $I_{mn} = 0$ for $m \neq n$, $m = 1, 2, \cdots$, $n = 1, 2, \cdots$. Thus it remains to show that $I_{m0} = 0$, $m = 1, 2, \cdots$. To do this, we recall the identity (2.26) and the definition (3.2) which together yield the expression

$$
(3.18) \qquad I_{m0} = -\pi \log \frac{2}{k'} \int_{-1}^{-k} + \int_{k}^{1} \phi_m(x)\, dx, \qquad m = 0, 1, \cdots.
$$

Upon making the change of variables (3.10), (3.11), we find that

$$
\begin{aligned}
&I_{10} = -\pi \log \frac{2}{k'} \int_{-K(k)-K(k')}^{-K(k)} + \int_{K}^{K(k)+K(k')} \operatorname{sgn}\theta\, d\theta = 0, \qquad m = 1, \\
(3.19)\quad & \\
&I_{m0} = -\pi \log \frac{2}{k'} \int_{K(k)}^{K(k)+K(k')} \cos\left\{\frac{m\pi}{K(k')}[\theta - K(k)]\right\} d\theta = 0, \qquad m = 2, 3, \cdots.
\end{aligned}
$$

This establishes the desired orthogonality.

The constants $M_0 = -\pi^2 \log(2/k')$ and $M_1 = -2\pi K(k)K(k')$ are easily found by straightforward calculation of the integrals I_{00} and I_{11} respectively. We show in § 5 that $M_{2n} = M_{2n+1} = -K^2(k')/(2n)$, $n = 1, 2, \cdots$.

Now that our digression into the determination of the $\{\phi_n(x)\}$ is finished, we return to the task of demonstrating that the series $S(x)$ provides the solution to the integral equation (1.1).

As our first step toward showing that $u(x) = S(x)$ we consider the function $(AS)(x)$ generated by action of the integral operator on the series. We assume that the summation and integration operations can be interchanged so that

$$(AS)(x) = \sum_{n=0}^{\infty} \left[\frac{1}{M_n} \int_{-1}^{-k} + \int_k^1 f(s)\phi_n(s)\, ds \right] \int_{-1}^{-k} + \int_k^1 \log|x - t|\phi_n(t)\, dt, \tag{3.20}$$
$$k < |x| < 1.$$

Then by virtue of the orthogonality relation (3.3) for the $\{\phi_n(x)\}$ we have

$$\int_{-1}^{-k} + \int_k^1 \phi_n(x)(AS)(x)\, dx = \int_{-1}^{-k} + \int_k^1 \phi_n(x) f(x)\, dx, \tag{3.21}$$
$$n = 0, 1, \cdots.$$

Given any $f(x)$ for which there exists a $u(x)$ such that (1.1) is satisfied, we have

$$\int_{-1}^{-k} + \int_k^1 \phi_n(x)(AS)(x)\, dx = \int_{-1}^{-k} + \int_k^1 \phi_n(x)(Au)(x)\, dx, \tag{3.22}$$
$$n = 0, 1, \cdots.$$

Since the operators involved are linear, this is equivalent to the condition

$$\int_{-1}^{-k} + \int_k^1 \phi_n(x)[A(S - u)(x)]\, dx = 0, \qquad n = 0, 1, \cdots. \tag{3.23}$$

Our final step is to show that (3.23) implies that $S(x) - u(x) = 0$. We make the change of variables defined by (3.10) and use the explicit forms for $\phi_{2n}(x)$ and $\phi_{2n+1}(x)$, $n = 1, 2, \cdots$, to obtain the expressions

$$\begin{aligned} &\int_{K(k)}^{K(k)+K(k')} \cos\left\{\frac{n\pi}{K(k')}[\theta - K(k)]\right\} [A(S - u)(x(\theta))]\, d\theta = 0, \\ &\qquad\qquad n = 1, 2, \cdots, \\ &\int_{-K(k)-K(k')}^{-K(k)} \cos\left\{\frac{n\pi}{K(k')}[\theta + K(k)]\right\} [A(S - u)(x(\theta))]\, d\theta = 0, \\ &\qquad\qquad n = 1, 2, \cdots. \end{aligned} \tag{3.24}$$

The well-known property of the completeness of the cosine functions enables us to conclude from (3.24) that the function $A(S - u)(x(\theta))$ is constant on each of the indicated intervals. That is, we are dealing with a function of the form

$$A(S - u)(x(\theta)) = a + b \operatorname{sgn} \theta, \qquad K(k) < |\theta| < K(k) + K(k'), \tag{3.25}$$

where a and b are arbitrary constants.

It remains to satisfy the conditions (3.23) for $\phi_0(x)$ and $\phi_1(x)$. This gives us

$$\begin{aligned} 0 &= \int_{-1}^{-k} + \int_k^1 (a + b \operatorname{sgn} x)[R(x)]^{-1/2}\, dx = a \int_{-1}^{-k} + \int_k^1 |x|[R(x)]^{-1/2}\, dx, \\ 0 &= \int_{-1}^{-k} + \int_k^1 (a + b \operatorname{sgn} x)[R(x)]^{-1/2} \operatorname{sgn} x\, dx = b \int_{-1}^{-k} + \int_k^1 [R(x)]^{-1/2}\, dx, \end{aligned} \tag{3.26}$$

whereupon we conclude that $a = b = 0$, and hence

$$A(S - u)(x) = 0, \qquad k < |x| < 1. \tag{3.27}$$

But the explicit form (1.2) of the solution to the integral equation shows that (3.27) is satisfied only by

$$S(x) - u(x) = 0, \qquad k < |x| < 1, \tag{3.28}$$

which establishes the desired result.

4. Correction of Tricomi's solution. The integral equation

$$\int_{-1}^{-k} + \int_k^1 \frac{v(t)}{t - x}\, dt = g(x), \qquad k < |x| < 1, \quad 0 < k < 1, \tag{1.9}$$

has been treated by Tricomi [4], who gives two formulas for the solution. Unfortunately one is in error. The integral identity found at the foot of p. 405 in his paper has been substituted with a loss of sign into the equation which precedes it. This leads to the presence of two terms in his equation (9) which should cancel instead. Upon correction, his result verifies the one given here.

To solve (1.9), we differentiate (1.1) to find that

$$\int_{-1}^{-k} + \int_k^1 \frac{u(t)}{t - x}\, dt = -f'(x), \qquad k < |x| < 1. \tag{4.1}$$

Then formula (1.2) applies to (4.1) and (1.9) with the identification $v(x) = u(x)$, $g(x) = -f'(x)$, $k < |x| < 1$. Of course, $f(x) = -\int g(x)\, dx$ is now arbitrary to the extent of an additive constant, and so the definite integrals involving $f(x)$ in (1.2) must also be considered as arbitrary constants. Thus we are led to the formula (1.10) as the solution.

5. Proof of two integral identities. From (1.1) and (2.23), we consider the special case in which $f(x) \equiv 1$, $k < |x| < 1$. The corresponding solution $u_1(x)$ is given by

$$\pi^2 u_1(x) = -D_1 |x| [R(x)]^{-1/2}, \qquad k < |x| < 1, \tag{5.1}$$

where D_1 is unknown. Moreover, we have

$$\int_{-1}^{-k} + \int_k^1 \log|x - t|\, |t| [R(t)]^{-1/2}\, dt = -\frac{\pi^2}{D_1}, \qquad k < |x| < 1. \tag{5.2}$$

Then by letting $x \to \pm 1$ in (5.2) and combining the results, we find

$$-\frac{\pi^2}{D_1} = \frac{1}{2} \int_{-1}^{-k} + \int_k^1 \log(1 - t^2) |t| [R(t)]^{-1/2}\, dt = -\pi \log \frac{2}{k'} \tag{5.3}$$

which establishes (2.26).

To derive (2.27) we consider (1.1) and (2.32) for the special case $f(x) \equiv x$, $k < |x| < 1$. The solution $u_2(x)$ then follows from (2.32) as

$$\pi^2[R(x)]^{1/2} \operatorname{sgn} x\, u_2(x) = \int_{-1}^{-k} + \int_k^1 \frac{[R(t)]^{1/2} \operatorname{sgn} t}{t - x}\, dt - \frac{N_2 + 2\pi k}{2K(k)}, \qquad k < |x| < 1,$$

(5.4)

$$N_2 = \int_{-k}^{k} [-R(x)]^{-1/2}\left[\int_{-1}^{-k} + \int_k^1 \frac{[R(t)]^{1/2} \operatorname{sgn} t}{t - s}\, dt\right] ds.$$

The evaluation of the integrals in (5.4) is effected by use of the identity

$$\frac{2}{\pi}\int_{-1}^{-k} + \int_k^1 \frac{[R(t)]^{1/2} \operatorname{sgn} t}{t - y}\, dt = \begin{cases} 1 - k^2 - 2(k^2 - y^2)^{1/2}[(1 - y^2)^{1/2} \\ \quad - (k^2 - y^2)^{1/2}], & |y| < k, \\ 1 - k^2 - 2(y^2 - k^2), & k < |y| < 1, \end{cases} \tag{5.5}$$

which follows from identities found in [2]. It follows that

$$\pi u_2(x) = \operatorname{sgn} x\, [R(x)]^{1/2}[1 - E(k)/K(k) - x^2], \qquad k < |x| < 1. \tag{5.6}$$

Then substitution into (1.1) yields (2.27).

Next, we evaluate the normalizing constants

$$M_{2n} = M_{2n+1} = -K^2(k')/(2n), \qquad n = 1, 2, \cdots. \tag{5.7}$$

From the form of the $\{\phi_n(x)\}$, it is clear that $M_{2n} = M_{2n+1}$, $n = 1, 2, \cdots$. We consider the special case of (1.1) in which $f(x) = [R(x)]^{1/2}\phi_{2n}(x)$, $k < |x| < 1$. After some manipulation, the corresponding solution $u_3(x)$ is found from (1.2) to be

$$\begin{aligned} [R(x)]^{1/2} \operatorname{sgn} x\, u_3(x) = {} & \frac{n^2}{K^2(k')}\int_{-1}^{-k} + \int_k^1 \log|x - t|\phi_{2n}(t)\, dt \\ & + \frac{x}{M_0}\int_{-1}^{-k} + \int_k^1 [R(t)]^{1/2}\phi_0(t)\phi_{2n}(t)\, dt \\ & - \frac{1}{\pi^2}\int_{-1}^{-k} + \int_k^1 t|t|\phi_{2n}(t)\, dt, \\ & \qquad k < |x| < 1, \quad n = 1, 2, \cdots. \end{aligned} \tag{5.8}$$

Alternatively, from (1.4) we find that

$$\begin{aligned} [R(x)]^{1/2} \operatorname{sgn} x\, u_3(x) = {} & \frac{K(k')}{2M_n}[R(x)]^{1/2} \operatorname{sgn} x\, \phi_{2n}(x) \\ & + \frac{x}{M_0}\int_{-1}^{-k} + \int_k^1 [R(t)]^{1/2}\phi_0(t)\phi_{2n}(t)\, dt, \\ & \qquad k < |x| < 1, \quad n = 1, 2, \cdots. \end{aligned} \tag{5.9}$$

Comparison of these results yields the relation

$$
\begin{aligned}
(5.10)\qquad \frac{n^2}{K^2(k')}\left(\int_{-1}^{-k} + \int_k^1\right) \log|x-t|\phi_{2n}(t)\,dt - \frac{1}{\pi^2}\left(\int_{-1}^{-k} + \int_k^1\right) t|t|\phi_{2n}(t)\,dt \\
= \frac{K(k')}{2M_{2n}}[R(x)]^{1/2}\operatorname{sgn} x\,\phi_{2n}(x), \\
k < |x| < 1, \quad n = 1, 2, \cdots.
\end{aligned}
$$

Upon multiplying (5.9) by $\phi_{2n}(x)$ and integrating over the indicated interval, we find that

$$
(5.11)\qquad \frac{n^2}{K^2(k')}\left(\int_{-1}^{-k} + \int_k^1\right) \phi_{2n}(x)\left(\int_{-1}^{-k} + \int_k^1\right) \log|x-t|\phi_{2n}(t)\,dt\,dx = \frac{K^2(k')}{4M_{2n}}, \qquad n = 1, 2, \cdots,
$$

which gives

$$
(5.12)\qquad M_{2n}^2 = \frac{K^4(k')}{4n^2}, \qquad n = 1, 2, \cdots.
$$

Then (3.3) and (3.5) show that $M_n = I_{nn} < 0$, $n = 1, 2, \cdots$, which leads to the desired result (5.7).

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SIAM J. MATH. ANAL.
Vol. 2, No. 3, August 1971

ON LIE ALGEBRAS OF DIFFERENCE OPERATORS AND THE SPECIAL FUNCTIONS OF MATHEMATICAL PHYSICS*

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Abstract. A method is given whereby special functions of hypergeometric type and their generalizations can be derived in a systematic fashion from Lie algebras of difference and differential operators in one complex variable. The method automatically yields contour integral representations, generating functions, recurrence formulas, integral identities, and infinite sum identities for each function treated. This approach unites Vilenkin's integral transform method for special functions and the older factorization method into a single flexible tool.

Introduction. In the study of special functions via group theory two basic methods have been found useful. The older approach, related to the factorization method [1], considers special functions as basis vectors for models of Lie algebra representations and leads to generating functions and series identities. The author's book [2] probably presents this approach in its purest form, although the idea is an old one. The newer approach due primarily to Vilenkin [3] involves a Fourier or Mellin transform of simple multiplier representations of certain Lie groups. The group action in the transformed representation is via integral operators whose kernels are special functions. The group multiplication property of these operators yields integral identities for special functions.

Both methods agree in considering the groups with Lie algebras $\mathscr{T}_3$, $\mathscr{G}(0, 1)$ and sl(2) (to be defined later) as fundamental for the study of special functions of hypergeometric type. However, beyond this the relationship of the methods is not entirely clear. The older method is more systematic than Vilenkin's, but it seems incapable of obtaining by itself his integral identities. On the other hand, Vilenkin's approach does yield some series identities (through the evaluation of some contour integrals by residues) but their group theoretic significance is not clear and the identities are obtained for very restricted values of the parameters. Finally, the factorization method isolates five classes of special functions while Vilenkin's method (as applied to $\mathscr{T}_3$, $\mathscr{G}(0, 1)$, and sl(2)) yields only three of these.

In the present paper we outline a theory which is capable of obtaining all the results of [2] and [3] for functions associated with the above three Lie algebras, and relating the results to one another. Furthermore, the theory can easily be generalized to apply to new classes of special functions.

The theory involves a classification of all realizations of $\mathscr{T}_3$, $\mathscr{G}(0, 1)$ and sl(2) by second order *difference* operators in one complex variable. Using these operators to construct models of irreducible Lie algebra representations we show that the basis functions for each model are special functions of hypergeometric type. Then using (formally) the Fourier transform we map our model into a new model involving first order *differential* operators. Local Lie theory is employed to extend this Lie algebra model to a Lie group representation and the results are mapped back to the difference operator space via the inverse Fourier transform. The effect of this procedure is to exponentiate a Lie algebra representation by difference

* Received by the editors September 14, 1970.

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operators into a local Lie group representation. In the process we obtain both series and integral identities for special functions as well as combinations of the two.

The difference operator approach yields three classes of functions—those obtained by Vilenkin. We show that the remaining two classes are related to Lie algebra models in terms of second order differential operators, which can also be exponentiated via the Fourier transform.

All of the special function identities obtained in this paper are known, although we could have obtained new results by considering sufficiently complicated examples. The point we wish to emphasize is the pedagogical and logical simplicity of a reorganization of special function theory along the lines indicated here.

1. Lie algebras of difference operators. We begin by constructing realizations of the Lie algebra $\mathscr{T}_3$. This Lie algebra has a basis J^+, J^-, J^3 with commutation relations

$$[J^3, J^\pm] = \pm J^\pm, \qquad [J^+, J^-] = 0. \tag{1.1}$$

We require that the realizations consist of linear difference operators acting on a space of functions of one complex variable x. In particular,

$$\begin{aligned} J^+ &= aE + b + cL, \\ J^- &= gE + h + jL, \\ J^3 &= sE + u + vL, \end{aligned} \tag{1.2}$$

where the lower-case letters denote functions of x, and E, L are the difference operators

$$Ef(x) = f(x + 1), \qquad Lf(x) = f(x - 1). \tag{1.3}$$

Here, f is a function of x.

We shall classify all operators (1.2) satisfying the commutation relations (1.1). (We shall not be specific as to the domains of the functions involved since our computations are purely formal.) These operators will then be used to construct models of those irreducible representations of $\mathscr{T}_3$ classified in [2] or [4]. For all such representations the Casimir operator J^+J^- is a nonzero multiple of the identity operator. Therefore, we shall also subject the operators (1.2) to the requirement

$$J^+J^- = \lambda I, \tag{1.4}$$

where λ is a nonzero complex constant and I is the identity operator.

To simplify the statement of the solution of this problem note that if the operators $J^\pm, J^3$ provide a realization of (1.1), (1.4), then so do the operators

$$\tilde{J}^\pm = \rho(x)^{-1}J^\pm\rho(x), \qquad \tilde{J}^3 = \rho(x)^{-1}J^3\rho(x), \tag{1.5}$$

where $\rho(x)$ is a nonzero function of x. Here,

$$\tilde{J}^3 = \rho(x)^{-1}\rho(x + 1)s(x)E + u(x) + \rho(x)^{-1}\rho(x - 1)v(x)L \tag{1.6}$$

and

$$s(x)v(x + 1) = \tilde{s}(x)\tilde{v}(x + 1)$$

with similar results for $\tilde{J}^{\pm}$. We shall regard two realizations J, $\tilde{J}$ as *equivalent* if they are related by (1.5) for some function $\rho(x)$. (Note that by an appropriate choice of $\rho(x)$ we can show that every realization is equivalent to a realization for which either $s(x) \equiv 1$ or $s(x) \equiv 0$.)

Furthermore, if $J^{\pm}$, J^3 provide a realization of the commutation relations, then so do the operators $\hat{J}^{\pm} = J^{\pm}$, $\hat{J}^3 = -J^3$. We shall identify the realizations J and $\hat{J}$.

THEOREM 1. *Every realization of relations* (1.1) *and* (1.4) *by difference operators* (1.2) *is equivalent to a realization of the form*

$$J^+ = k_1 E, \qquad J^- = k_2 L,$$
$$J^3 = k_3 E + (k_4 - x) + k_5 L, \qquad k_j \in \mathbb{C},$$

where $k_1 k \ = \lambda$ *and* $k_3 = 0$ *or* 1.

The proof of this theorem, though tedious, is completely elementary, so we omit it.

The four-dimensional Lie algebra $\mathscr{G}(0, 1)$ has a basis $J^{\pm}$, J^3, I with commutation relations

$$\begin{aligned} &[J^3, J^{\pm}] = \pm J^{\pm}, \qquad [J^+, J^-] = -I, \\ &[J^3, I] = [J^{\pm}, I] = 0. \end{aligned} \tag{1.7}$$

We look for realizations of this Lie algebra such that $J^{\pm}$, J^3 take the form (1.2) and I is the identity operator. In order to successfully construct models of the irreducible representations of $\mathscr{G}(0, 1)$ listed in [2] or [4] we require that the Casimir operator be a multiple of the identity

$$J^+J^- - J^3 = \lambda I, \tag{1.8}$$

where λ is a complex constant. Two realizations of (1.7) and (1.8) are equivalent if they are related by (1.5) for some nonzero function $\rho(x)$. Furthermore, noting that the operators $\hat{J}^+ = J^-$, $\hat{J}^- = -J^+$, $\hat{J}^3 = -J^3$, $\hat{I} = I$ provide a realization whenever $J^{\pm}$, J^3 do so, we identify these realizations.

THEOREM 2. *Every realization of* (1.7) *and* (1.8) *by difference operators is equivalent to a realization of the form*

$$J^+ = k_1 E + k_2, \qquad J^- = k_3 + (k_4 - x/k_1)L,$$
$$J^3 = k_5 E + (k_6 - x) + k_2(k_4 - x/k_1)L, \qquad k_j \in \mathbb{C},$$

where $k_1 \neq 0$, $k_1 k_3 = k_5$ *and* $k_2 k_3 + k_1 k_4 - k_6 = \lambda + 1$. *We can assume* $k_5 = 0$ *or* 1 *if necessary.*

The three-dimensional Lie algebra sl(2) has a basis $J^{\pm}$, J^3 with commutation relations

$$[J^3, J^{\pm}] = \pm J^{\pm}, \qquad [J^+, J^-] = 2J^3. \tag{1.9}$$

We look for realizations of this Lie algebra such that $J^{\pm}$, J^3 are difference operators (1.2) and the Casimir operator

$$J^+J^- + J^3J^3 - J^3 = \lambda(\lambda + 1)I \tag{1.10}$$

is a multiple of the identity operator. We define the equivalence of two realizations by (1.5) and identify two realizations J, $\hat{J}$ if $\hat{J}^+ = J^-$, $\hat{J}^- = J^+$, $\hat{J}^3 = -J^3$.

THEOREM 3. *Every realization of* (1.9), (1.10) *for which* $c(x) \not\equiv 0$ *is equivalent to a realization of the form*

$$
\begin{aligned}
J^+ &= k_1E + (2k_1x + k_2) + (k_1x(x-1) + k_2x + k_3)L,\\
J^- &= -\frac{k_4^2E}{k_1} - \frac{k_4(k_4-1)}{k_1^2}(2k_1x + k_2)\\
&\quad - \frac{(k_4-1)^2}{k_1^2}(k_1x(x-1) + k_2x + k_3)L,\\
J^3 &= k_4E + \left((2k_4-1)x + (2k_4-1)\frac{k_2}{2k_1}\right)\\
&\quad + \frac{k_4-1}{k_1}(k_1x(x-1) + k_2x + k_3)L,\\
&k_j \in \mathbb{C},\quad k_1 \neq 0,\quad \lambda(\lambda+1) = -\frac{k_2}{2k_1}\left(-\frac{k_2}{2k_1} + 1\right) - \frac{k_3}{k_1}.
\end{aligned}
\tag{1.11}
$$

(*We can assume* $k_4 = 0$ *or* 1.) *Every realization for which* $c(x) \equiv 0$ *is equivalent to a realization of the form*

$$
\begin{aligned}
J^+ &= k_1E,\\
J^- &= -\frac{k_4^2}{k_1}E + \frac{2k_4}{k_1}(x - k_2) + \left(\frac{x}{k_1}(1-x) + \frac{2k_2}{k_1}x + k_3\right)L,\\
J^3 &= k_4E + (k_2 - x),\\
&k_j \in \mathbb{C},\quad k_1 \neq 0,\quad \lambda(\lambda+1) = k_2(k_2+1) + k_1k_3.
\end{aligned}
\tag{1.12}
$$

(*We can assume* $k_4 = 0$ *or* 1.)

If $k_3 = 0$ it is easy to show that (1.11) is equivalent to the realization

$$
\begin{aligned}
J^+ &= k_1(k_1x + k_2)E + (2k_1x + k_2) + xL,\\
J^- &= -\frac{k_4^2}{k_1}(k_1x + k_2)E - \frac{k_4(k_4-1)}{k_1^2}(2k_1x + k_2) - \frac{(k_4-1)^2}{k_1^2}xL,\\
J^3 &= k_4(k_1x + k_2)E + \left((2k_4-1)x + (2k_4-1)\frac{k_2}{2k_1}\right) + \frac{k_4-1}{k_1}xL,
\end{aligned}
\tag{1.13}
$$

which is first order in x. Similarly, if $k_3 = 0$, then (1.12) is equivalent to the realization

$$
\begin{aligned}
J^+ &= k_1(2k_2 - x)E,\\
J^- &= -\frac{k_4^2}{k_1}(2k_2 - x)E + \frac{2k_4}{k_1}(x - k_2) + \frac{x}{k_1}L,\\
J^3 &= k_4(2k_2 - x)E + (k_2 - x),\qquad \lambda(\lambda+1) = k_2(k_2+1).
\end{aligned}
\tag{1.14}
$$

2. Models of $\mathcal{T}_3$ representations. We now use Theorem 1 to construct models of the irreducible representations of $Q(\omega, m_0)$ of $\mathcal{T}_3$, listed in [2] and [4]. Here, the representations space W has a basis $\{f_m\}$, $m \in S = \{m_0 + n : n = 0, \pm 1, \cdots\}$, where $0 \leqq \operatorname{Re} m_0 < 1$. The action of the operators $J^{\pm}$, J^3 on this basis is

$$J^3 f_m = m f_m, \quad J^{\pm} = \omega f_{m \pm 1}, \quad J^+ J^- f_m = \omega^2 f_m. \tag{2.1}$$

If the J-operators are the difference operators of Theorem 1, then the basis functions f_m are Bessel functions or their degenerate limits. Indeed, if we choose

$$\begin{aligned} J^+ &= \omega E, \qquad J^- = \omega L, \\ J^3 &= \frac{z}{2}E - x - \frac{z}{2}L, \qquad z, \omega \in \mathbb{C}, \end{aligned} \tag{2.2}$$

and

$$f_m(x) = K_{m+x}(z), \tag{2.3}$$

we obtain a model of (2.1). A linearly independent set of solutions is given by

$$f_m(x) = I_{m+x}(-z). \tag{2.4}$$

Here $K_m(z)$ and $I_m(z)$ are modified Bessel functions [5, Chap. 7].

We can exponentiate our model of a Lie algebra representation to construct a Lie group representation. One way to accomplish this is via the Fourier transform. Let $\mathcal{H}$ be the space of C^∞-functions $h(y)$ on the real line such that $e^{\alpha|y|} y^n \, dy^k h(y)$ is integrable for all nonnegative integers n and k and for some $\alpha > 0$. We define the Fourier transform of $h \in \mathcal{H}$ by

$$H(x) = \mathcal{F}[h(y)] = \int_{-\infty}^{\infty} h(y) e^{xy} \, dy. \tag{2.5}$$

Then $H(x)$ is an analytic function of x in the strip $-\alpha < \operatorname{Re} x < \alpha$ and

$$h(y) = \mathcal{F}^{-1}[H(x)] = \frac{1}{2\pi i} \int_{c - i\infty}^{c + i\infty} H(x) e^{-xy} \, dx, \qquad -\alpha < c < \alpha. \tag{2.6}$$

Furthermore, we have the following relations:

$$\begin{aligned} &\mathcal{F}(e^y h(y)) = H(x+1), \qquad \mathcal{F}(y h(y)) = \frac{dH}{dx}(x), \\ &\mathcal{F}(e^{-y} h(y)) = H(x-1), \\ &\mathcal{F}\left(\frac{dh}{dy}\right) = -xH(x). \end{aligned} \tag{2.7}$$

Proceeding formally, we can regard the basis functions $f_m(x)$ as the Fourier transforms of functions $h_m(y)$ on the real line. The induced action of the Lie algebra representation on the functions $h(y)$ is given by differential operators $K = \mathcal{F}^{-1} J \mathcal{F}$, where the J-operators are determined by (2.2):

$$K^+ = \omega e^y, \qquad K^- = \omega e^{-y}, \tag{2.8}$$
$$K^3 = z \sinh y + \frac{d}{dy}.$$

(Note that the K-operators satisfy the same commutation relations as the J-operators.) Using the K-operators to construct a model of $Q(\omega, m_0)$ we easily find the basis

$$h_m(y) = c_0 \exp[-z \cosh y + my], \tag{2.9}$$

where c_0 is a nonzero constant. (For convenience we set $c_0 = \frac{1}{2}$.) Here $h_m(y) \in \mathscr{H}$ for $\operatorname{Re} z > 0$.

Since the operators (2.8) are first order differential operators we can use Lie theory to compute the local Lie group representation which they induce. Consider the *real* Lie algebra with basis $K^\pm$, K^3. This is the Lie algebra of the group of motions MH(2) of the pseudo-Euclidean plane [3, Chap. V]. Here MH(2) has the 3×3 matrix realization

$$A(\theta, b_1, b_2) = \begin{bmatrix} e^\theta & 0 & b_1 \\ 0 & e^{-\theta} & b_2 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.10}$$

and multiplication rule

$$A(\theta, b_1, b_2)A(\theta', b_1', b_2') = A(\theta + \theta', b_1 + e^\theta b_1', b_2 + e^{-\theta} b_2'). \tag{2.11}$$

Furthermore, we can make the identifications $\exp \theta K^3 = A(\theta, 0, 0)$, $\exp b_1 K^+ = A(0, b_1, 0)$, and $\exp b_2 K^- = A(0, 0, b_2)$. A straightforward computation using local Lie theory [2] shows that the action of MH(2) on $\mathscr{H}$ induced by the K-operators is

$$[T(A)h](y) = \exp\left[\omega b_1 e^y + \omega b_2 e^{-y} + 2z \sinh\frac{\theta}{2} \sinh\left(y + \frac{\theta}{2}\right)\right] h(y + \theta), \qquad h \in \mathscr{H}. \tag{2.12}$$

These operators leave $\mathscr{H}$ invariant provided

$$\operatorname{Re}(\omega b_1 + (z/2)(-1 + e^\theta)) < 0 \tag{2.13a}$$

and

$$\operatorname{Re}(\omega b_2 + (z/2)(-1 + e^{-\theta})) < 0. \tag{2.13b}$$

However, for fixed $h(y)$ these operators may make sense for a larger parameter domain. From the group property we have

$$T(A)T(A') = T(AA'), \qquad A, A' \in \mathrm{MH}(2). \tag{2.14}$$

Expression (2.14) makes sense on $\mathscr{H}$ provided the coordinates of A, A' and AA' each satisfy the inequalities (2.13). (Here again for fixed $h(y)$ the restrictions on A and A' may be relaxed considerably.) The matrix elements of the operators $T(A)$

with respect to the basis $h_m = h_{m_0+k}$, $k = 0, \pm 1, \cdots$, are uniquely determined by (2.1) and are given by

$$[T(A)h_{m_0+k}](y) = \sum_{l=-\infty}^{\infty} T(A)_{lk} h_{m_0+l}(y),$$
$$T(A)_{lk} = \frac{e^{(m_0+k)\theta}(b_2\omega)^{(k-l+|k-l|)/2}(b_1\omega)^{(l-k+|k-l|)/2}}{|k-l|!}\ {}_0F_1(|k-l|+1;\omega^2 b_1 b_2) \tag{2.15}$$

(see [2, Chap. 3]).

Now we transfer this representation to the space $\mathscr{F}\mathscr{H}$. The basis vectors are

$$\mathscr{F}[h_m(y)] = f_m(x) = \frac{1}{2}\int_{-\infty}^{\infty} \exp[-z\cosh y + (x+m)y]\,dy, \quad \operatorname{Re} z > 0. \tag{2.16}$$

It is easily verified that the operators $J = \mathscr{F}K\mathscr{F}^{-1}$ on $\mathscr{F}\mathscr{H}$ are given by (2.2) and that the relations (2.1) are satisfied by the basis functions (2.16). Moreover,

$$\mathscr{F}[h_m(y)] = K_{x+m}(z) \tag{2.17}$$

(see [5, vol. 2, p. 82]).

The operators $U(A) = \mathscr{F}T(A)\mathscr{F}^{-1}$ define a local representation of MH(2) on $\mathscr{F}\mathscr{H}$. Here,

$$\begin{aligned} U(A)f(x) &= \mathscr{F}T(A)\mathscr{F}^{-1}f \\ &= \frac{1}{2\pi i}\int_{-\infty}^{\infty} dy e^{yx} \exp\left[\omega b_1 e^y + \omega b_2 e^{-y} + 2z\sinh\frac{\theta}{2}\sinh\left(y+\frac{\theta}{2}\right)\right] \\ &\quad \cdot \int_{c-i\infty}^{c+i\infty} e^{-t(y+\theta)} f(t)\,dt, \end{aligned} \tag{2.18}$$

where $A = A(b_1, b_2, \theta)$. If the integrals are absolutely convergent, we can interchange the order of integration and write

$$U(A)f(x) = \int_{c-i\infty}^{c+i\infty} K(x,t;A)f(t)\,dt, \tag{2.19}$$

$$\begin{aligned} K(x,t;A) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \exp\bigg[&\omega b_1 e^y + \omega b_2 e^{-y} + 2z\sinh\frac{\theta}{2}\sinh\left(y+\frac{\theta}{2}\right) \\ &+ y(x-t) - t\theta\bigg]\,dy. \end{aligned} \tag{2.20}$$

(These expressions are valid if the inequalities (2.13) hold.) The group multiplication law implies

$$K(x,t;A_1A_2) = \int_{c-i\infty}^{c+i\infty} K(x,y;A_1)K(y,t;A_2)\,dy \tag{2.21}$$

provided it is permissible to interchange the orders of integration in $U(A_1A_2)f = U(A_1)[U(A_2)f]$ (see [3, Chap. V]).

Formulas (2.20) and (2.21) lead to a number of identities for Bessel functions which are identical with those derived in [3] so we shall not repeat them here. The novelty in our treatment is that we also have the relations

$$(2.22)\qquad U(A)f_{m_0+k}(x) = \int_{c-i\infty}^{c+i\infty} K(x,t;\,A)f_{m_0+k}(t)\,dt = \sum_{l=-\infty}^{\infty} T_{lk}(A)f_{m_0+l}(x)$$

which permit us to exponentiate the Lie algebra representation (2.2), (2.3). For example let $A = A(b/2, b/2, 0)$, $\omega = i\rho$. Then

$$(2.23)\qquad \begin{aligned} K(x,t;\,A) &= \frac{1}{\pi i}K_{x-t}(-i\rho b), \qquad \operatorname{Im}\rho b > 0, \\ T_{lk}(A) &= i^{k-l}J_{k-l}(\rho b), \end{aligned}$$

where $J_n(z)$ is a Bessel function [5, Chap. 7]. Then if $|\rho b| < |z|$, $\operatorname{Re} z > 0$,

$$(2.24)\qquad \begin{aligned} U(A)f_{m_0+k}(x) &= \frac{1}{2}\int_{-\infty}^{+\infty} T(A)h_{m_0+k}(y)e^{xy}\,dy \\ &= \frac{1}{2}\int_{-\infty}^{+\infty} \exp\left[(i\rho b - z)\cosh y + (m_0 + k + x)y\right]dy. \end{aligned}$$

Making use of (2.15), (2.23) and changing the order of integration and summation, we obtain

$$U(A)f_{m_0+k}(x) = \sum_{l=-\infty}^{\infty} i^{k-l}J_{k-l}(\rho b)f_{m_0+l}(x)$$

(see [6, p. 45]). On the other hand we can evaluate the integral (2.24) directly to obtain

$$U(A)f_{m_0+k}(x) = K_{m_0+k+x}(z - i\rho b).$$

Thus,

$$(2.25)\qquad K_m(z - i\rho b) = \sum_{l=-\infty}^{\infty} i^l J_l(\rho b)K_{m+l}(z), \qquad |\rho b| < |z|, \quad \operatorname{Re} z > 0.$$

Furthermore,

$$(2.26)\qquad \begin{aligned} U(A)f_{m_0+k}(x) &= \frac{1}{\pi i}\int_{c-i\infty}^{c+i\infty} K_{x-t}(-i\rho b)K_{m_0+k+t}(z)\,dt \\ &= K_{m_0+k+x}(z - i\rho b), \qquad \operatorname{Re} z > 0, \quad \operatorname{Im}\rho b > 0. \end{aligned}$$

The group theoretic significance of both integral and infinite series identities are obvious from this approach. Choosing other group elements we can derive a wide variety of such formulas. Furthermore, by choosing contours in (2.5) other than the real axis we can extend the domain of validity of the summation identities and establish them for other types of Bessel functions.

Vilenkin establishes the identities (2.21) in terms of the degenerate basis

$$(2.2')\qquad J^+ = E, \quad J^- = L, \quad J^3 = -x.$$

This basis is convenient for integral identities but it fails to show the group theoretic significance of the identities (2.22). By applying our method to other operators listed in Theorem 1, such as $J^+ = E$, $J^- = L$, $J^3 = zE - x$, we could obtain additional series identities for Bessel functions.

3. Models of representations of $\mathscr{G}(0, 1)$. Theorem 2 can be used to construct models of the irreducible representations $R(-2, m_0, 1)$ of $\mathscr{G}(0, 1)$ (see [2, Chap. 4]). For the representation $R(\omega, m_0, 1)$ the space W has basis $\{f_m\}$, $m \in S = \{m_0 + n : n = 0, \pm 1, \pm 2, \cdots\}$, where $0 \leqq \operatorname{Re} m_0 < 1$. The action of the operators $J^\pm$, J^3, I is given by

$$\begin{aligned} &J^3 f_m = m f_m, \quad I f_m = f_m, \quad J^+ f_m = f_{m+1}, && \omega \in \mathbb{C}, \quad \omega + m_0 \\ &J^- f_m = (m + \omega) f_{m-1}, \qquad (J^+J^- - J^3) f_m = \omega f_m, && \text{not an integer.} \end{aligned} \tag{3.1}$$

If the J-operators are the difference operators of Theorem 2, then the basis functions for $R(-2, m_0, 1)$ are confluent hypergeometric functions or degenerate limits of these functions. Indeed, the operators

$$\begin{aligned} J^+ &= E + 1, \qquad J^- = z + (1 - x)L, \\ J^3 &= zE + (z + 2 - x) + (1 - x)L \end{aligned} \tag{3.2}$$

and basis functions

$$f_m(x) = \Gamma(x)\psi(x, x + m - 1; z) \tag{3.3}$$

satisfy (3.1). The basis functions

$$e^{i\pi x}\frac{\Gamma(x)\Gamma(m - 1)}{\Gamma(m + x - 1)}{}_1F_1(x; m + x - 1; z) \tag{3.4}$$

provide another solution. Here ${}_1F_1$ and ψ are confluent hypergeometric functions [5, vol. I], and $\Gamma(x)$ is a gamma function.

Just as in § 2, we exponentiate our Lie algebra representation via the Fourier transform. The operators $K = \mathscr{F}^{-1}J\mathscr{F}$ are given by

$$\begin{aligned} K^+ &= u + 1, \qquad K^- = \frac{d}{du} + z, \\ K^3 &= u\frac{d}{du} + \frac{d}{du} + zu + z + 2, \end{aligned} \tag{3.5}$$

where $u = e^y$, i.e., we are using the Mellin transform.

Let $\mathscr{H}$ be the space of all C^∞-functions $h(u)$ on $(0, \infty)$ such that $h(u) = o(u^\beta)$, $u \to 0$, for some $\beta < -1$, and $h(u) = O(e^{-\gamma u})$, $u \to \infty$, for some $\gamma > 0$. Let $\mathscr{H}_0 \subset \mathscr{H}$ be the space of C^∞-functions on $(0, \infty)$ with compact support. It is easy to construct a model of $R(-2, m_0, 1)$ in terms of the K-operators. The basis is

$$h_m(u) = c(1 + u)^{m-2}e^{-zu}, \tag{3.6}$$

where c is a constant. For convenience we set $c = 1$. These functions belong to $\mathscr{H}$ provided $\operatorname{Re} z > 0$. However, many of the identities which we derive will make sense for less restricted values of the parameters.

We can use local Lie theory to compute the Lie group representation induced by the operators (3.5). The *real* Lie algebra spanned by $K^{\pm}$, K^3, 1 is the Lie algebra of a four-parameter group G [3, Chap. VIII]. The elements of G are triangular matrices

$$A(a,b,c,\alpha) = \begin{pmatrix} 1 & b & c \\ 0 & \alpha & a \\ 0 & 0 & 1 \end{pmatrix}, \qquad a,b,c \text{ real}, \quad \alpha > 0. \tag{3.7}$$

The group multiplication rule is

$$A(a,b,c,\alpha)A(a',b',c',\alpha') = A(a + \alpha a', b\alpha' + b', c + c' + ba', \alpha\alpha'), \tag{3.8}$$

and we make the identification

$$\begin{aligned} \exp aK^+ &= A(a,0,0,1), & \exp bK^- &= A(0,b,0,1), \\ \exp cI &= A(0,0,c,1), & \exp \varphi K^3 &= A(0,0,0,e^{\varphi}). \end{aligned} \tag{3.9}$$

A straightforward computation [2] shows that the action of G on $\mathscr{H}$ induced by the K-operators is

$$[T(A)h](u) = \exp[(u+1)(ze^{\varphi} - z + a) + 2\varphi + zb + c]h[(u+1)e^{\varphi} + b - 1]. \tag{3.10}$$

This expression defines a semigroup of operators on $\mathscr{H}_0$. Indeed, (3.10) makes sense only if $(u + 1)e^{\varphi} + b - 1 > 0$ for all $u > 0$, i.e., only if

$$e^{\varphi} + b \geqq 1. \tag{3.11}$$

The operators are defined for all z, $\omega \in C$. Furthermore the group property $T(A)T(A') = T(AA')$ holds whenever A and A' satisfy (3.11), and $\mathscr{H}_0$ is invariant under these operators. If $h \in \mathscr{H}$ and $h(u) = O(e^{-\gamma u})$ as $u \to \infty$, then in order that $T(A)h \in \mathscr{H}$ we must require that (3.11) hold and in addition $\operatorname{Re}(ze^{\varphi} - z + a) < \gamma$.

The matrix elements of the operators $T(A)$ with respect to the basis (3.6) are uniquely determined by relations (3.1):

$$\begin{aligned} T(A)h_{m_0+k}(u) &= \sum_{l=-\infty}^{\infty} T_{lk}(A)h_{m_0+l}(u), \qquad |be^{-\varphi}| < |u+1|, \\ T_{lk}(A) &= e^c\alpha^{m_0+l}b^{k-l}L^{(k-l)}_{m_0+l+\omega}(-ab/\alpha), \end{aligned} \tag{3.12}$$

where

$$L_{\nu}^{(\mu)}(z) = \frac{\Gamma(\mu+\nu+1)}{\Gamma(\mu+1)\Gamma(\nu+1)}\,{}_1F_1(-\nu;\mu+1;z)$$

is a generalized Laguerre function [5, vol. I].

We can now transfer this representation to the spaces $\mathscr{M}\mathscr{H}_0$ and $\mathscr{M}\mathscr{H}$, where $\mathscr{M}$ stands for the Mellin transform. Here

$$H(x) = \mathscr{M}[h(u)] = \mathscr{F}[h(e^y)] = \int_0^\infty h(u)u^{x-1}\,du, \qquad h(u) \in \mathscr{H}_0, \tag{3.13}$$
$$h(u) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} H(x)u^{-x}\,dx.$$

If $h \in \mathscr{H}$, then (3.13) is still valid provided the requirements $\operatorname{Re} x > -\beta$ and $\gamma > -\beta$ are met. The basis vectors $\mathscr{M}[h_m(u)] \in \mathscr{M}\mathscr{H}$ are

$$\mathscr{M}[h_m(u)] = f_m(x) = \int_0^\infty (1 + u)^{m-2}u^{x-1}e^{-zu}\,du, \tag{3.14}$$
$$\operatorname{Re} x > 0, \quad \operatorname{Re} z > 0.$$

It is straightforward to check that the functions $f_m(x)$ and the operators $J^{\pm}$, J^3 satisfy (3.1), thus providing us with a model of $R(-2, m_0, 1)$. (The integral can be analytically continued into the half-plane $\operatorname{Re} x < 0$ with the exception of simple poles at $x = 0, -1, -2, \cdots$.) In fact, $f_m(x)$ is given by (3.3) as should have been expected [5, vol. I].

The operators $U(A) = \mathscr{M}T(A)\mathscr{M}^{-1}$ define a local semigroup representation of G on $\mathscr{M}\mathscr{H}_0$:

$$U(A)f(x) = \mathscr{M}T(A)\mathscr{M}^{-1}f$$
$$= \frac{1}{2\pi i}\int_0^\infty du\, u^{x-1}\exp\left[(u + 1)(ze^\varphi - z + a) + 2\varphi + zb + c\right] \tag{3.15}$$
$$\cdot\int_{\gamma - i\infty}^{\gamma + i\infty} \left[(u + 1)e^\varphi + b - 1\right]^{-t}f(t)\,dt.$$

Corresponding to those values of the group parameters for which the iterated integral is absolutely convergent we can interchange the order of integration and obtain

$$U(A)f(x) = \int_{\gamma - i\infty}^{\gamma + i\infty} K(x, t; A)f(t)\,dt, \tag{3.16}$$

$$K(x, t; A) = \frac{1}{2\pi i}\int_0^\infty \exp\left[(u + 1)(ze^\varphi - z + a) + 2\varphi + zb + c\right]u^{x-1} \tag{3.17}$$
$$\cdot\left[(u + 1)e^\varphi + b - 1\right]^{-t}\,du.$$

In particular these expressions are valid if

$$e^\varphi + b > 1, \quad \operatorname{Re} x > 0, \quad \operatorname{Re}(ze^\varphi - z + a) < 0. \tag{3.18}$$

The group multiplication rule implies the identity

$$K(x, t; A_1A_2) = \int_{\gamma - i\infty}^{\gamma + i\infty} K(x, y; A_1)K(y, t; A_2)\,dy \tag{3.19}$$

is valid if the coordinates of A_1, A_2, and A_1A_2 each satisfy (3.18). Furthermore we have the identity

$$\begin{aligned} U(A)f_{m_0+k}(x) &= \int_{\gamma-i\infty}^{\gamma+i\infty} K(x,t;A)f_{m_0+k}(t)\,dt \\ &= \sum_{l=-\infty}^{\infty} T_{lk}(A)f_{m_0+l}(x) \end{aligned} \tag{3.20}$$

valid whenever each portion of the equality converges. (The basis functions $f_m(x)$ belong to $\mathscr{M}\mathscr{H}$ but not to $\mathscr{M}\mathscr{H}_0$. Thus, the domain of validity of (3.20) is more restricted than that of (3.19).)

The kernel functions $K(x,t;A)$ can be expressed in terms of confluent hypergeometric functions and the identities (3.19) are equivalent to those derived in [3, Chap. VIII] (see also [7]). However, (3.20) is new. As in § 2, we prove this formula directly for a single example: $A = A(-b,b,0,1)$, $1 > b > 0$. Here

$$\begin{aligned} T_{lk}(A) &= b^{k-l}L_{m_0+l-2}^{(k-l)}(b^2), \\ K(x,t;A) &= \frac{\Gamma(x)e^{b(z-1)}}{2\pi i}\Psi(x,x-t+1;b), \qquad \operatorname{Re} x > 0. \end{aligned} \tag{3.21}$$

Furthermore,

$$\begin{aligned} U(A)f_{m_0+k}(x) &= \int_0^\infty [T(A)h_{m_0+k}](u)u^{x-1}\,du \\ &= \int_0^\infty \exp[-b(u+1)-zu](u+1+b)^{m_0+k-2}u^{x-1}\,du \\ &= \sum_{l=-\infty}^{\infty} b^{k-l}L_{m_0+l-2}^{(k-l)}(b^2)\Gamma(x)\Psi(x,m_0+l+x-1;z), \end{aligned} \tag{3.22}$$

where we have used (3.12) and interchanged the order of summation and integration. Furthermore, direct evaluation of the integral yields

$$U(A)f_{m_0+k}(x) = e^{-b}(1+b)^{m_0+k+x-2}\Gamma(x)\Psi(x,m_0+k+x-1,(1+b)(b+z)). \tag{3.23}$$

Finally, from (3.16) and (3.21) we have

$$U(A)f_{m_0+k}(x) = \frac{e^{b(z-1)}\Gamma(x)}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \Psi(x,x-t+1;b)\Gamma(t)\Psi(t,t+m-1;z)\,dt, \qquad \operatorname{Re}\gamma > 0. \tag{3.24}$$

One can obtain many such identities by varying the group parameters. Moreover, by changing the contour in (3.13) to a path other than the positive real axis it is possible to derive series identities for other confluent hypergeometric functions. One need only require that the new contour integrals converge and that the duality between (3.2) and (3.5) be maintained. Finally, we can get additional results by changing our choice of operators from Theorem 2 and our selection of an irreducible representation from [2]. Vilenkin's choice of operators is

$$J^+ = (x-1)L, \quad J^- = E, \quad J^3 = z - x$$

which is convenient for integral identities but not for identities involving series.

These results can be extended in another direction by allowing u to vary over the whole real axis. Then (3.10) defines a local group of operators on $\mathscr{H}_0$, not just a semigroup. One takes the Mellin transform on the positive real numbers and on the negative real numbers simultaneously, thus associating with each $h \in \mathscr{H}$ a pair of Mellin transforms $\mathscr{M}^{\pm}h$. This is the approach followed in [3].

4. Models of representations of sl(2). In analogy with the work of the previous sections we use Theorem 3 to construct models of the irreducible representations $D(\rho, m_0)$ of sl(2), [2]. Here, the representation space W has a basis $\{f_m\}$, $m \in S = \{m_0 + n : n = 0, \pm 1, \cdots\}$, where $\rho, m_0 \in \mathbb{C}$, $0 \leqq \operatorname{Re} m_0 < 1$ and

$$\begin{aligned} &J^3 f_m = m f_m, \qquad J^+ f_m = (m - \rho) f_{m+1}, \\ &J^- f_m = -(m+\rho) f_{m-1}, \qquad (J^+J^- + J^3J^3 - J^3) f_m = \rho(\rho+1) f_m. \end{aligned} \tag{4.1}$$

If the J-operators are the difference operators of Theorem 3, then the functions f_m are hypergeometric functions or their degenerate limits. For example, the operators

$$\begin{aligned} J^+ &= (z-1)^{-1}[(x+1-\gamma)E + (2x-\gamma) + (x-1)L], \\ J^- &= (z-1)^{-1}\{(x+1-\gamma)E - z(2x-\gamma) - z^2(x-1)L\}, \\ J^3 &= (1-z)^{-1}\left\{(x+1-\gamma)E + (1+z)\left(x - \frac{\gamma}{2}\right) + z(x-1)L\right\} \end{aligned} \tag{4.2}$$

and basis functions

$$f_m(x) = \frac{(-1)^m \Gamma(\gamma - x)\Gamma(x)}{\Gamma(\gamma)} F\left(m + \frac{\gamma}{2}, \gamma - x; \gamma; 1 - z\right) \tag{4.3}$$

satisfy (4.1) with $\rho = -\gamma/2$. Here the operators (4.2) are obtained from (1.13) by replacing x with $x + 1$, setting $k_1 = 1$, $k_2 = 2 - \gamma$, $k_4 = (1 - z)^{-1}$, and renormalizing the operators for the sake of symmetry. The $F(a, b; c; z)$ is a hypergeometric function [5, vol. I].

As in the preceding section we apply the Mellin transform and obtain differential operators $K = \mathscr{M}^{-1} J \mathscr{M}$ on $\mathscr{H}$:

$$\begin{aligned} K^+ &= (z-1)^{-1}\left\{-(u+1)^2 \frac{d}{du} - \gamma(u+1)\right\}, \\ K^- &= (z-1)^{-1}\left\{(u+z)^2 \frac{d}{du} + \gamma(u+z)\right\}, \\ K^3 &= (z-1)^{-1}\left\{(u+z)(u+1)\frac{d}{du} + \gamma\left(u + \frac{1+z}{2}\right)\right\}. \end{aligned} \tag{4.4}$$

It is easy to verify that the functions

$$h_m(u) = (-1)^m (u+1)^{m-\gamma/2}(u+z)^{-m-\gamma/2}, \qquad u \geqq 0, \tag{4.5}$$

together with the K-operators define a model of $D(-\gamma/2, m_0)$. To be definite we require z real, $z > 1$, but we could also make sense of these formulas for complex z. The $h_m(u)$ belong to $\mathscr{H}$ provided $\operatorname{Re}\gamma > 0$.

The real Lie algebra generated by $K^{\pm}$, K^3 is isomorphic to the Lie algebra of SL$(2, R)$. The elements of SL$(2, R)$ are 2×2 real unimodular matrices

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad ad - bc = 1. \tag{4.6}$$

Here we make the identification

$$\begin{aligned} \exp bK^+ &= \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix}, \qquad \exp cK^- = \begin{pmatrix} 1 & 0 \\ -c & 1 \end{pmatrix}, \\ \exp \varphi K^3 &= \begin{pmatrix} e^{\varphi/2} & 0 \\ 0 & e^{-\varphi/2} \end{pmatrix}. \end{aligned} \tag{4.7}$$

It follows from these expressions and (4.4) that the action of SL$(2, R)$ on $\mathscr{H}$ induced by the K-operators is

$$\begin{aligned} [T(A)h](u) &= (z-1)^{\gamma}[-(a+b)(u+1) + (c+d)(u+z)]^{-\gamma} \\ &\quad \cdot h\left[\frac{az(u+1) + b(u+1) - cz(u+z) - d(u+z)}{-a(u+1) - b(u+1) + c(u+z) + d(u+z)}\right], \\ &\qquad\qquad \left|\frac{cz}{a}\right|, \quad \left|\frac{b}{c}\right| < 1. \end{aligned} \tag{4.8}$$

Just as (3.10) this expression defines a local semigroup of operators on $\mathscr{H}_0$. In particular $\mathscr{H}_0$ is invariant under the operators provided $c + d \geqq a + b$, $az + b \geqq z(cz + d)$.

The matrix elements of these operators with respect to the basis (4.5) are given by

$$\begin{aligned} [T(A)h_{m_0+k}](u) &= \sum_{l=-\infty}^{\infty} T_{lk}(A) h_{m_0+l}(u), \\ T_{lk}(A) &= a^{\rho+m_0+l} d^{\rho-m_0-k} c^{k-l} \frac{\Gamma(\rho + m_0 + k + 1)}{\Gamma(\rho - m_0 + l + 1)} \\ &\quad \cdot \frac{F(-\rho - m_0 - l, -\rho + m_0 + k; k - l + 1; bc/ad)}{\Gamma(k - l + 1)}, \\ &\qquad\qquad d = \frac{1 + bc}{a}. \end{aligned} \tag{4.9}$$

(See [2, Chap. 5] for a precise determination of the domain of definition of these operators and matrix elements.)

Using (3.13) we can tranfer our representation to the spaces $\mathscr{M}\mathscr{H}_0$ and $\mathscr{M}\mathscr{H}$. The basis vectors $\mathscr{M}[h_m(u)]$ are

$$\mathscr{M}[h_m(u)] = f_m(x) = (-1)^m \int_{-\infty}^{\infty} u^{x-1}(u+1)^{m-\gamma/2}(u+z)^{-m-\gamma/2}\,du, \qquad \operatorname{Re}\gamma > \operatorname{Re} x > 0. \tag{4.10}$$

It is straightforward to check that these functions and the operators (4.2) satisfy relations (4.1). Furthermore, the integrals (4.10) are equal to the hypergeometric functions (4.3), [5, vol. I].

We can define a local semigroup representation of SL(2, R) on $\mathscr{M}\mathscr{H}_0$:

$$\begin{aligned} U(A)f(x) &= \mathscr{M}T(A)\mathscr{M}^{-1}f \\ &= \frac{(z-1)^\gamma}{2\pi i}\int_0^\infty du\, u^{x-1}\int_{\delta-i\infty}^{\delta+i\infty}[-(a+b)(u+1)+(c+d)(u+z)]^{-\gamma+t} \\ &\qquad \cdot[(az+b)(u+1)-(cz+d)(u+z)]^{-t}f(t)\,dt. \end{aligned} \tag{4.11}$$

Whenever this iterated integral converges absolutely we can interchange the order of integration and obtain

$$U(A)f(x) = \int_{\delta-i\infty}^{\delta+i\infty} K(x,t;A)f(t)\,dt, \tag{4.12}$$

$$\begin{aligned} K(x,t;A) &= \frac{(z-1)^\gamma}{2\pi i}\int_0^\infty u^{x-1}[-(a+b)(u+1)+(c+d)(u+z)]^{-\gamma+t} \\ &\qquad \cdot[(az+b)(u+1)-(cz+d)(u+z)]^{-t}\,du, \\ &\qquad c+d \geqq a+b, \quad az+b \geqq z(cz+d). \end{aligned} \tag{4.13}$$

The kernel functions are hypergeometric functions or their limits. The group multiplication rule leads to integral identities of the form (3.19). These are worked out in detail in [3, Chap. VII]. In addition we have infinite sum identities of the form (3.20).

For example, let $a = d = \cosh\alpha$, $b = -z^{1/2}\sinh\alpha$, $c = -z^{1/2}\sinh\alpha$, $\alpha > 0$. Then

$$K(x,t;A) = \frac{z^{(x-t)/2}}{2\pi i}\,\frac{\Gamma(x)\Gamma(-x+\gamma)}{\Gamma(\gamma)}\,\frac{(\cosh\alpha)^{x+t-\varphi}}{(\sinh\alpha)^{x+t}}F\left(x,t;\gamma;\frac{-1}{\sinh^2\alpha}\right), \qquad \operatorname{Re}\gamma > \operatorname{Re}x > 0, \tag{4.14}$$

$$\begin{aligned} T_{lk}(A) &= (\cosh\alpha)^{-\gamma+l-k}(-z^{1/2}\sinh\alpha)^{k-l} \\ &\quad \cdot\frac{\Gamma\left(-\frac{\gamma}{2}+m_0+k+1\right)}{\Gamma\left(-\frac{\gamma}{2}+m_0+l+1\right)}\,\frac{F\left(\frac{\gamma}{2}-m_0-l,\frac{\gamma}{2}+m_0+k;k-l+1;\tanh^2\alpha\right)}{\Gamma(k-l+1)}. \end{aligned} \tag{4.15}$$

Also,

$$
\begin{aligned}
U(A)f_{m_0+k}(x) &= \int_0^\infty [T(A)h_{m_0+k}](u)u^{x-1}\,du \\
&= (-1)^{m_0+k}(\cosh\alpha)^{-\gamma}\int_0^\infty (u+z)^{-m_0-k-\gamma/2}(u+1)^{m_0+k-\gamma/2} \\
&\quad \cdot\left[1+z^{-1/2}\tanh\alpha\left(\frac{u+z}{u+1}\right)\right]^{m_0+k-\gamma/2} \\
&\quad \cdot\left[1+z^{1/2}\tanh\alpha\left(\frac{u+1}{u+z}\right)\right]^{-m_0-k-\gamma/2} du \\
&= \sum_{l=-\infty}^{\infty} T_{lk}(A)(-1)^{m_0+l}\frac{\Gamma(\gamma-z)\Gamma(x)}{\Gamma(\gamma)}F\bigg(m_0+l+\frac{\gamma}{2}, \\
&\qquad \gamma-x;\gamma;1-z\bigg),\quad |z^{1/2}\tanh\alpha|<1,
\end{aligned}
\tag{4.16}
$$

and

$$
\begin{aligned}
U(A)f_{m_0+k}(x) &= \frac{z^{x/2}}{2\pi i}\frac{\Gamma(x)\Gamma(-x+\gamma)}{\Gamma(\gamma)^2}\int_{\delta-i\infty}^{\delta+i\infty}\frac{(\cosh\alpha)^{x+t-\gamma}}{(\sinh\alpha)^{x+t}}F\left(x,t;\gamma;\frac{-1}{\sinh^2\alpha}\right) \\
&\quad \cdot\Gamma(\gamma-t)\Gamma(t)F\left(m_0+k+\frac{\gamma}{2},\gamma-t;\gamma;1-z\right)dt, \\
&\qquad \operatorname{Re}\gamma>\xi>0.
\end{aligned}
\tag{4.17}
$$

Finally a direct computation of the integral in (4.6) yields

$$
\begin{aligned}
U(A)f_{m_0+k}(x) &= \frac{(-1)^{m_0+k}}{2\pi i}(\cosh\alpha+z^{1/2}\sinh\alpha)^{-\gamma}\xi^{m_0+k+\gamma/2-x} \\
&\quad \cdot\frac{\Gamma(\gamma-x)\Gamma(x)}{\Gamma(\gamma)}F\left(m_0+k+\frac{\gamma}{2},\gamma-x;\gamma;1-z\xi^2\right), \\
&\qquad \xi=\frac{1+z^{-1/2}\tanh\alpha}{1+z^{1/2}\tanh\alpha}.
\end{aligned}
\tag{4.18}
$$

These results can be extended in many directions (the discussion at the end of § 3 also applies here). In particular by choosing contours other than the positive u-axis we can obtain many additional summation identities. Furthermore, this technique can be used to construct models of irreducible unitary representations of SL$(2, R)$ and SU(2). Such models lead to orthogonality relations for special functions. (See [8] and [9] for some examples.)

5. Representations by differential operators of second order. In [2] we classified all models of the Lie algebras $\mathscr{T}_3$, $\mathscr{G}(0,1)$, and sl(2) by first order differential operators in one complex variable. We have made use of these models in this paper.

Now we classify all models of these Lie algebras by differential operators of at most second order:

$$J^+ = a\frac{d^2}{dx^2} + b\frac{d}{dx} + c, \qquad J^- = g\frac{d^2}{dx^2} + h\frac{d}{dx} + j,$$
(5.1)
$$J^3 = s\frac{d^2}{dx^2} + u\frac{d}{dx} + v, \qquad E = 1,$$

where the lower-case letters denote analytic functions of x. We look for all models of the three Lie algebras such that at least one of a, s, g is not identically zero and such that the Casimir operator is a constant. Two models J, $\hat{J}$ are *equivalent* if

(5.2) $$\hat{J}^{\pm} = \rho(x)^{-1}J^{\pm}\rho(x), \qquad \hat{J}^3 = \rho(x)^{-1}J^3\rho(x)$$

for some nonzero analytic function $\rho(x)$. Here

$$\rho(x)^{-1}J^+\rho(x) = a\frac{d^2}{dx^2} + \left(2a\rho^{-1}\frac{d\rho}{dx} + b\right)\frac{d}{dx} + \left(a\rho^{-1}\frac{d^2\rho}{dx^2} + b\rho^{-1}\frac{d\rho}{dx} + c\right)$$
(5.3)

with similar results for $\tilde{J}^-$, $\tilde{J}^3$. Two models will be identified if one can be obtained from the other by an analytic change of variable. We also identify the realizations J and $\hat{J}$ discussed in § 1.

THEOREM 4. *There are no models of $\mathscr{T}_3$ by second order ordinary differential operators. Every realization of $\mathscr{G}(0, 1)$ is equivalent to*

$$J^+ = -\frac{d}{dx} + \frac{x}{c}, \qquad J^- = c\frac{d}{dx},$$
5.4)
$$J^3 = -c\frac{d^2}{dx^2} = x\frac{d}{dx} - \lambda,$$
$$J^+J^- - J^3 = \lambda I, \qquad \lambda, c \in \mathbb{C}.$$

Every realization of sl(2) *is equivalent to*

$$J^+ = -x\frac{d^2}{dx^2} + 2(x - \rho - 1)\frac{d}{dx} - x + 2(\rho + 1),$$
$$J^- = x\frac{d^2}{dx^2} + 2(\rho + 1)\frac{d}{dx},$$
(5.5)
$$J^3 = -x\frac{d^2}{dx^2} + (x - 2\rho - 2)\frac{d}{dx} + \rho + 1,$$
$$J^+J^- + J^3J^3 - J^3 = \rho(\rho + 1), \qquad \rho \in \mathbb{C}.$$

The proof of this theorem is tedious but straightforward, so we omit it.

We can use the operators (5.4) ($c = 1$) to construct a model of the representation $R(0, m_0, 1)$, (3.1), of $\mathcal{G}(0, 1)$. (Consideration of $R(\omega, m_0, 1)$ for $\omega \neq 0$ leads to no new results.) As basis functions we can take

$$f_m(x) = 2^{-m/2} H_m(x/\sqrt{2}) = e^{x^2/4} D_m(x) \tag{5.6}$$

or

$$f_m(x) = e^{x^2/2} e^{i\pi(m-1)} \Gamma(m+1) 2^{(m+1)/2} H_{-m-1}(ix/\sqrt{2}), \tag{5.7}$$

where $H_m(x)$ is a Hermite function and $D_m(x)$ is a parabolic cylinder function [5, Chap. 8].

Since $J^{\pm}$ are first order operators we could use local Lie theory and (3.12) to derive a series of summation identities for these basis vectors. This is essentially what is done in [2, Chap. 4]. It is more instructive, however, to formally take the inverse Fourier transform as in the preceding sections of this paper. The transformed operators $K = \mathscr{F}^{-1} J \mathscr{F}$ are

$$K^+ = -\left(\frac{d}{du} + u\right), \quad K^- = u, \quad K^3 = -\left(u\frac{d}{du} + u^2 + 1\right). \tag{5.8}$$

In terms of the K-operators, a model for $R(0, m_0, 1)$ is provided by the basis vectors

$$h_m(u) = c\Gamma(m+1) u^{-(m+1)} e^{-u^2/2}, \tag{5.9}$$

where we set $c = 1$ for convenience. Since the K-operators are first order we can use local Lie theory to compute the induced group action of G:

$$[T(A)h](u) = \left\{\exp\left[\frac{(u-a)}{\alpha}\left(\frac{u-a}{2\alpha} + b\right) - \frac{u^2}{2} + c\right]\right\} \alpha^{-1} h\left(\frac{u-a}{\alpha}\right). \tag{5.10}$$

Here, $A = A(a, b, c, \alpha)$ is defined by (3.7). The matrix elements of $T(A)$ with respect to the basis (5.9) are given by (3.12) ($\omega = 0$). To transfer these results to the operators (5.4) we take the Fourier transform:

$$\mathscr{F}[h_m(u)] = \Gamma(m+1) \int_{-\infty}^{\infty} e^{-u^2/2 + ux} u^{-(m+1)}\, du. \tag{5.11}$$

It is easy to verify that these functions and (5.4) define a model of $R(0, m_0, 1)$ at least for $\operatorname{Re} m < 0$. The $\mathscr{F}[h_m(u)]$ are linear combinations of the basis vectors (5.6) and (5.7). Since these functions are defined only for $\operatorname{Re} m < 0$ they are not suitable for the application of operators $U(A) = \mathscr{F} T(A) \mathscr{F}^{-1}$. We get around this difficulty by choosing a new contour for the integral transform $\mathscr{F}$ which preserves the formal properties (2.7) and such that $\mathscr{F}[h_m(u)]$ converges for all $m \in S$. A suitable candidate is

$$\mathscr{F}[h(u)] = \frac{1}{2\pi i} \int_{\infty}^{(0+)} e^{ux} h(u)\, du. \tag{5.12}$$

Here the contour starts at $+\infty$ on the real axis, circles the origin counterclockwise and returns to $+\infty$. Now the functions

$$\begin{aligned}\mathscr{F}[h_m(u)] &= \frac{\Gamma(m+1)}{2\pi i}\int_{\infty}^{(0+)} e^{-u^2/2+ux}u^{-(m+1)}\,du \\ &= e^{i\pi m}D_m(-x)e^{x^2/4} = f_m(x)\end{aligned} \tag{5.13}$$

are defined for all $m \in \mathbb{C}$ [5, Chap. 8] and yield a basis for $R(0, m_0, 1)$. Furthermore, from (5.10) and (3.12) we have

$$\begin{aligned}U(A)f_m(x) &= \mathscr{F}(T(A)h_m(u)) \\ &= \alpha^{-m-2}\frac{\Gamma(m+1)}{2\pi i}\int_{\infty}^{(0+)} \\ &\quad \cdot\left\{\exp\left[-\frac{u^2}{2} + u\left(x + \frac{b}{\alpha}\right) - \frac{ab}{\alpha} + c\right]\right\}(u-z)^{-m-1}\,du \\ &= \sum_{l=-\infty}^{\infty} e^c\alpha^{m_0+l}b^{k-l}L_{m_0+l}^{(k-l)}\left(-\frac{ab}{\alpha}\right)e^{i\pi(m_0+l)}D_{m_0+l}(-x)e^{x^2/4}, \\ &\qquad\qquad m = m_0 + k,\end{aligned} \tag{5.14}$$

where the contour C has been chosen such that $|a/u| < c_0 < 1$ for all u and C. Furthermore, by direct evaluation of the integral we find

$$U(A)f_m(x) = e^{i\pi m}D_m\left(-x + a - \frac{b}{a}\right)\exp\left[\frac{x^2}{4} - \frac{a^2}{4} + \frac{b^2}{4\alpha^2} + \frac{ax}{2} + \frac{xb}{2\alpha} - \frac{ab}{2\alpha} + c\right]. \tag{5.15}$$

These results hold for all values of the parameters x, m and make sense for complex a, b, c, and $\alpha \neq 0$; i.e., we can extend G to a complex Lie group G^c.

The operators (5.4) can also be used to construct unitary irreducible representations of S_4, another real form of the group G^c. This approach, which leads to Hermite polynomials and their orthogonality relations, is discussed in [2, Chap. 4].

The operators (5.5) can be used to construct models of the representations of $D(\rho, m_0)$ of sl(2). In fact, the Laguerre functions

$$f_m(x) = L_{m-\rho-1}^{(2u+1)}(x) \tag{5.16}$$

form a basis for such a model [2, p. 185]. Taking an inverse Fourier transform we can define operators $K = \mathscr{F}^{-1}J\mathscr{F}$, where

$$\begin{aligned}K^+ &= -(u+1)^2\frac{d}{du} + 2\rho u + 2(\rho+1), \\ K^- &= u^2\frac{d}{du} - 2\rho u, \\ K^3 &= -u(u+1)\frac{d}{du} + 2\rho u + \rho.\end{aligned} \tag{5.17}$$

(Note that these are virtually the operators (4.4) with $z = 0$.) As usual we can construct a model of $D(\rho, m_0)$ in terms of the K-operators, extend the model to a group representation using local Lie theory, and map the results back to x-space. The details are similar to the preceding example and lead to various contour integrals and series identities for Laguerre functions. The identities are essentially those derived in [2, Chap. 5].

The operators (5.5) can also be used to construct unitary irreducible representations of SL(2, R). This approach leads to the orthogonality relations for Laguerre polynomials and is briefly discussed in [9].

6. Extensions of the method. To illustrate that our method can be considerably extended we first consider an example which leads to familiar functions. Suppose we search for all realizations of $\mathscr{G}(0, 1)$ by difference operators such that (1.2) and (1.8) hold except that the condition (1.2) for J^3 is dropped. Then there are new solutions in addition to those of Theorem 2. One such is

$$\begin{aligned} J^+ &= E, \qquad J^- = E + x + (1 - x)L, \\ J^3 &= E^2 + sE - x - \lambda, \qquad s, \lambda \in \mathbb{C}. \end{aligned} \tag{6.1}$$

Let us use these operators to find a model of the representation $\uparrow_0$. Here $\lambda = 0$ and W has a basis f_m, $m \in S = \{0, 1, 2, \cdots\}$ such that

$$\begin{aligned} &J^+ f_m = f_{m+1}, \quad J^- f_m = m f_{m-1}, \quad J^3 f_m = m f_m, \\ &J^+ J^- - J^3 = 0, \qquad m \in S. \end{aligned} \tag{6.2}$$

To discover the functions $f_m(x)$ we formally take the inverse Mellin transform and obtain operators

$$K^+ = u, \quad K^- = \frac{d}{du} + u + s, \quad K^3 = u\frac{d}{du} + u^2 + su. \tag{6.3}$$

A straightforward computation shows that the K-operators and the functions

$$h_m(u) = u^m \exp\left[-u^2/2 - su\right], \qquad m = 0, 1, 2, \cdots, \tag{6.4}$$

yield a model of $\uparrow_0$. Taking the Mellin transform we see that the J-operators and the functions

$$\begin{aligned} f_m(x) &= \int_0^\infty u^{m+x-1} \exp\left[-u^2/2 - su\right] du \\ &= e^{s^2/4}\Gamma(m + x) D_{-m-x}(s), \qquad \operatorname{Re}(m + x) > 0, \end{aligned} \tag{6.5}$$

yield another model of $\uparrow_0$. (Since $m \geqq 0$, for convergence of the integral it is enough to require $\operatorname{Re} x > 0$.) The K-operators induce the semigroup representation

$$\begin{aligned} [T(A)h](u) = \Bigg\{ \exp\Bigg[&\frac{u^2}{2}(\alpha^2 - 1) + u(s\alpha - s + a + b\alpha) \\ &+ c + bs - \frac{b^2}{2}\Bigg]\Bigg\} h(\alpha u + b), \qquad A \in G, \quad h \in \mathscr{H}. \end{aligned} \tag{6.6}$$

The matrix elements of these operators with respect to the basis $h_m(u)$ are given by (3.12) with $m_0 = \omega = 0$. Just as in § 3 we can define integral operators $U(A) = \mathscr{M}T(A)\mathscr{M}^{-1}$ with kernels

$$K(x, t; A) = \frac{1}{2\pi i}\int_0^\infty \left\{\exp\left[\frac{u^2}{2}(\alpha^2 - 1) + u(s\alpha - s + a + b\alpha) + c + bs + \frac{b^2}{2}\right]\right\} \cdot u^{x-1}(\alpha u + b)^{-t}\,du, \qquad \alpha^2 < 1, \quad b \geqq 0, \quad \mathrm{Re}\,(x - t) > 0. \tag{6.7}$$

If $\alpha = 1$ and $a + b < 0$, we still get convergence. In fact by (3.3), (3.14) the kernel is then a confluent hypergeometric function. If $\alpha < 1$, $b = 0$, then

$$K(x, t; A) = \frac{\alpha^{-t}(1 - \alpha^2)^{(t-x)/2}}{2\pi i}\Gamma(x - t) \cdot \exp\left[\frac{(s(\alpha - 1) + a)^2}{4(1 - \alpha)^2}\right] D_{t-x}\left(\frac{s(1 - \alpha) - a}{\sqrt{1 - \alpha^2}}\right), \tag{6.8}$$

where $A = A(a, 0, 0, \alpha)$ and $\sqrt{1 - \alpha^2} > 0$. The formulas (3.19) and (3.20) (with l summed from 0 to ∞) lead to integral and series identities for the parabolic cylinder functions whose explicit derivation is left to the reader.

To obtain identities for new classes of functions one need only find models of our Lie algebras in terms of higher order difference and differential operators whose coefficients are at most first order in x. For example the methods of this paper apply to the model

$$J^+ = \frac{d^2}{dx^2} + \frac{x}{k_1}, \qquad J^- = k_1\frac{d}{dx} + k_2, \qquad J^3 = k_1\frac{d^3}{dx^3} + k_2\frac{d^2}{dx^2} + x\frac{d}{dx} + \frac{k_2}{k_1}x - \lambda \tag{6.9}$$

of $\mathscr{G}(0, 1)$ in terms of third order differential operators.

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SIAM J. MATH. ANAL.
Vol. 2, No. 3, August 1971

ASYMPTOTICS FOR A CLASS OF NONANALYTIC SECOND ORDER DIFFERENTIAL EQUATIONS*

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Abstract. New formulas giving asymptotic behavior (at ∞) of the solutions to $y'' + py' + qy = 0$ on $[a, \infty)$ are developed. The coefficient functions p and q must satisfy certain regularity and relative rate of growth conditions, but it is not required that they be analytic nor that they possess asymptotic power series representations.

1. Introduction. If $p(t) = \pm t^{\mu}$, $q(t) = \pm t^{\nu}$ (μ, ν real), and $a > 0$, a result of Ghizzetti ([5] or [1, Theorem 3, p. 92]) yields information about the solutions of

$$y'' + py' + qy = 0 \quad \text{on } [a, \infty) \tag{1.1}$$

provided $\mu < -1$ and $\nu < -2$. In this case the solutions are asymptotic to the solutions to $y'' = 0$. By using the multiplying factor, $\exp\{\int_a^t p\,dt\}$, information about the solutions of (1.1) can be derived from the WKB approximations ([1, Theorem 13, p. 120] or [6, Theorem 1, p. 592] with $m = k = 1$) provided $\nu > -2$ and $\nu > 4\mu + 2$. The results so obtained are in agreement with those indicated by Corollary 1 below. Other less explicit information concerning the solutions to (1.1) may be found in [4]; and recent related papers include [3], [6], [8], [9] and [10].

We shall state all our hypotheses and conclusions explicitly in terms of the coefficient functions of the differential equation under consideration.

2. Applicability. If $a > 0$, $q(t) > 0$ for $t \geqq a$, $p(t) = t^{\mu}[k_1 + o(1)]$, $p'(t) = t^{\mu-1}[k_2 + o(1)]$, $q(t) = t^{\nu}[k_3 + o(1)]$, $q'(t) = t^{\nu-1}[k_4 + o(1)]$, $q''(t) = t^{\nu-2}[k_5 + o(1)$, ($\mu$, ν, k_j real), $k_1 \neq 0$, $0 < k_3$, and all $o(1)$'s represent continuous functions, then conditions (i) through (iv) of Theorem 1 and condition (iii) of Corollary 1 will be satisfied provided $\nu > -2$ and $\nu > 2\mu$. If also k_1, $p(t) > 0$, then conditions (i) through (vi) of Theorem 2 and, for some positive integer l, condition (iii)′ of Corollary 2 will be satisfied provided $\mu > -1$ and $\nu < 2\mu$.

3. Results. The solutions to (1.1) have essentially different behaviors depending on whether q dominates p^2 (the case considered in the first theorem and corollary) or p^2 dominates q (the case considered in the second theorem and corollary).

THEOREM 1. *Suppose that each of p and q is a function defined on $[a, \infty)$ with p continuously differentiable and real-valued and q twice continuously differentiable*

* Received by the editors August 4, 1970, and in revised form November 13, 1970.

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and positive-valued. If $n = 1$ or $n = 2$, and if

(i) $$\int_a^\infty \left|\frac{q''}{q^{3/2}}\right| dt < \infty,$$

(ii) $$\int_a^\infty \left|\frac{pq'}{q^{3/2}}\right| dt < \infty,$$

(iii) $$\left(\frac{p}{q^{1/2}}\right)(t) \to 0 \quad \text{as } t \to \infty,$$

(iv) $$\int_a^\infty \left|\frac{p'}{q^{1/2}}\right| dt < \infty,$$

then there exists a pair of linearly independent solutions (y_1, y_2) to

(3.1) $$y'' - py' - (-1)^n qy = 0 \quad \text{on } [a, \infty)$$

and a number $b \geqq a$ such that for $j = 1, 2$,

$$y_j(t) = q^{-1/4}(t) \exp\left\{\int_b^t f_j\, dt\right\}(1 + o(1))$$

and

$$y_j'(t) = (-\mathbf{i})^n(-1)^j q^{1/4}(t) \exp\left\{\int_b^t f_j\, dt\right\}(1 + o(1)),$$

where

$$f_j = (-\mathbf{i})^n(-1)^j q^{1/2}(1 + o(1)).$$

Explicitly,

$$f_j = q^{1/2} \sum_{k=0}^{\infty} c_{jk}\left(\frac{p}{q^{1/2}}\right)^k \quad \text{on } [b, \infty),$$

where

$$c_{j0} = (-\mathbf{i})^n(-1)^j, \quad c_{j1} = \tfrac{1}{2}, \quad c_{j2} = 1/(8c_{j0}), \quad c_{j3} = 0$$

and

$$c_{jk} = \frac{-1}{2c_{j0}} \sum_{l=2}^{k-2} c_{jl}c_{j,k-l} \quad \text{for } k > 3.$$

The sum for f_j converges uniformly and absolutely on $[b, \infty)$.

COROLLARY 1. *Suppose that all the hypotheses of Theorem 1 are satisfied, except perhaps condition* (iii); *and*

(iii)′ $$\int_a^\infty \left|\left(\frac{p}{q^{1/2}}\right)^{l+1} q^{1/2}\right| dt < \infty$$

for some nonnegative integer l. Then the conclusion to the theorem remains valid if f_j is defined by

$$f_j = q^{1/2} \sum_{k=0}^{l} c_{jk}\left(\frac{p}{q^{1/2}}\right)^k \quad \text{on } [b, \infty).$$

THEOREM 2. *Suppose that each of p and q is a positive-valued function defined on $[a, \infty)$, p is continuously differentiable and q is twice continuously differentiable. If $m, n \in \{1, 2\}$, and if*

$$\text{(i)} \qquad \int_a^t p\,dt \to \infty \quad \text{as } t \to \infty,$$

$$\text{(ii)} \qquad \int_a^\infty \left| \left(\frac{q^{1/2}}{p} \right)' \right| dt < \infty,$$

$$\text{(iii)} \qquad \frac{q^{1/2}}{p}(t) \to 0 \quad \text{as } t \to \infty,$$

$$\text{(iv)} \qquad \int_a^\infty \left| \left(\frac{q'}{pq} \right)' \right| dt < \infty,$$

$$\text{(v)} \qquad \int_a^\infty \left(\frac{q'}{q} \right)^2 \frac{1}{p}\,dt < \infty,$$

$$\text{(vi)} \qquad \int_a^\infty \left| \frac{q'}{p^2} \right| dt < \infty,$$

then there exists a pair of linearly independent solutions (y_1, y_2) to

$$y'' - (-1)^m p y' - (-1)^n q y = 0 \quad \text{on } [a, \infty) \tag{3.2}$$

and a number $b \geqq a$ such that

$$y_1(t) = \exp\left\{ -\frac{(-1)^m}{2} \int_b^t (pf)\,dt \right\} (1 + o(1)),$$

$$y_1'(t) = q^{1/2}(t) \exp\left\{ -\frac{(-1)^m}{2} \int_b^t (pf)\,dt \right\} (o(1)),$$

$$y_2(t) = q^{-1/2}(t) \exp\left\{ \frac{(-1)^m}{2} \int_b^t p(2 + f)\,dt \right\} (o(1))$$

and

$$y_2'(t) = \exp\left\{ \frac{(-1)^m}{2} \int_b^t p(2 + f)\,dt \right\} (1 + o(1)),$$

where $f(t) \to 0$ as $t \to \infty$. Explicitly,

$$f = \sum_{k=1}^\infty c_k \left(\frac{q}{p^2} \right)^k \quad \text{on } [b, \infty),$$

where $c_1 = 2(-1)^n$ and $c_k = -\frac{1}{2}\sum_{j=1}^{k-1} c_j c_{k-j}$ for $k > 1$. The sum for f converges uniformly and absolutely on $[b, \infty)$.

COROLLARY 2. *Suppose that all the hypotheses of Theorem 2 are satisfied, except perhaps condition* (iii); *and*

$$\text{(iii)}' \qquad \int_a^\infty \left(\frac{q}{p^2} \right)^l p\,dt < \infty$$

for some positive integer l. Then the conclusion to the theorem remains valid if f is defined by

$$f = \sum_{k=1}^{l-1} c_k \left(\frac{q}{p^2}\right)^k$$

($f \equiv 0$ *if* (iii)′ *holds for* $l = 1$).

4. Proofs. Verification of the above results will be facilitated by the following lemma.

LEMMA. *Suppose that Y_1 is a fundamental matrix for the n-dimensional equation*

$$y'(t) = A(t)y(t) \quad \text{on } [a, \infty), \tag{4.1}$$

there exist numbers $a_1, \cdots, a_n$ with $a_i \geqq a$ for $1 \leqq i \leqq n$, functions ξ_i and *ζ_i defined on $[a_i, \infty)$ for $1 \leqq i \leqq n$ with each ξ_i locally integrable on $[a_i, \infty)$ and each ζ_i integrable on $[a_i, \infty)$, and there is a constant matrix L such that*

$$Y_1(t) \exp\left\{\operatorname{diag}\left[\int_{a_1}^t (\xi_1 + \zeta_1)\,dt, \cdots, \int_{a_n}^t (\xi_n + \zeta_n)\,dt\right]\right\} \to L \quad \text{as } t \to \infty.$$

Then there are a number $b \geqq a$ and a fundamental matrix Y_2 for (4.1) *such that*

$$Y_2(t) \exp\left\{\operatorname{diag}\left[\int_b^t (\xi_1)\,dt, \cdots, \int_b^t (\xi_n)\,dt\right]\right\} \to L \quad \text{as } t \to \infty.$$

Proof. Let $b = \max\{a_1, \cdots, a_n\}$ and let $Y_2 = Y_1 D$, where D is the constant nonsingular diagonal matrix whose jth diagonal entry is

$$\exp\left\{\int_{a_j}^\infty (\zeta_j)\,dt - \int_b^{a_j} (\xi_j)\,dt\right\}.$$

Then

$$\begin{aligned} Y_2(t) &\exp\left\{\operatorname{diag}\left[\int_b^t (\xi_1)\,dt, \cdots, \int_b^t (\xi_n)\,dt\right]\right\} \\ &= Y_1(t) \exp\left\{\operatorname{diag}\left[\int_{a_1}^t (\xi_1 + \zeta_1)\,dt, \cdots, \int_{a_n}^t (\xi_n + \zeta_n)\,dt\right]\right\} \\ &\quad \cdot \exp\left\{\operatorname{diag}\left[\int_t^\infty (\zeta_1)\,dt, \cdots, \int_t^\infty (\zeta_n)\,dt\right]\right\} \\ &\to LI = L \quad \text{as } t \to \infty \qquad (I \text{ is the } n \times n \text{ identity matrix}). \end{aligned}$$

Proof of Theorem 1. We begin by examining the behavior of certain functions which will arise when "changes of variable" are made in (3.1). Let

$$h(t) = \int_a^t q^{1/2}\,dt \quad \text{for } t \geqq a.$$

By [1, Lemma 6, p. 121] (with $\alpha = 0$) it follows that $h(t) \to \infty$ as $t \to \infty$, and since q is positive-valued we may let g be the function inverse to h so that $h(g(s)) = s$ for $s \geqq 0$. Let

$$\alpha(s) = [q'/q^{3/2}]\,(g(s)) \quad \text{and} \quad \gamma(s) = [p/q^{1/2}]\,(g(s))$$

for $s \geqq 0$. Each of α', α^2, γ' and $\alpha\gamma$ is in $\mathbf{L}(0, \infty)$ (absolutely integrable on $[0, \infty)$) if and only if each of $[q'/q^{3/2}]'$, $(q')^2/q^{5/2}$, $[p/q^{1/2}]'$ and $pq'/q^{3/2}$ is in $\mathbf{L}(a, \infty)$ respectively. Condition (i) of Theorem 1 and the Corollary on p. 594 of [6] imply that each of $[q'/q^{3/2}]'$ and $(q')^2/q^{5/2}$ is in $\mathbf{L}(a, \infty)$. Since $[p/q^{1/2}]' = p'/q^{1/2} - pq'/(2q^{3/2})$, it follows from (ii) and (iv) of Theorem 1 that $[p/q^{1/2}]'$ is in $\mathbf{L}(a, \infty)$. Thus

$$\alpha', \alpha^2, \gamma', \alpha\gamma \in \mathbf{L}(0, \infty). \tag{4.2}$$

In view of (iii) and the fact that each of α' and α^2 is in $\mathbf{L}(0, \infty)$, we have

$$\alpha(s) \to 0 \quad \text{and} \quad \gamma(s) \to 0 \quad \text{as } s \to \infty. \tag{4.3}$$

The conclusion to the theorem will be immediate once we have shown that the standard vector matrix formulation,

$$y' = \begin{bmatrix} 0 & 1 \\ (-1)^n q & p \end{bmatrix} y \quad \text{on } [a, \infty), \tag{4.4}$$

of (3.1) has a fundamental matrix Y_0 such that

$$Q(t)Y_0(t)E(t) \to L \quad \text{as } t \to \infty, \tag{4.5}$$

where

$$Q = \operatorname{diag}[q^{1/4}, q^{-1/4}],$$

$$E(t) = \operatorname{diag}\left[\exp\left\{-\int_b^t f_1\,dt\right\}, \exp\left\{-\int_b^t f_2\,dt\right\}\right],$$

for some $b \geqq a$, and

$$L = \begin{bmatrix} 1 & 1 \\ -(-\mathbf{i})^n & (-\mathbf{i})^n \end{bmatrix}.$$

For, we may then take y_1 to be the (1, 1) entry of Y_0 and y_2 to be the (1, 2) entry of Y_0.

To establish (4.5) we begin by letting Y be a fundamental matrix for (4.4) and defining Z by

$$Z(s) = Q(g(s))Y(g(s)) \quad \text{for } s \geqq 0.$$

(Recall $g = h^{-1}$ so $g'(s) = 1/h'(g(s)) = 1/q^{1/2}(s)$.) Computation shows that Z is a fundamental matrix for

$$z' = [A_0 + \gamma A_1 + \alpha A_2]z \quad \text{on } [0, \infty), \tag{4.6}$$

where

$$A_0 = \begin{bmatrix} 0 & 1 \\ (-1)^n & 0 \end{bmatrix},$$

$$A_1 = \operatorname{diag}[0, 1] \quad \text{and} \quad A_2 = \operatorname{diag}[\tfrac{1}{4}, -\tfrac{1}{4}].$$

(See [9, Lemma 1.3] for details.) The characteristic polynomial, $\det[A_0 + \gamma A_1 + \alpha A_2 - \lambda I]$ (I is the identity matrix), is

$$\mathscr{P}(\lambda, \gamma, \alpha) = \lambda^2 - \gamma\lambda + \tfrac{1}{4}\alpha\gamma - \tfrac{1}{16}\alpha^2 - (-1)^n.$$

In light of (4.3) we may let $\lambda_j(s)$ be the root of $\mathscr{P}(\lambda, \gamma(s), \alpha(s))$ such that

$$\lambda_j(s) \to (-\mathbf{i})^n(-1)^j \quad \text{as } s \to \infty$$

for $j = 1, 2$. From the limiting values of the λ_j it is clear that

$$\text{(4.7)} \qquad \operatorname{Re}(\lambda_j(s) - \lambda_k(s)) \quad \text{does not change sign}$$

for all large s whenever $1 \leqq j,\ k \leqq 2$ when $n = 2$, and since $\mathscr{P}(\lambda, \gamma(s), \alpha(s))$ is a polynomial with real coefficients we see that $\lambda_1(s) = \overline{\lambda_2(s)}$ for all large s when $n = 1$. Thus (4.7) holds in both cases. The eigenvalues of A_0 are $(-\mathbf{i})^n(-1)^j, j = 1, 2$, and corresponding eigenvectors are $\begin{bmatrix} 1 \\ (-\mathbf{i})^n(-1)^j \end{bmatrix}$, $j = 1, 2$. Examining (4.7), (4.3) and (4.2), we see that all the hypotheses of [2, Theorem 8.1, p. 92] or the theorem of [7] are satisfied. Hence there exist $s_1, s_2 \geqq 0$ and a fundamental matrix Z_1 for (4.6) such that

$$\text{(4.8)} \qquad Z_1(s)E_1(s) \to L \quad \text{as } s \to \infty,$$

where L is as in (4.5) and

$$E_1(s) = \operatorname{diag}\left[\exp\left\{-\int_{s_1}^{s} \lambda_1\, ds\right\}, \exp\left\{-\int_{s_2}^{s} \lambda_2\, ds\right\}\right].$$

Returning to the definition of $\mathscr{P}$ and using the quadratic formula we see that

$$\text{(4.9)} \qquad \lambda_j(s) = \tfrac{1}{2}[\gamma(s) + (-\mathbf{i})^n(-1)^j F(\gamma(s), \alpha(s))],$$

where

$$F(z, w) = [4 + (-1)^n z^2 - (-1)^n zw + ((-1)^n/4)w^2]^{1/2}.$$

($x^{1/2}$ denotes $\exp\{\frac{1}{2}\ln x\}$ for x complex and off the nonpositive real axis.) F is clearly an analytic function for $|z|$ and $|w|$ sufficiently small. Hence there exists a sequence $\{a_{lm}\}$ such that

$$F(z, w) = \sum_{l,m=0}^{\infty} a_{lm} z^l w^m$$

for $|z|$ and $|w|$ sufficiently small. Since $F(0, 0) = 2$ we see that $a_{00} = 2$; and since

$$\sum_{l,m=0}^{\infty} \sum_{\substack{u=0 \\ v=0}}^{\substack{u=l \\ v=m}} a_{uv} a_{l-u,m-v} z^l w^m = 4 + (-1)^n z^2 - (-1)^n zw + \frac{(-1)^n}{4} w^2,$$

it follows from "equating coefficients on like powers" that

$$a_{01} = 0 = a_{10}, \quad a_{20} = (-1)^n/4, \quad a_{30} = 0,$$

$$a_{l0} = \frac{-1}{4} \sum_{u=2}^{l-2} a_{u0} a_{l-u,0} \quad \text{for } l > 3.$$

Since the power series for F is absolutely convergent we have

$$F(z, w) = 2 + \frac{(-1)^n}{4} z^2 + \sum_{l=3}^{\infty} a_{l,0} z^l + w^2 \sum_{m=2}^{\infty} a_{0m} w^{m-2} + zw \sum_{l,m=1}^{\infty} a_{lm} z^{l-1} w^{m-1}$$

for $|z|$ and $|w|$ sufficiently small.

Returning to (4.9) and the definition of $\{c_{jk}\}$ in the conclusion to the theorem we see that

$$\lambda_j = \sum_{k=0}^{\infty} c_{jk}\gamma^k + \left[\alpha^2 \sum_{m=2}^{\infty} a_{0m}\alpha^{m-2}\right]\tfrac{1}{2}(-\mathbf{i})^n(-1)^j$$
$$+ \left[\alpha\gamma \sum_{l,m=1}^{\infty} a_{lm}\alpha^{m-1}\gamma^{l-1}\right]\tfrac{1}{2}(-\mathbf{i})^n(-1)^j$$

for all large s. Since each of the sums in brackets represents a bounded function for all large s (note (4.3)), we see from (4.2) that each of the terms in brackets represents a function in $\mathbf{L}(s_3, \infty)$ for some s_3 sufficiently large.

It now follows from (4.8) and the lemma that there are an s_4 and a fundamental matrix Z_2 for (4.6) such that

$$Z_2(s)E_2(s) \to L \quad \text{as } s \to \infty, \tag{4.10}$$

where

$$E_2(s) = \operatorname{diag}\left[\exp\left\{-\int_{s_4}^{s} \xi_1\, ds\right\}, \exp\left\{-\int_{s_4}^{s} \xi_2\, ds\right\}\right]$$

and $\xi_j = \sum_{k=0}^{\infty} c_{jk}\gamma^k$.

Since each of Z_2 and Z is a fundamental matrix for (4.6) there is a constant nonsingular matrix C such that $Z_2 = ZC$. Let Y_0 be YC; then from the definition of Z and (4.10) we have that Y_0 is a fundamental matrix for (4.4) such that

$$Q(g(s))Y_0(g(s))E_2(s) \to L \quad \text{as } s \to \infty.$$

Hence

$$Q(t)Y_0(t)E_2(h(t)) \to L \quad \text{as } t \to \infty.$$

By noting $\int_{s_4}^{h(t)} \xi_j\, ds = \int_{g(s_4)}^{t} f_j\, dt$, it follows (letting $b = g(s_4)$) that $E_2(h(t)) = E(t)$ and (4.5) has been established.

Proof of Corollary 1. Condition (iii)′ ensures that $\gamma^{l+1} \in \mathbf{L}(0, \infty)$, where γ is as in the proof of Theorem 1. This together with the fact that $\gamma' \in \mathbf{L}(0, \infty)$ (see (4.2)) implies $\gamma(s) \to 0$ as $s \to \infty$, and this implies that condition (iii) of Theorem 1 is satisfied. By noting that

$$q^{1/2} \sum_{k=0}^{\infty} c_{jk}\left(\frac{p}{q^{1/2}}\right)^k = q^{1/2} \sum_{k=0}^{l} c_{jk}\left(\frac{p}{q^{1/2}}\right)^k + \left[q^{1/2}\left(\frac{p}{q^{1/2}}\right)^{l+1} \sum_{k=l+1}^{\infty} c_{jk}\left(\frac{p}{q^{1/2}}\right)^{k-l-1}\right],$$

that the sum in brackets represents a bounded function, and that $q^{1/2}(p/q^{1/2})^{l+1} \in \mathbf{L}(a, \infty)$, it follows that the term in brackets represents a function in $\mathbf{L}(a, \infty)$, and the conclusion to the corollary follows immediately from the lemma.

Proof of Theorem 2. We begin by redefining some of the symbols used in the proof of Theorem 1. Let

$$h(t) = \int_a^t p\,dt \quad \text{for } t \geqq a,$$

and let g be the function inverse to h so that $h(g(s)) = s$ for $s \geqq 0$. Let

$$\alpha(s) = [q'/(pq)](g(s)) \quad \text{and} \quad \gamma(s) = [q^{1/2}/p](g(s))$$

for $s \geqq 0$. From conditions (ii), (iv) and (v) of the theorem it follows that

$$\gamma', \alpha', \alpha^2 \in \mathbf{L}(0, \infty). \tag{4.11}$$

From (4.11) and condition (iii) it follows that

$$\alpha(s) \to 0 \quad \text{and} \quad \gamma(s) \to 0 \quad \text{as } s \to \infty. \tag{4.12}$$

In order to establish the theorem we shall show that the standard vector matrix formulation of (3.2),

$$y' = \begin{bmatrix} 0 & 1 \\ (-1)^n q & (-1)^m p \end{bmatrix} y \quad \text{on } [a, \infty), \tag{4.13}$$

has a fundamental matrix Y_0 such that

$$Q(t) Y_0 Q^{-1}(t) E(t) \to I \quad \text{as } t \to \infty, \tag{4.14}$$

where

$$Q(t) = \operatorname{diag}[q^{1/4}, q^{-1/4}]$$

and

$$E(t) = \exp\left\{ -\frac{(-1)^m}{2} \operatorname{diag}\left[\int_b^t (-pf)\,dt, \int_b^t p(2 + f)\,dt \right] \right\}$$

for some $b \geqq a$ with f as in the conclusion to the theorem.

Let Y be a fundamental matrix for (4.13) and let Z be given by

$$Z(s) = Q(g(s)) Y(g(s)) \quad \text{for } s \geqq 0.$$

Computation shows that Z is a fundamental matrix for

$$z' = [A + V] z \quad \text{on } [0, \infty), \tag{4.15}$$

where

$$A = \begin{bmatrix} 0 & 0 \\ 0 & (-1)^m \end{bmatrix}$$

and

$$V = \gamma \begin{bmatrix} 0 & 1 \\ (-1)^n & 0 \end{bmatrix} + \alpha \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & -\frac{1}{4} \end{bmatrix}.$$

In view of (4.12) we may let $\lambda_1(s)$ and $\lambda_2(s)$ be the roots of det $[A + V(s) - \lambda I]$ such that $\lambda_1(s) \to 0$ and $\lambda_2(s) \to (-1)^m$ as $s \to \infty$. Clearly Re $(\lambda_j(s) - \lambda_k(s))$ does not change sign for all large s when $1 \leqq j, k \leqq 2$. This together with (4.11) and (4.12) enables us to conclude from [2, Theorem 8.1, p. 92] or [7] that there exist s_1, $s_2 \geqq 0$ and a fundamental matrix Z_1 for (4.15) such that

$$Z_1(s)E_1(s) \to I, \tag{4.16}$$

where $E_1(s) = \operatorname{diag}\,[\exp\{-\int_{s_1}^s \lambda_1\,ds\}, \exp\{-\int_{s_2}^s \lambda_2\,ds\}]$. From the quadratic formula it follows that

$$\lambda_j = ((-1)^m/2)[1 + (-1)^j F(\gamma^2, \alpha)],$$

where

$$F(z, w) = [1 + 4(-1)^n z - (-1)^m w + \tfrac{1}{4}w^2]^{1/2}.$$

Since F is analytic there exists a sequence $\{a_{kl}\}$ such that

$$F(z, w) = \sum_{k,l=0}^{\infty} a_{kl} z^k w^l$$

for $|z|$ and $|w|$ sufficiently small. Proceeding as in the proof of Theorem 1 we find that

$$a_{00} = 1, \quad a_{01} = -(-1)^m/2, \quad a_{10} = 2(-1)^n$$

and

$$a_{k0} = -\tfrac{1}{2}\sum_{u=1}^{k-1} a_{u0}a_{k-u,0} \qquad \text{for } k > 1,$$

Thus

$$\lambda_j = \frac{(-1)^m}{2}\{1 + (-1)^j[\xi + \zeta]\}$$

for all large s, where

$$\xi = 1 - \frac{(-1)^m}{2}\alpha + \sum_{k=1}^{\infty} c_k \gamma^{2k}$$

with $\{c_k\}$ as in the conclusion to the theorem and

$$\zeta = \alpha\gamma^2 \sum_{k,l=1}^{\infty} a_{kl}\gamma^{2k-2}\alpha^{l-1} + \alpha^2 \sum_{l=2}^{\infty} a_{0l}\alpha^{l-2}.$$

Condition (vi) of the theorem ensures that $\alpha\gamma^2 \in \mathbf{L}(0, \infty)$, and we have noted $\alpha^2 \in \mathbf{L}(0, \infty)$. So, since each of the last two sums represents a bounded function, it follows that $\zeta \in \mathbf{L}(s_3, \infty)$ for some s_3.

Applying the lemma we see that there exist a fundamental matrix Z_2 for (4.15) and an s_4 such that

$$Z_2(s)E_2(s) \to I \quad \text{as } s \to \infty,$$

where

$$E_2(s) = \exp\left\{\operatorname{diag}\left[\int_{s_4}^{s} \frac{-\alpha}{4}\,ds, \int_{s_4}^{s} \frac{\alpha}{4}\,ds\right]\right\}$$
$$\cdot \exp\left\{-\frac{(-1)^m}{2}\operatorname{diag}\left[\int_{s_4}^{s} (-\eta)\,ds, \int_{s_4}^{s} (2+\eta)\,ds\right]\right\}$$

and

$$\eta = \sum_{k=1}^{\infty} c_k \gamma^{2k}.$$

Noting that $Z_2(h(t))E_2(h(t)) \to I$ as $t \to \infty$, evaluating $E_2(h(t))$, and letting $b = g(s_4)$, we have

$$Z_2(h(t))DQ^{-1}(t)E(t) \to I,$$

where

$$D = \operatorname{diag}[q^{1/4}(b), q^{-1/4}(b)]$$

and E and Q are as in (4.14). Since each of Z_2D and Z is a fundamental matrix for (4.15) there is a constant nonsingular matrix C such that $Z_2D = ZC$. Letting Y_0 be YC we have

$$Q(t)Y_0(t)Q^{-1}(t)E(t) = Z_2(h(t))DQ^{-1}(t)E(t)$$
$$\to I \quad \text{as } t \to \infty,$$

and (4.14) is established.

Proof of Corollary 2. Condition (iii)′ implies that $\gamma^{2l} \in \mathbf{L}(0, \infty)$, where γ is as in the proof of Theorem 2. Since γ' is also in $\mathbf{L}(0, \infty)$, it follows that $\gamma(s) \to 0$ as $s \to \infty$; hence condition (iii) of Theorem 2 is satisfied. The conclusion to the corollary now follows from the lemma as did the conclusion to Corollary 1.

5. Remarks. We noted in §1 that (1.1) may be transformed into

$$(uy')' + vy = 0 \quad \text{on } [a, \infty) \tag{5.1}$$

by setting $u(t) = \exp\{\int_a^t p\,dt\}$ and $v = uq$. Under certain circumstances results such as [1, Theorem 13, p. 120] or the theorem of [6] may then be used to determine asymptotic behavior of the solutions of (5.1), hence of (1.1).

Another approach is to transform (1.1) into

$$z'' + fz = 0 \quad \text{on } [a, \infty), \tag{5.2}$$

where

$$f = q - \tfrac{1}{2}[p' + \tfrac{1}{2}p^2] \tag{5.3}$$

by letting $mz = y$, where $m(t) = \exp\{-\frac{1}{2}\int_a^t p\,dt\}$, and to apply one of the known results such as [1, Theorem 14, p. 122], [8, Theorems 3 and 4], or the main result of [10] to (5.2). This procedure (compared with the direct approach of using one of our theorems) has the disadvantage of requiring more stringent smoothness conditions on p and q in (1.1) and where it is applicable produces more complicated asymptotic formulas involving $f^{-1/4}$ and $\int_a^t f^{1/2}\,dt$ where f is given by (5.3).

We wish to point out a specific example where our Corollary 1 gives results for (1.1) but where no results are obtainable by applying previously known asymptotic formulas to (1.1) or to either of the transformed equations (5.1) and (5.2).

Let $p_0:[1,\infty) \to R$ be given by: $p_0(t) = 0$ for all $t \in [1,\infty)$ such that $t \notin [n - 1/(4n^3), n + 1/(4n^3)]$ for each positive integer n; on each interval $[n - 1/(4n^3), n]$ let p_0 be strictly increasing with $p_0(n - 1/(4n^3)) = 0$, $p_0(n) = 1/n^2$, p_0' continuous, and $p_0'(n - 1/(4n^3)) = 0 = p_0'(n)$; and on each interval $[n, n + 1/(4n^3)]$ let p_0 be strictly decreasing with $p_0(n) = 1/n^2$, $p_0(n + 1/(4n^3)) = 0$, p_0' continuous, and $p_0'(n) = 0 = p_0'(n + 1/(4n^3))$. Note p_0' is continuous on $[a,\infty)$,

$$\int_a^\infty |p_0|\,dt < \sum_{n=1}^\infty \frac{1}{2n^6} < \infty, \tag{5.4}$$

and

$$\int_a^\infty |p_0'|\,dt = 1 + 2\sum_{n=2}^\infty \frac{1}{n^2} < \infty. \tag{5.5}$$

Using the mean value theorem we see that

$$\begin{array}{l} \forall \text{ integer } n \geqq 2\ \exists\ \alpha_n \text{ and } \beta_n \text{ in} \\ [n - 1(4n^3), n + 1/(4n^3)] \cdot t \cdot\ p_0'(\alpha_n) = 4n \\ \text{and } p_0'(\beta_n) = -4n. \end{array} \tag{5.6}$$

It is easily verified that the hypotheses of Corollary 1 are satisfied for

$$y'(t) - (1 + p_0(t))y'(t) - ty'(t) = 0 \quad \text{on } [1,\infty) \tag{5.7}$$

with $l = 3$, and from Corollary 1, the technique indicated in the lemma, and the fact that $\int_1^t p_0^k\,dt$ has a limit as $t \to \infty$ for $k = 1, 2$, we conclude that (5.7) has a pair of linearly independent solutions y_1 and y_2 such that

$$y_j(t) = t^{-1/4}e^{f_j(t)}(1 + o(1)), \tag{5.8}$$

where $f_j(t) = -(-1)^j\frac{2}{3}t^{3/2} + \frac{1}{2}t - (-1)^j\frac{1}{4}t^{1/2}$.

Ghizzetti's theorem [1, Theorem 3, p. 92] clearly does not apply to (5.7). Transforming (5.7) into the form of (5.2) we have

$$z''(t) - \{t + \tfrac{1}{2}[\tfrac{1}{2}(1 + p_0)^2 - p_0']\}z(t) = 0. \tag{5.9}$$

Ghizzetti's theorem does not apply to (5.9), for if it did it would predict a solution of the form

$$z(t) = t(1 + o(1)),$$

hence a solution of the form

$$y(t) = te^{-t/2}(1 + o(1))$$

for (5.7) (again we use the fact that $\int_a^t p_0\,dt$ has a limit as $t \to \infty$) which is not consistent with (5.8). Since $(1 + p_0)^2(t) \to 1$ as $t \to \infty$ we see from (5.6) that the function in braces in (5.9) is oscillatory; its value at β_n will be positive and its value at α_n will be negative for all sufficiently large n. Hence none of the results such as [8, Theorems 3 and 4], [1, Theorem 14, p. 122], and the main result of [10] will apply to (5.9).

Transforming (5.7) into the form of (5.1) we have

$$(uy')' - tuy = 0, \tag{5.10}$$

where $u(t) = \exp\{-\int_1^t (1 + p_0)\,dt\}$. Neither [1, Theorem 13, p. 120] nor the theorem of [6] is applicable to (5.10), for if one were it would predict (again using the technique of the lemma) two linearly independent solutions y_1 and y_2 of (5.10), hence of (5.7), such that

$$y_j(t) = t^{-1/4}(1 + o(1)) \exp\{(-1)^j \tfrac{2}{3} t^{3/2} + \tfrac{1}{2}t\}.$$

This is clearly not consistent with (5.8).

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SIAM J. MATH. ANAL.
Vol. 2, No. 3, August 1971

ORTHOGONAL EXPANSIONS WITH POSITIVE COEFFICIENTS. II*

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Abstract. A sufficient condition is given for writing a set of orthogonal polynomials as a linear combination of a second set of orthogonal polynomials with nonnegative coefficients. Some inequalities for Pollaczek polynomials and associated ultraspherical polynomials follow from this result.

1. Introduction. Recently a number of problems have reduced to the problem of expanding one set of orthogonal polynomials in a second set of orthogonal polynomials and proving that the coefficients are nonnegative. These problems have ranged from differential geometry [2] to numerical analysis [10], and this property has also been useful in harmonic analysis [6] and in the investigation of an interesting new set of discrete orthogonal polynomials [12]. Thus it seems worthwhile to try to find some general theorems of this type. Wilson [11] has one theorem, but in many cases it is impossible to verify his conditions and many classical results do not follow from his theorem. One other small result and an interesting conjecture are given in [1]. These two results are theorems with assumptions on the weight functions. We shall give a theorem of a different type with the assumptions on the coefficients in the recurrence formulas. Only a few of the many results for the classical polynomials (see [5]) are contained in this theorem, but there are some corollaries which have not been obtained by any other method.

2. Positive coefficient expansion theorems. Any set of orthogonal polynomials $\{p_n(x)\}$ satisfies

$$xp_n(x) = p_{n+1}(x) + \alpha_n p_n(x) + \beta_n p_{n-1}(x), \qquad n = 0, 1, \cdots, \tag{1}$$

where $p_{-1}(x) = 0$, $p_0(x) = 1$, α_{n-1} real, $\beta_n > 0$, $n = 1, 2, \cdots$, and the polynomials are normalized by

$$p_n(x) = x^n + \cdots .$$

Conversely by a theorem of Favard [9] if $p_n(x)$ satisfies (1) with α_{n-1} real, $\beta_n > 0$, $n = 1, 2, \cdots$, $p_{-1}(x) = 0$, $p_0(x) = 1$, then the $p_n(x)$ are orthogonal with respect to a nonnegative measure $d\alpha(x)$; i.e.,

$$\int_{-\infty}^{\infty} p_n(x)p_m(x)\, d\alpha(x) = 0, \qquad n \neq m.$$

In general the measure $d\alpha(x)$ cannot be constructed (in fact it is not always unique) and it often is impossible to even give the set which supports this measure. The most complete survey of what has been said is in [8].

* Received by the editors March 2, 1970, and in final revised form November 30, 1970.

† Department of Mathematics, University of Wisconsin, Madison, Wisconsin 53706. This work was supported in part by a fellowship from the John Simon Guggenheim Foundation, in part by the Office of Naval Research under Contract N 00014-67-A-0128-0012 and in part by the Mathematisch Centrum, Amsterdam.

We are given two sets of orthogonal polynomials, $p_n(x)$ and $q_n(x)$, with $p_n(x)$ defined by (1) and $q_n(x)$ by

$$xq_n(x) = q_{n+1}(x) + \gamma_n q_n(x) + \delta_n q_{n-1}(x), \qquad n = 0, 1, \cdots, \tag{2}$$

where again $q_{-1}(x) = 0$, $q_0(x) = 1$, γ_{n-1} real, $\delta_n > 0$, $n = 1, 2, \cdots$.

THEOREM 1. *Let $p_n(x)$ and $q_n(x)$ be defined by* (1) *and* (2) *and set*

$$q_n(x) = \sum_{k=0}^{n} a(k, n)p_k(x).$$

Then $a(k, n) \geqq 0$ if

$$\alpha_k \geqq \gamma_n, \qquad k = 0, 1, \cdots, n, \quad n = 0, 1, \cdots, \tag{3}$$

$$\beta_k \geqq \delta_n, \qquad k = 0, 1, \cdots, n, \quad n = 0, 1, \cdots. \tag{4}$$

Proof. We have $q_{n+1}(x) = \sum_{k=0}^{n+1} a(k, n+1)p_k(x)$. From (1) and (2),

$$\begin{aligned}
q_{n+1}(x) &= xq_n(x) - \gamma_n q_n(x) - \delta_n q_{n-1}(x) \\
&= x\sum_{k=0}^{n} a(k, n)p_k(x) - \gamma_n \sum_{k=0}^{n} a(k, n)p_k(x) - \delta_n \sum_{k=0}^{n-1} a(k, n-1)p_k(x) \\
&= \sum_{k=0}^{n} a(k, n)[p_{k+1}(x) + \alpha_k p_k(x) + \beta_k p_{k-1}(x)] - \gamma_n \sum_{k=0}^{n} a(k, n)p_k(x) \\
&\quad - \delta_n \sum_{k=0}^{n-1} a(k, n-1)p_k(x) \\
&= p_{n+1}(x) + [a(n-1, n) + (\alpha_n - \gamma_n)]p_n(x) \\
&\quad + \sum_{k=1}^{n-1} [a(k-1, n) + (\alpha_k - \gamma_n)a(k, n) + \beta_{k+1}a(k+1, n) \\
&\quad - \delta_n a(k, n-1)]p_k(x) + [(\alpha_0 - \gamma_n)a(0, n) + \beta_1 a(1, n) - \delta_n a(0, n-1)].
\end{aligned}$$

Thus

$$a(n+1, n+1) = 1, \tag{5}$$

$$a(n, n+1) = a(n-1, n) + (\alpha_n - \gamma_n), \tag{6}$$

$$\begin{aligned}
a(k, n+1) &= a(k-1, n) + (\alpha_k - \gamma_n)a(k, n) + (\beta_{k+1} - \delta_n)a(k+1, n) \\
&\quad + \delta_n[a(k+1, n) - a(k, n-1)], \qquad k = 1, \cdots, n-1,
\end{aligned} \tag{7}$$

$$a(0, n+1) = (\alpha_0 - \gamma_n)a(0, n) + (\beta_1 - \delta_n)a(1, n) + \delta_n[a(1, n) - a(0, n-1)]. \tag{8}$$

If we adopt the convention that $a(n+1, n) = a(-1, n) = 0$, then (6) and (8) are just (7) for $k = n$ and $k = 0$.

We shall show that $a(k, n) \geqq 0$ by an induction on n. Assume that $a(k, m) \geqq 0$ has been proved for $k \leqq m$, $m \leqq n$, and consider $a(k, n+1)$. If $k = n+1$, then $a(n+1, n+1) = 1 > 0$. If $k = n$, then $a(n, n+1) = a(n-1, n) + (\alpha_n - \gamma_n)$ and $\alpha_n \geqq \gamma_n$ by hypothesis and $a(n-1, n) \geqq 0$ by the inductive assumption. Thus $a(n, n+1) \geqq 0$. If $k \leqq n-1$, then

$$
\begin{aligned}
a(k, n+1) = a(k-1, n) &+ (\alpha_k - \gamma_n)a(k, n) + (\beta_{k+1} - \delta_n)a(k+1, n) \\
&+ \delta_n[a(k+1, n) - a(k, n-1)].
\end{aligned} \tag{9}
$$

Each of the terms on the right-hand side is nonnegative except possibly the last term $a(k+1, n) - a(k, n-1)$. Equation (9) gives

$$
\begin{aligned}
&a(k, n+1) - a(k-1, n) \\
&\quad \geqq \delta_n[a(k+1, n) - a(k, n-1)] \\
&\quad \geqq \delta_n\delta_{n-1}[a(k+2, n-1) - a(k+1, n-2)] \\
&\quad \geqq \delta_n\delta_{n-1} \cdots \delta_{n-j}[a(k+j+1, n-j) - a(k+j, n-j-1)].
\end{aligned}
$$

Choosing $j = [(n-k-1)/2]$ we either have $k+j+1 = n-j$ or $k+j+2 = n-j$. In the first case $a(k+j+1, n-j) - a(k+j, n-j-1) = 1 - 1 = 0$, and in the second case using (6) we have

$$
\begin{aligned}
&a(k+j+1, n-j) - a(k+j, n-j-1) \\
&\quad = a\left(\frac{n+k}{2}, \frac{n+k}{2} + 1\right) - a\left(\frac{n+k}{2} - 1, \frac{n+k}{2}\right) \\
&\quad = \alpha_{(n+k)/2} - \gamma_{(n+k)/2} \geqq 0,
\end{aligned}
$$

and so

$$
a(k, n+1) - a(k-1, n) \geqq \delta_n[a(k+1, n) - a(k, n-1)] \geqq 0.
$$

This completes the induction and so completes the proof of Theorem 1.

One possible generalization is to assume only $\alpha_n \geqq \gamma_n$, $\beta_n \geqq \delta_n$. The following example shows that this does not work. Let $\alpha_n = \gamma_n = 0$, $\beta_1 = 2$, $\delta_1 = 1$, $\beta_n = \delta_n = n+1$, $n = 2, 3, \cdots$. Then $\beta_n \geqq \delta_n$, $\alpha_n \geqq \gamma_n$ and

$$
q_4(x) = p_4(x) + p_2(x) - 2p_0(x).
$$

This proof is similar to the proof [3] which gives

$$
p_n(x)p_m(x) = \sum a(k, m, n)p_k(x), \qquad a(k, m, n) \geqq 0, \tag{10}
$$

under certain conditions on the coefficients in (1). Equation (10) is actually a corollary of a maximum principle for difference equations [4] and it is natural to try to find the maximum principle connected with Theorem 1. Let Δ_n and ∇_n be defined by

$$\Delta_n k(n) = k(n+1) + \alpha_n k(n) + \beta_n k(n-1), \quad n = 0, 1, \cdots, \quad k(-1) = \beta_0 = 0, \tag{11}$$
$$\beta_n > 0, \quad n = 1, 2, \cdots,$$

$$\nabla_n k(n) = k(n+1) + \gamma_n k(n) + \delta_n k(n-1), \quad n = 0, 1, \cdots, \quad k(-1) = \delta_0 = 0, \tag{12}$$
$$\delta_n > 0, \quad n = 1, 2, \cdots.$$

THEOREM 2. (i) *$a(m, n)$ satisfies*

$$\nabla_n a(m, n) = \Delta_m a(m, n). \tag{13}$$

(ii) *Relations* (3) *and* (4) *hold, i.e.,* $\alpha_m \geqq \gamma_n, \beta_m \geqq \delta_n$, $m = 0, 1, \cdots, n$, $n = 0, 1, \cdots$.

(iii)

$$a(0, 0) \geqq 0, \tag{14}$$

$$a(n+1, n) = 0, \quad a(n+2, n) = 0, \quad n = 0, 1, 2, \cdots. \tag{15}$$

Then

$$a(m, n) \geqq 0, \qquad m = 0, 1, \cdots, n, \quad n = 0, 1, \cdots. \tag{16}$$

Proof. Equation (13) is

$$a(m, n+1) + \gamma_n a(m, n) + \delta_n a(m, n-1)$$
$$= a(m+1, n) + \alpha_m a(m, n) + \beta_m a(m-1, n).$$

Letting $m = n + 1$ and using (15), we find that

$$a(n+1, n+1) = \beta_{n+1} a(n, n) = \cdots$$
$$= \beta_{n+1}\beta_n \cdots \beta_1 a(0, 0) \geqq 0.$$

Assume that $a(m, k) \geqq 0$ for $m \leqq k \leqq n$ has been proved and consider $a(m, n+1)$. We have just shown that $a(n+1, n+1) \geqq 0$. For $m \leqq n$ use (13) to obtain

$$\begin{aligned} a(m, n+1) &= \beta_m a(m-1, n) + (\alpha_m - \gamma_n) a(m, n) + a(m+1, n) - \delta_n a(m, n-1) \\ &= \beta_m a(m-1, n) + (\alpha_m - \gamma_n) a(m, n) + [\beta_{m+1} - \delta_n] a(m, n-1) \\ &\quad + [a(m+1, n) - \beta_{m+1} a(m, n-1)]. \end{aligned} \tag{17}$$

The first two terms are nonnegative and the third is also, for $\beta_{m+1} \geqq \delta_n$ if $m + 1 \leqq n$ and $a(m, n-1) = 0$ if $m + 1 > n$. From (17),

$$a(m, n+1) - \beta_m a(m-1, n) \geqq a(m+1, n) - \beta_{m+1} a(m, n-1),$$

and continuing in this fashion we have

$$a(m, n+1) - \beta_m a(m-1, n) \geqq a(m+1, n) - \beta_{m+1} a(m, n-1)$$
$$\geqq a(m+j, n+1-j) - \beta_{m+j} a(m+j-1, n-j).$$

This last term is zero for $m + j > n + 1 - j$ or $2j > n + 1 - m$.

Theorem 1 is an easy corollary of Theorem 2. The boundary conditions are satisfied since the expansion of a polynomial of degree n in a series of polynomials has zero coefficients for all terms that involve polynomials of degree k for $k > n$. Also the difference equation (13) is satisfied if we define

$$a(m, n) = \int_E q_n(x)p_m(x)\, d\alpha(x).$$

The expansion is then

$$q_n(x) = \sum_{m=0}^{n} \frac{a(m, n)p_m(x)}{\int_E p_m^2(x)\, d\alpha(x)},$$

where $p_n(x)$ are orthogonal on a set E with respect to the measure $d\alpha(x)$.

3. Applications. Theorem 1 has a few applications which we will now examine. Consider first some of the Pollaczek polynomials. Let $R_n^\lambda(x, a)$ satisfy

$$xR_n^\lambda(x, a) = R_{n+1}^\lambda(x, a) + \frac{n(n + 2\lambda - 1)}{4(n + \lambda + a)(n + \lambda + a - 1)} R_{n-1}^\lambda(x, a),$$

$R_0^\lambda(x, a) = 1$, $R_1^\lambda(x, a) = x$. For $a = 0$, $R_n^\lambda(x, a)$ reduces to the ultraspherical polynomials. In [4] it was shown that

$$|R_n^\lambda(x, a)| \leqq R_n^\lambda(1, a), \qquad -1 \leqq x \leqq 1,$$

for $a \geqq 0$, $a \geqq (\lambda - \lambda^2)/(1 + \lambda)$, $\lambda > 0$. For $0 < \lambda < 1$ we shall now remove the restriction $a \geqq (\lambda - \lambda^2)/(1 + \lambda)$.

For $a = 0$, $0 < \lambda < 1$,

$$\beta_n = \frac{n(n + 2\lambda - 1)}{4(n + \lambda)(n + \lambda - 1)}$$

is a decreasing sequence. Thus

$$\beta_1 \geqq \cdots \geqq \beta_n \geqq \delta_n = \frac{n(n + 2\lambda - 1)}{4(n + \lambda + a)(n + \lambda + a - 1)}$$

for $a > 0$. From Theorem 1 we have

$$R_n^\lambda(x, a) = \sum_{k=0}^{n} a(k, n)R_k^\lambda(x, 0),$$

with $a(k, n) \geqq 0$ for $0 < \lambda < 1$, $a > 0$. Since $|R_k^\lambda(x, 0)| \leqq |R_k^\lambda(1, 0)| = R_k^\lambda(1, 0)$, this implies

$$|R_n^\lambda(x, a)| \leqq \sum_{k=0}^{n} a(k, n)R_k^\lambda(1, 0) = R_n^\lambda(1, a).$$

Another interesting application is to the associated polynomials. If $p_n(x)$ satisfies

$$xp_n(x) = p_{n+1}(x) + \alpha_n p_n(x) + \beta_n p_{n-1}(x), \tag{18}$$

the associated polynomial $p_n(x, \mu)$ is defined by

(19) $$xp_n(x,\mu) = p_{n+1}(x,\mu) + \alpha_{n+\mu}p_n(x,\mu) + \beta_{n+\mu}p_{n-1}(x,\mu),$$

$p_{-1}(x, \mu) = 0$, $p_0(x, \mu) = 1$. For $\mu = 1$ we may use $p_n(x)$ and $p_{n-1}(x, 1)$ to obtain the general solution of the second order difference equation (18). For general polynomials we are restricted to considering $\mu = 1, 2, \cdots$, but for some of the classical polynomials (19) defines $p_n(x, \mu)$ for $\mu > 0$ and even at times for some $\mu < 0$. In particular, for the Legendre polynomials ($\alpha_n = 0$, $\beta_n = n^2/(4n^2 - 1)$ in (18)) Barrucand and Dickinson [7] explicitly computed the coefficients in

$$p_n(x,\mu) = \sum_{k=0}^{n} a(k,n)p_k(x).$$

Their expression for $a(k, n)$ is complicated (the product of about 20 gamma functions) but a moment's reflection shows that the coefficients are nonnegative for $\mu > 0$.

Using Theorem 1 we can prove the following. Let $C_n^\nu(x, \mu)$, the associated ultraspherical polynomials, be defined by

$$xC_n^\nu(x,\mu) = C_{n+1}^\nu(x,\mu) + \frac{(n+2\nu+\mu-1)(n+\mu)}{4(n+\nu+\mu)(n+\nu+\mu-1)}C_{n-1}^\nu(x,\mu),$$

i.e., let $\alpha_n = 0$ and $\beta_n = (n + 2\nu - 1)n/[4(n + \nu)(n + \nu - 1)]$ in (18), and define $C_n^\nu(x, \mu)$ by (19). Then

(20) $$C_n^\nu(x,\mu) = \sum_{k=0}^{n} a(k,n)C_k^1(x,\mu)$$

with $a(k,n) \geqq 0$ for $\nu > 1$, $\mu > -1$. Since $C_k^1(x,\mu) = C_k^1(x,0) = C_k^1(x)$ and $|C_k^1(x)| \leqq C_k^1(1)$ as above, we have

$$|C_n^\nu(x,\mu)| \leqq C_n^\nu(1,\mu), \qquad -1 \leqq x \leqq 1, \quad \nu \geqq 1, \quad \mu > -1.$$

Letting $\mu \to \infty$ in $\beta_{n+\mu}$ we obtain $\lim_{\mu\to\infty} \beta_{n+\mu} = \frac{1}{4}$ and so

$$C_n^\nu(x,\infty) = C_n^1(x,\mu).$$

Thus (20) is also

(21) $$C_n^\nu(x,\mu) = \sum_{k=0}^{n} a(k,n)C_k^\nu(x,\infty), \qquad a(k,n) \geqq 0 \quad \text{for } \nu > 1.$$

For $0 < \nu < 1$ there is a more general result of this type:

$$C_n^\nu(x,\mu) = \sum_{k=0}^{n} a(k,n)C_k^\nu(x,\lambda)$$

with $a(k,n) \geqq 0$ for $0 < \nu < 1$, $\mu \geqq \lambda \geqq 0$. This follows immediately from Theorem 1. Letting $\lambda = 0$ and using

$$|C_n^\nu(x)| \leqq C_n^\nu(1), \qquad \nu > 0,$$

we obtain

$$|C_n^\nu(x,\mu)| \leqq C_n^\nu(1,\mu), \qquad -1 \leqq x \leqq 1, \quad 0 < \nu < 1, \quad \mu \geqq 0.$$

There is probably a more general result than (21) with $C_n^\nu(x, \lambda)$, $\lambda > \mu$, on the right-hand side. Thus, there should be a generalization of Theorem 1. One other useful result for the classical polynomials which does not follow from Theorem 1 is

$$L_n^\beta(x) = \sum_{k=0}^{n} L_{n-k}^{\beta-\alpha-1}(0) L_k^\alpha(x),$$

where $L_{n-k}^{\beta-\alpha-1}(0) > 0$ if $\beta > \alpha$. A general theorem which implied this result would be very interesting.

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SIAM J. MATH. ANAL.
Vol. 2, No. 3, August 1971

NEW PROOF OF THE ADDITION THEOREM FOR GEGENBAUER POLYNOMIALS*

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Abstract. The quantity $(\lambda - z)^\rho$ is expanded in Jacobi polynomials $P_n^{(\alpha,\beta)}(z)$, where α, β, and ρ are unrelated. The known case $\alpha = \beta = -\rho - 1$ is then used in a short proof of the addition theorem for Gegenbauer polynomials. The only other ingredients of the proof are well-known generating relations for these polynomials.

1. Introduction. Proofs of the addition theorem for Gegenbauer polynomials C_n^ν of general order ν have been given by Gegenbauer [7], Henrici [8], and Manocha [10]. The cases in which 2ν is a positive integer are treated in [6, vol. 2, p. 244] and [16] as part of the theory of spherical harmonics in $2\nu + 2$ dimensions. Related addition theorems for Legendre functions and Gegenbauer functions of the first and second kinds are discussed by Robin [13, Chap. 7] and Henrici [8]. Gegenbauer (or ultraspherical) polynomials are special Jacobi polynomials, and an addition theorem for general Jacobi polynomials is not known. The possibility of such a theorem is one reason for seeking new proofs in the Gegenbauer case, another reason being the desire for a proof that is simple, transparent, and reasonably elementary.[1]

The first part of this note is concerned with a new relation (2.4) which has some interest by itself, apart from its use in proving an addition theorem. It is the expansion of $(\lambda - z)^\rho$ in Jacobi polynomials $P_n^{(\alpha,\beta)}(z)$, where α, β, and ρ are unrelated. In the literature of Jacobi polynomials one finds the limiting case $\lambda = 1$ (misprinted in some standard reference books), the case $\rho = -1$ [14, p. 251], and the case $\alpha = \beta$ [8, (107)]. The case $\alpha = \beta$ and ρ a positive integer is given in [16, p. 487]. The simplest generating relation for the Gegenbauer polynomials is the case $\alpha = \beta = -\rho - \frac{1}{2}$, and for the addition theorem we shall want $\alpha = \beta = -\rho - 1$.

2. An expansion in Jacobi polynomials. Let λ and ρ be (possibly complex) constants, and assume that λ is not a real number in the closed interval $[-1, 1]$. Let D denote the interior of the ellipse in the complex plane which passes through the point λ and has foci at -1 and 1. By requiring $\arg(\lambda - z)$ to be continuous on D and to coincide with a fixed value of $\arg \lambda$ at $z = 0$, we determine a single-valued branch of $(\lambda - z)^\rho$ which we denote by $f(z)$, $z \in D$. Because f is holomorphic on D, it can be expanded [14, p. 245] in a series of Jacobi polynomials,

$$f(z) = \sum_{n=0}^{\infty} A_n P_n^{(\alpha,\beta)}(z), \qquad z \in D, \tag{2.1}$$

with any real α, β such that $\alpha > -1$ and $\beta > -1$.

* Received by the editors September 22, 1970, and in revised form December 2, 1970.

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[1] See note added in proof.

Leaving for a moment the special choice of f and taking (2.1) to be the Jacobi series of any function holomorphic on D, we use the orthogonality of the Jacobi polynomials to represent A_n by an integral, which is then transformed by substituting Rodrigues' formula and integrating by parts n times. The result of this familiar procedure can be written in the unfamiliar form

$$\begin{aligned} f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} F^{(n)}(1 + \alpha + n, 1 + \beta + n; -1, 1) \\ \cdot R_n(-\alpha-n, -\beta-n; z + 1, z - 1). \end{aligned} \tag{2.2}$$

The structure of (2.2) shows the kinship between Jacobi series and Taylor series, both of which are special cases of [3, (3.16a)]. The R-polynomial is a Jacobi polynomial normalized so that the coefficient of z^n is unity:

$$P_n^{(\alpha,\beta)}(z) = \frac{(1 + \alpha + \beta + n)_n}{2^n n!} R_n(-\alpha - n, -\beta - n; z + 1, z - 1), \tag{2.3a}$$

where we define $(a)_0 = 1$ and $(a)_n = a(a + 1) \cdots (a + n - 1)$. The coefficient $F^{(n)}$ is a weighted average of $f^{(n)} = d^n f/dz^n$ over a line segment:

$$\begin{aligned} F^{(n)}(b, b'; x, y) &= \int_0^1 f^{(n)}[ux + (1 - u)y] \, d\mu(u), \\ d\mu(u) &= \left[\int_0^1 u^{b-1}(1 - u)^{b'-1} \, du\right]^{-1} u^{b-1}(1 - u)^{b'-1} \, du, \end{aligned} \tag{2.3b}$$

where b and b' are assumed to have positive real parts.

In the special case at hand, we substitute

$$f^{(n)}(z) = (-\rho)_n(\lambda - z)^{\rho-n}$$

and find

$$F^{(n)}(b, b'; x, y) = (-\rho)_n R_{\rho-n}(b, b'; \lambda - x, \lambda - y),$$

where

$$R_t(b, b'; x, y) = \int_0^1 [ux + (1 - u)y]^t \, d\mu(u),$$

$$\operatorname{Re} b > 0, \quad \operatorname{Re} b' > 0.$$

The R-function [2], [3], [4] is connected with standard notation by

$$R_t(b_x, b_y; x, y) = y^t \, {}_2F_1(-t, b_x; b_x + b_y; 1 - x/y),$$

both sides being symmetric in x and y. (The R-polynomial in (2.3a) is a special case of R_t but does not have the preceding integral representation because its parameters are negative.)

Specializing f and $F^{(n)}$ in (2.2), we find

$$\begin{aligned} (\lambda - z)^\rho = \sum_{n=0}^{\infty} \frac{(-\rho)_n}{n!} R_{\rho-n}(1 + \alpha + n, 1 + \beta + n; \lambda + 1, \lambda - 1) \\ \cdot R_n(-\alpha - n, -\beta - n; z + 1, z - 1), \end{aligned} \tag{2.4}$$

where $\alpha > -1$, $\beta > -1$, and $z \in D$. If $\rho = 0, 1, 2, \cdots$, the series terminates and (2.4) is valid for all finite λ and z.

In view of (2.1) and (2.3) we have

$$\begin{aligned} A_n &= \frac{2^n(-\rho)_n}{(1+\alpha+\beta+n)_n} R_{\rho-n}(1+\alpha+n, 1+\beta+n; \lambda+1, \lambda-1) \\ &= \frac{2^n(-\rho)_n}{(1+\alpha+\beta+n)_n}(\lambda+1)^{\rho-n} \\ &\quad \cdot {}_2F_1\left(n-\rho, 1+\beta+n; 2+\alpha+\beta+2n; \frac{2}{\lambda+1}\right). \end{aligned} \tag{2.5}$$

Although we have excluded the limiting case $\lambda = 1$ in which the ellipse degenerates to the line segment with endpoints -1 and 1, the coefficient A_n must still be given by (2.5) if it exists. The hypergeometric function now has unit argument, and we find

$$A_n = 2^\rho \Gamma(1+\alpha+\rho)(-\rho)_n \frac{(1+\alpha+\beta+2n)\Gamma(1+\alpha+\beta+n)}{\Gamma(1+\alpha+n)\Gamma(2+\alpha+\beta+\rho+n)}, \quad \lambda = 1, \tag{2.6}$$

if $\alpha > -\operatorname{Re}\rho - 1$. (Questions of convergence of the Jacobi series are more difficult if $\lambda = 1$ and lead to the further condition $\alpha > -2\operatorname{Re}\rho - \frac{3}{2}$.) This result is given correctly by Tricomi [15, p. 250] but is reproduced with the factor $(1+\alpha+\beta+2n)$ misprinted as $\Gamma(1+\alpha+\beta+2n)$ in [6, (10.20(3))] and [9, p. 217].

Putting $\lambda = A/B$ and $\alpha = \beta = \nu - \frac{1}{2}$, where $\nu > -\frac{1}{2}$, we find

$$\begin{aligned} (A - Bz)^\rho &= \sum_{n=0}^{\infty} \frac{(-\rho)_n}{(\nu)_n}\left(\frac{B}{2}\right)^n \\ &\quad \cdot R_{\rho-n}\left(\frac{1}{2}+\nu+n, \frac{1}{2}+\nu+n; A+B, A-B\right) C_n^\nu(z). \end{aligned} \tag{2.7}$$

We use here the customary notation C_n^ν for the Gegenbauer polynomial,

$$\begin{aligned} C_n^\nu(\cos\theta) &= \frac{2^n(\nu)_n}{n!} R_n\left(\frac{1}{2}-\nu-n, \frac{1}{2}-\nu-n; \cos\theta+1, \cos\theta-1\right) \\ &= \frac{(2\nu)_n}{n!} R_n(\nu, \nu; e^{i\theta}, e^{-i\theta}). \end{aligned} \tag{2.8}$$

The last expression comes from comparing the generating relation

$$(1 - 2t\cos\theta + t^2)^{-\nu} = \sum_{n=0}^{\infty} t^n C_n^\nu(\cos\theta), \qquad |te^{i\theta}| < 1, \quad |te^{-i\theta}| < 1, \tag{2.9}$$

with the generating relation of the R-polynomials [3, (3.11)]

$$(1 - tx)^{-b}(1 - ty)^{-b'} = \sum_{n=0}^{\infty} t^n \frac{(b+b')_n}{n!} R_n(b, b'; x, y), \qquad |tx| < 1, \quad |ty| < 1. \tag{2.10}$$

If $A = r^2 + r'^2$, $B = 2rr'$, and $z = \cos\theta = \mathbf{r}\cdot\mathbf{r}'/(rr')$, (2.7) is an expansion of $|\mathbf{r} - \mathbf{r}'|^{2\rho}$. If $\rho = -\nu$, (2.7) can be shown equivalent to (2.9) by using a quadratic transformation of the R-function. Putting $\rho = -\nu - \frac{1}{2}$ and then replacing ν by $\nu - \frac{1}{2}$, we get the case needed later:

$$\begin{aligned}(A - Bz)^{-\nu} = \sum_{n=0}^{\infty} \frac{(\nu)_n}{(\nu - \frac{1}{2})_n}\left(\frac{B}{2}\right)^n \\ \cdot R_{-\nu-n}(\nu + n, \nu + n; A + B, A - B)C_n^{\nu - 1/2}(z),\end{aligned} \tag{2.11}$$

where $\nu > 0$.

3. The addition theorem. We cite first a result more general than the one we shall use. A very simple proof [5] shows that

$$\begin{aligned}&\sum_{n=0}^{\infty} \frac{(a + a')_n}{n!} R_n(a, a'; x, y)R_n(b, b'; z, w) \\ &= (1 - yz)^{a-b}(1 - yw)^{a-b'}R_{-a}[b, b'; (1 - xz)(1 - yw), (1 - xw)(1 - yz)],\end{aligned} \tag{3.1}$$

where xz, xw, yz, yw all lie inside the unit circle and $a + a' = b + b' \neq 0, -1, -2, \cdots$. This is the case $k = 2$ of [5, (2.4)]. Putting $a = a' = b = b' = \nu$, $x = te^{i\theta}$, $y = te^{-i\theta}$, $z = e^{i\varphi}$, $w = e^{-i\varphi}$, we have

$$\begin{aligned}&\sum_{n=0}^{\infty} \frac{t^n n!}{(2\nu)_n} C_n^\nu(\cos\theta)C_n^\nu(\cos\varphi) \\ &= R_{-\nu}[\nu, \nu; 1 - 2t\cos(\theta + \varphi) + t^2, 1 - 2t\cos(\theta - \varphi) + t^2].\end{aligned} \tag{3.2}$$

This generating relation was written explicitly for Gegenbauer polynomials by Ossicini [12], although (3.1) had been found earlier by Meixner [11], [6, (2.5(12))]. A different and still earlier generalization of (3.2) is due to Bailey [17, p. 102], [18, (2.1)].[2] Another proof of Ossicini's formula has been given by Carlitz [1].

Returning to (2.11) we define

$$x = \cos\theta\cos\varphi + \sin\theta\sin\varphi\cos\psi, \tag{3.3}$$

and we put $z = \cos\psi$ and $A - Bz = 1 - 2tx + t^2$, whence $A \pm B = 1 - 2t\cos(\theta \pm \varphi) + t^2$. For any complex θ and φ, $|A/B|$ can be made arbitrarily large by choosing $|t|$ sufficiently small, and the ellipse passing through A/B can thus be made to encircle $\cos\psi$ for any given complex ψ. Hence we find, for $\nu > 0$,

$$\begin{aligned}(1 - 2tx + t^2)^{-\nu} = \sum_{m=0}^{\infty} \frac{(\nu)_m}{(\nu - \frac{1}{2})_m}(t\sin\theta\sin\varphi)^m C_m^{\nu - 1/2}(\cos\psi) \\ \cdot R_{-\nu-m}[\nu + m, \nu + m; 1 - 2t\cos(\theta + \varphi) \\ + t^2, 1 - 2t\cos(\theta - \varphi) + t^2].\end{aligned} \tag{3.4}$$

Insertion of Ossicini's formula (3.2) gives

$$\begin{aligned}(1 - 2tx + t^2)^{-\nu} = \sum_{m=0}^{\infty}\sum_{s=0}^{\infty} t^{m+s} \frac{(\nu)_m s!}{(\nu - \frac{1}{2})_m(2\nu + 2m)_s} \sin^m\theta\sin^m\varphi \\ \cdot C_s^{\nu+m}(\cos\theta)\, C_s^{\nu+m}(\cos\varphi)\, C_m^{\nu - 1/2}(\cos\psi).\end{aligned} \tag{3.5}$$

[2] For these two references I thank an associate editor.

Comparing coefficients of t^n with the help of (2.9), we have the addition theorem,

$$\begin{aligned} C_n^\nu(x) = \sum_{m=0}^{n} \frac{(\nu)_m (n-m)!}{(\nu - \frac{1}{2})_m (2\nu + 2m)_{n-m}} \sin^m \theta \, C_{n-m}^{\nu+m}(\cos\theta) \sin^m \varphi \\ \cdot C_{n-m}^{\nu+m}(\cos\varphi)\, C_m^{\nu-1/2}(\cos\psi). \end{aligned} \tag{3.6}$$

The restriction $\nu > 0$ can be dropped by analytic continuation because the left-hand side is an entire function of ν and the right-hand side is analytic in ν except for singularities at $2\nu = 1, 0, -1, \cdots, -2n + 2$. These singularities must be removable by continuity, and (3.6) is then valid for all complex ν, θ, φ and ψ.

Note added in proof. R. A. Askey informs me that T. Koornwinder has recently found an addition theorem for general Jacobi polynomials.

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SIAM J. MATH. ANAL.
Vol. 2, No. 3, August 1971

AN EXTENSION OF A CLASS OF POLYNOMIALS. II*

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Abstract. In this paper we study the algebraic structure of the class of polynomials $\{u_n!H_n(x)\}$ in x, where $\{H_n(x)\}$ satisfies the functional equation $D_uH_n(x) = H_{n-1}(x)$ for $n = 0, 1, 2, \cdots$, and where D_u is a general operator, linear and distributive, which transforms a polynomial of degree n in x into one of degree $n - 1$; in particular, $D_ux^n = u_nx^{n-1}$ where (u) is a given sequence of real or complex numbers subject to the restrictions $u_0 = 0$, $u_1 = 1$, $u_n \neq 0$ for $n \geqq 1$. Some of the algebraic properties of this class of polynomials are then used to study an important particular example.

1. Introduction. In this paper we continue the study of the Appell set of polynomials to the base (u), that is, the class of polynomials $\{H_n(x)\}$ in x which satisfies the functional equation (see [5])

$$D_uH_n(x) = H_{n-1}(x), \qquad n = 0, 1, 2, \cdots, \tag{1.1}$$

where D_u is a general operator, linear and distributive, which transforms a polynomial of degree n in x into one of degree $n - 1$. In particular, $D_ux^n = u_nx^{n-1}$, where (u) is a given sequence of real or complex numbers subject to the restrictions that $u_0 = 0$, $u_1 = 1$ and $u_n \neq 0$ for $n \geqq 1$ (see Ward [17]). Incidentally, we know that (see [5] with Δu_n instead of u^n) $D_u \equiv D_q$ if and only if the sequence $\{\Delta u_n\}$ obeys the law $\Delta u_n \cdot \Delta u_m = \Delta u_{n+m}$ for $n, m = 0, 1, 2, \cdots$, where $u_n \equiv u_{n+1} - u_n$ and the q-difference operator of Jackson is given by

$$D_qf(x) = \frac{f(qx) - f(x)}{(q-1)x} \quad \text{or} \quad D_q \equiv \frac{q^{x(d/dx)} - 1}{(q-1)x}; \tag{1.2}$$

the q-difference operator D_q itself tends to the ordinary differential operator $D \equiv d/dx$, as $q \to 1$.

References to the literature on the subject of generalized binomial coefficients similar to those studied by Ward [17] are given at the end of this paper; the chronological ordering is [7], [13], [10], [11], [12], [5], [16], [3], [8], [9]. It is interesting to note that recently Chihara [6] studied the orthogonality of polynomials with Brenke-type generating functions, of which ours is a special case; more recently, Ismail [9] generalized the work of Sheffer [15] on polynomial sets of type zero by replacing the ordinary differential operator D by Ward's operator D_u.

In the present paper we study the algebraic structure of the class of polynomials $\{u_n!H_n(x)\}$ in x, where $\{H_n(x)\}$ is an Appell set to the base (u); that is, it satisfies the functional equation (1.1). Some of these algebraic properties are then used to study an important example.

* Received by the editors January 8, 1970, and in final revised form January 11, 1971.

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2. Algebraic structure. Let $P \equiv P_n(x)$, $n = 0, 1, 2, \cdots$, be a simple set of polynomials whose nth member $P_n(x)$ is of proper degree n. Appell [2] in 1880 defined (see also [1] and [14]) on the set $\mathscr{I}$ of all polynomial sets two operations $+$ and $\divideontimes$ as follows:

(i) $$(P + Q)_n = P_n(x) + Q_n(x),$$

provided that $p(n, n) + q(n, n) \neq 0$, where

$$P_n(x) = \sum_{k=0}^{n} p(n, k)x^k \quad \text{and} \quad Q_n(x) = \sum_{k=0}^{n} q(n, k)x^k;$$

(ii) $$(P \divideontimes Q)_n = \sum_{k=0}^{n} p(n, k)Q_k(x).$$

He also defined

(iii) $$(\alpha P)_n = \alpha P_n(x),$$

where α is a real or a complex number.

We observe that $(\alpha P \divideontimes Q) = (P \divideontimes \alpha Q) = \alpha(P \divideontimes Q)$ and that the operation $+$ is obviously commutative but $\divideontimes$ is not commutative. It is interesting to note that one commutative subclass is the set $\mathscr{A}$ of all polynomials $\{n!P_n(x)\}$, where $P_n(x)$ satisfies the Appell property

$$\frac{d}{dx}P_n(x) = P_{n-1}(x), \qquad n = 0, 1, 2, \cdots. \tag{2.1}$$

Our Appell set of polynomials to the base (u) is an extremely general class. Let us denote by $\mathscr{A}(u)$ the class of all polynomial sets $\{u_n!G_n(x)\}$, where $G_n(x)$ is an Appell polynomial set to the base (u). In $\mathscr{A}(u)$ the identity element, with respect to $\divideontimes$, is the set $I = \{x^n\}$ and 1 is the determining function of I; (see [5, Theorem II]). We give below some easily proved properties of the class of polynomials $\mathscr{A}(u)$ and also show that the system $\{\mathscr{A}(u), \divideontimes\}$ is a commutative group.

Let $P, Q, R \in \mathscr{A}(u)$ and have $A(t), B(t), C(t)$, respectively, as their determining functions. Then:

(a) $P + Q \in \mathscr{A}(u)$ if $A(0) + B(0) \neq 0$ and has the determining function $A(t) + B(t)$;

(b) $P + (Q + R) = (P + Q) + R = P + Q + R$;

(c) $P \divideontimes Q = Q \divideontimes P$ both belong to $\mathscr{A}(u)$ and have the determining function $A(t)B(t)$;

(d) $P \divideontimes (Q \divideontimes R) = (P \divideontimes Q) \divideontimes R$ and both belong to $\mathscr{A}(u)$.

The important property (c) can be proved quite easily if we use Theorems I and II of Chak [5].

It is obvious that for every $P \in \mathscr{A}(u)$ there exists a set $Q \in \mathscr{A}(u)$ such that $P \divideontimes Q = Q \divideontimes P = I$. Indeed $B(t) = (A(t))^{-1}$. We may denote the element Q by P^{-1} and define $P^0 = I$, $P^n = P \divideontimes (P^{n-1})$, where n is a nonnegative integer, and $P^{-n} = P^{-1} \divideontimes (P^{-n+1})$. Since the system $\{\mathscr{A}(u), \divideontimes\}$ is a commutative group, we observe in passing that if $P \divideontimes Q = R$ and any two of the elements belong to $\mathscr{A}(u)$, then the third also belongs to $\mathscr{A}(u)$.

Sheffer [14] has shown that the system $(\mathscr{I}, \divideontimes)$ is a noncommutative group. We have seen that $\{\mathscr{A}(u), \divideontimes\}$ is a commutative subgroup. It can further be shown that if $P \in \mathscr{A}(u)$ and $Q \in \mathscr{I}$ and if $P \divideontimes Q = Q \divideontimes P$, then $Q \in \mathscr{A}(u)$.

As a simple example of a polynomial $H_n(x)$ belonging to the class $\mathscr{A}(u)$ let us define

$$H_n(x, u) \equiv H_n(x) = \sum_{r=0}^{n} [n, r] x^r. \tag{2.2}$$

This was cited in the first paper of this series of papers but in a slightly different form (see [5]).

It is easy to see that the set of polynomials $H_n(x, u)$ is generated by

$$e_u(t) e_u(xt) = \sum_{n=0}^{\infty} H_n(x, u) \frac{t^n}{u_n!}, \tag{2.3}$$

where $e_u(x)$ is the u-basic exponential function given by

$$e_u(x) = \sum_{n=0}^{\infty} \frac{x^n}{u_n!} \quad \text{and} \quad [n, k] = \frac{u_n u_{n-1} \cdots u_{n-k+1}}{u_k u_{k-1} \cdots u_1}$$

for n, k positive integers and $n \geqq k$; $[n, 0] = 1$; $u_n! = u_n u_{n-1} \cdots u_1$ if $n > 0$; $u_n! = 1$ if $n = 0$; also $[n, k] = 0$ if $n < k$ or if $k < 0$.

Let

$$\frac{e_u(xt)}{e_u(t)} = \sum_{n=0}^{\infty} A_n(x, u) \frac{t^n}{u_n!}. \tag{2.4}$$

As an application of property (c) of our class of polynomials $\mathscr{A}(u)$ let us try to find the set of polynomials $A_n(x, u) \in \mathscr{A}(u)$ and generated by the inverse of $e_u(t)$. For this purpose let us define another function $E_u(x)$ by the relationship $e_u(x) E_u(-x) = 1$; if $E_u(x) = \sum_{n=0}^{\infty} \beta_n x^n$, then it is easy to see that

$$\beta_n = \beta_{n-1} - \frac{1}{u_2!} \beta_{n-2} + \cdots + (-1)^{n+1} \frac{1}{u_n!} \beta_0, \qquad \beta_0 = 1. \tag{2.5}$$

It is now easy to obtain the following expression for $A_n(x, u)$:

$$A_n(x, u) = \begin{cases} \displaystyle\prod_{r=0}^{n-1} (x - \Delta u_r) & \text{for } n \geqq 1, \\ 1 & \text{for } n = 0. \end{cases} \tag{2.6}$$

In the notation of the first paper of this series [5],

$$A_n(x, u) = \sum_{r=0}^{n} (-1)^r {}^n S_r x^{n-r} = \sum_{r=0}^{n} (-1)^r \frac{u_n!}{u_{n-r}!} \alpha_{n,r} x^{n-r}; \tag{2.7}$$

where ${}^n S_r(u) \equiv {}^n S_r$ means the sum of all the combinations of the products of $u_1 - u_0, u_2 - u_1, \cdots, u_n - u_{n-1}$ taken r at a time; also

$${}^n S_r(u) = \frac{u_n!}{u_{n-r}!} \alpha_{n,r}.$$

We now give two interesting expressions for x^n:

$$x^n = \sum_{r=0}^{n} (-1)^r \frac{u_n!}{u_{n-r}!} \alpha_{n,r} H_{n-r}(x, u); \tag{2.8}$$

$$x^n = \sum_{r=0}^{n} [n, r] A_r(x, u). \tag{2.9}$$

The last two relations are the u-analogues of those given by Carlitz [4] and Al-Salam [1] for the case in which $\Delta u_n \cdot \Delta u_m = \Delta u_{n+m}$. More generally, if C is an Appell set of polynomials to the base (u) and C^{-1} is its inverse and we write

$$(C^{-1})_n = a_0 \frac{x^n}{u_n!} + a_1 \frac{x^{n-1}}{u_{n-1}!} + \cdots + a_n, \tag{2.10}$$

then

$$\frac{x^n}{u_n!} = a_0 C_n(x) + a_1 C_{n-1}(x) + \cdots + a_n C_0(x). \tag{2.11}$$

Acknowledgment. The authors wish to express their thanks to the referee for many valuable suggestions.

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SIAM J. MATH. ANAL.
Vol. 2, No. 3, August 1971

ON THE ASYMPTOTIC BEHAVIOR OF THE SOLUTION OF A NONLINEAR INTEGRODIFFERENTIAL EQUATION*

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Abstract. A theorem concerning the asymptotic behavior of the solution of a nonlinear Volterra integrodifferential equation is proved.

1. Introduction. We consider the equations

$$x'(t) = \int_0^t b(t-\tau)g(x(\tau))\,d\tau + [g(x(t))]^{-1}y(t) + f(t), \tag{1.1a}$$

$$y'(t) = \alpha g(x(t)) - \beta y(t), \tag{1.1b}$$

where $0 \leqq t < \infty$, $|x(0)| < \infty$, $y(0) > 0$, $\alpha > 0$, $\beta > 0$, and prove the following theorem.

THEOREM. *Let*

$$g(x) \in C^1(-\infty, \infty), \quad g(x) > 0, \quad g'(x) \geqq 0, \quad |x| < \infty, \tag{1.2}$$

$$\lim_{x \to -\infty} g(x) = 0, \qquad \lim_{x \to \infty} g(x) = \infty, \tag{1.3}$$

$$g'(x) \leqq (1/\gamma)g(x), \quad |x| < \infty, \quad \textit{for some } \gamma, \quad 0 < \gamma < \infty, \tag{1.4}$$

$$f(t) \in C[0, \infty), \quad \sup_{0 \leqq t < \infty} |f(t)| < \infty, \tag{1.5}$$

$$\int_0^\infty |f(\tau) - F|\,d\tau < \infty \quad \textit{for some } F \geqq 0, \tag{1.6}$$

$$b(t) \in C^2[0, \infty) \cap L_1[0, \infty), \tag{1.7}$$

$$b(t) \leqq 0, \quad b'(t) \leqq -\rho b''(t), \quad 0 \leqq t < \infty, \quad b(0) < 0, \tag{1.8}$$

where

$$\rho = \begin{cases} \min(1/\beta, \gamma/F) & \textit{if } F > 0, \\ 1/\beta & \textit{if } F = 0. \end{cases}$$

Let $x(t), y(t)$ *be a solution of* (1.1) *on* $0 \leqq t < \infty$. *Then* $\lim_{t \to \infty} g(x(t))$, $\lim_{t \to \infty} y(t)$ *exist and satisfy*

$$\lim_{t \to \infty} g(x(t)) = \frac{\beta}{\alpha} \lim_{t \to \infty} y(t) = \left[\frac{\alpha}{\beta} + F\right]\left[\int_0^\infty |b(\tau)|\,d\tau\right]^{-1}. \tag{1.9}$$

While the existence of a solution $x(t), y(t)$ on $0 \leqq t < \infty$ is assumed, we note that the present hypothesis may also be used to obtain the a priori bounds necessary for an existence proof. Also observe that the assumptions above (in particular the first part of (1.2)) are sufficient to guarantee uniqueness of the solution. Clearly, by (1.1b) and the second part of (1.2), $y(t) > 0$, $0 \leqq t < \infty$.

* Received by the editors May 26, 1970, and in revised form November 27, 1970.

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A considerable literature [2], [3], [6], [12] concerning the asymptotic behavior of the solutions of

$$x'(t) = \int_0^t b(t - \tau)g(x(\tau))\,d\tau + f(t), \qquad 0 \leqq t < \infty, \tag{1.10}$$

exists under the hypothesis $xg(x) > 0$, $x \neq 0$. In [3], $(-1)^k b^{(k)}(t) \leqq 0$ ($k = 0, 1, 2, 3$; $0 < t < \infty$) is assumed, and in [6], $b(t)$ is taken completely monotonic on $0 < t < \infty$ (i.e., $(-1)^k b^{(k)}(t) \leqq 0$, $k = 0, 1, 2, \cdots$; $0 < t < \infty$). In [3] and [6], where $f(t) \equiv 0$, the result (obtained with the aid of certain Lyapunov functions) is that if $x(t)$ is a solution of (1.10) on $0 \leqq t < \infty$, then $\lim_{t \to \infty} x^{(k)}(t) = 0$, $k = 0, 1, 2$. In [2] a Popov-type condition is imposed on the kernel, and, in addition, $b(t)$, $\int_t^\infty b(\tau)\,d\tau \in L_1[0, \infty) \cap L_2[0, \infty)$; $f(t), f'(t) \in L_1[0, \infty)$ is assumed. The conclusion is that if $x(t)$ is a solution of (1.10) on $0 \leqq t < \infty$, then $\lim_{t \to \infty} x(t) = 0$. In [4] an integrated version of (1.9) is considered. If $b(t) \geqq 0$, $g(x) = x$, then (1.10) becomes an equation of renewal type [1, Chap. 7].

Under the hypothesis $g(x) \geqq -\lambda$, $|x| < \infty$, $\lambda < \infty$, (1.10) has earlier been studied in [5], where results concerning the existence of bounded solutions on $0 \leqq t < \infty$ under various assumptions on $b(t)$ were obtained. Under the same hypothesis on $g(x)$ (and making, in addition, certain growth assumptions), (1.1) has also been investigated [7]–[11], where different boundedness theorems were proved. (Note that $y(t)$ was taken more generally than here in [7]–[9].)

In the present paper a result concerning the asymptotic behavior of the solutions of (1.1) is obtained. This result—stronger than mere boundedness—obviously requires more restrictive hypotheses to be imposed on $b(t)$, (1.8). Note that the sign hypothesis $b(t) \leqq 0$, $0 \leqq t < \infty$, made in Theorems 3–8 in [7, Chap. 1], is in general not alone sufficient to guarantee that $\lim_{t \to \infty} x(t)$ exists (not even $b(t) < 0$, $0 \leqq t < \infty$, suffices) as it is not difficult to construct numerical counterexamples. As to the nonlinear function $g(x)$ we remark that (1.2) and (1.3) concern its monotonicity and (1.4) is a pointwise imposed growth condition.

We finally observe that (1.1)—with $g(x) = \exp\{x\}$—occurs in certain problems in nonlinear nuclear reactor dynamics if the delayed neutrons are taken into account; see [7, p. 11], for details.

2. Proof. Notice at first that we have the relation, $0 \leqq t < \infty$,

$$\begin{aligned} -\tfrac{1}{2}\int_0^t \int_0^\tau b'(\tau - s)[g(x(s)) - g(x(\tau))]^2\,ds\,d\tau + \tfrac{1}{2}\int_0^t b(t - \tau)g^2(x(\tau))\,d\tau \\ +\tfrac{1}{2}\int_0^t b(\tau)g^2(x(\tau))\,d\tau - b(0)\int_0^t g^2(x(\tau))\,d\tau \\ -\int_0^t g(x(\tau))\int_0^\tau b'(\tau - s)g(x(s))\,ds\,d\tau = 0, \end{aligned} \tag{2.1}$$

which can be verified by expanding $[g(x(s)) - g(x(\tau))]^2$ and performing an interchange of the order of integration.

We begin by examining the last term on the left side in (2.1),

$$-\int_0^t g(x(\tau))\int_0^\tau b'(\tau - s)g(x(s))\,ds\,d\tau, \tag{2.2}$$

and consider the case $F > 0$. By (1.1a),

$$\begin{aligned} g(x(t)) = F^{-1}[x'(t)g(x(t)) - y(t) - g(x(t))\int_0^t b(t-\tau)g(x(\tau))\,d\tau \\ - g(x(t))[f(t) - F]], \qquad 0 \leqq t < \infty. \end{aligned} \tag{2.3}$$

Substituting $g(x(t))$ from (2.3) into (2.2) one has

$$\begin{aligned} (2.2) = -F^{-1}\int_0^t \Big\{[x'(\tau)g(x(\tau)) - y(\tau) - g(x(\tau))\int_0^\tau b(\tau - s)g(x(s))\,ds \\ - g(x(\tau))[f(\tau) - F]]\int_0^\tau b'(\tau - s)g(x(s))\,ds\Big\}\,d\tau. \end{aligned} \tag{2.4}$$

Define, for some x_0, $|x_0| < \infty$,

$$G(x) = \int_{x_0}^x g(u)\,du, \qquad |x| < \infty. \tag{2.5}$$

We investigate the terms on the right side of (2.4) separately. Performing an integration by parts in the first term yields, by (1.7),

$$\begin{aligned} -F^{-1}\int_0^t x'(\tau)g(x(\tau))\int_0^\tau b'(\tau - s)g(x(s))\,ds\,d\tau \\ = -F^{-1}G(x(t))\int_0^t b'(t-\tau)g(x(\tau))\,d\tau \\ + F^{-1}\int_0^t\int_0^\tau b''(\tau - s)g(x(s))G(x(\tau))\,ds\,d\tau \\ + b'(0)F^{-1}\int_0^t g(x(\tau))G(x(\tau))\,d\tau \\ + F^{-1}\int_0^t\int_0^\tau b''(\tau - s)g(x(\tau))G(x(\tau))\,ds\,d\tau \\ - F^{-1}\int_0^t\int_0^\tau b''(\tau - s)g(x(\tau))G(x(\tau))\,ds\,d\tau \\ = -F^{-1}G(x(t))\int_0^t b'(t-\tau)g(x(\tau))\,d\tau \\ + F^{-1}\int_0^t\int_0^\tau b''(\tau - s)[g(x(s))G(x(\tau)) - g(x(\tau))G(x(\tau))]\,ds\,d\tau \\ + F^{-1}\int_0^t b'(\tau)g(x(\tau))G(x(\tau))\,d\tau. \end{aligned} \tag{2.6}$$

We need the following two lemmas.

LEMMA 1. *Let the hypothesis of the theorem hold. Then* $\sup_{0 \leqq t < \infty} |x(t)| < \infty$, $\sup_{0 \leqq t < \infty} y(t) < \infty$.

Proof. Choose $\varepsilon > 0$ sufficiently small. Solving (1.1b) for $y(t)$ gives

$$y(t) = y(0) \exp\{-\beta t\} + \int_0^t \exp\{-\beta[t-\tau]\}\alpha g(x(\tau))\,d\tau, \qquad 0 \leqq t < \infty, \tag{2.7}$$

and, if $\bar{t}$ is such that $g(x(t)) \leqq [1+\varepsilon]g(x(\bar{t}))$, $0 \leqq t \leqq \bar{t} < \infty$,

$$y(\bar{t}) \leqq K_1 g(x(\bar{t})) + K_2, \tag{2.8}$$

for some constants $K_1, K_2 < \infty$. Thus, after multiplying (1.1a) by $g(x(t))$, using (1.5) and (2.8), one obtains

$$\frac{d}{dt}G(x(\bar{t})) \leqq g(x(\bar{t})) \int_0^{\bar{t}} b(\bar{t}-\tau)g(x(\tau))\,d\tau + K_3 g(x(\bar{t})) + K_2 \tag{2.9}$$

$$\leqq K_3 g(x(\bar{t})) + K_2, \tag{2.10}$$

where the final step follows from (1.2) and (1.8).

Suppose $\sup_{0 \leqq t < \infty} x(t) = \infty$. Then, by (1.3), $\lim_{t\to\infty} \sup g(x(t)) = \infty$, and, from (1.2), $\lim_{t\to\infty} \sup G(x(t)) = \infty$. Thus there exists $\{t_n\}$ such that $G(x(t)) \leqq G(x(t_n))$, $0 \leqq t \leqq t_n$, $(d/dt)G(x(t_n)) \geqq 0$, $\lim_{n\to\infty} G(x(t_n)) = \infty$, $\lim_{n\to\infty} t_n = \infty$. By (2.9) we have that there exists a constant $K_4 < \infty$ such that

$$\int_0^{t_n} b(t_n - \tau)g(x(\tau))\,d\tau \geqq -K_4. \tag{2.11}$$

Let

$$\bar{t}_n = \max\{t \mid t < t_n, g(x(t)) = [1+\varepsilon]^{-1} g(x(t_n))\}.$$

Then, after integrating (1.4),

$$\begin{aligned} -K_4 &\leqq \int_0^{t_n} b(t_n-\tau)g(x(\tau))\,d\tau \leqq \int_{\bar{t}_n}^{t_n} b(t_n-\tau)g(x(\tau))\,d\tau \\ &\leqq \int_{\bar{t}_n}^{t_n} b(t_n-\tau)g(x(t_n))\,d\tau - \frac{1}{\gamma}\int_{\bar{t}_n}^{t_n} b(t_n-\tau)\int_{x(\tau)}^{x(t_n)} g(u)\,du\,d\tau \\ &\leqq g(x(t_n))\int_{\bar{t}_n}^{t_n} b(t_n-\tau)\,d\tau - \frac{1}{\gamma}g(x(t_n))\int_{\bar{t}_n}^{t_n} b(t_n-\tau)[x(t_n)-x(\tau)]\,d\tau. \end{aligned}$$

Also, from (1.4),

$$\tfrac{1}{2}\varepsilon g(x(t_n)) \leqq g(x(t_n)) - g(x(\bar{t}_n)) \leqq (1/\gamma)g(x(t_n))[x(t_n) - x(\bar{t}_n)],$$

and $x(t_n) - x(\bar{t}_n) \geqq \frac{1}{2}\gamma\varepsilon$. But, by (1.1a) and (2.8),

$$x'(t) \leqq K_5, \qquad \bar{t}_n \leqq t \leqq t_n. \tag{2.12}$$

Thus $t_n - \bar{t}_n$ is bounded away from zero and, as we may of course assume $b(t) < 0$, $0 \leqq t \leqq t_n - \bar{t}_n$, and also, by (2.12), that $x(t_n) - x(\tau)$ is sufficiently small, $\bar{t}_n \leqq \tau \leqq t_n$, we have

$$\lim_{n\to\infty} \int_{\bar{t}_n}^{t_n} b(t_n - \tau)g(x(\tau))\,d\tau = -\infty,$$

which violates (2.11). $\text{Sup}_{0\leqq t<\infty} x(t) < \infty$ follows. By (2.7), $\sup_{0\leqq t<\infty} y(t) < \infty$.

Rather obvious modifications of the final part of Theorem 3 [7, Chap. 1] then show that the condition

$$\text{(2.13)} \qquad \left[\lim_{x\to-\infty} \sup g(x)\right] \int_0^\infty |b(\tau)|\,d\tau - \liminf_{t\to\infty} \left[f(t) + \int_0^t \alpha\, d(\tau)\,d\tau\right] < 0$$

—which there, together with $\sup_{0\leqq t<\infty} x(t) < \infty$ and the eventual monotonicity of $g(x)$ (decreasing) as $x \to -\infty$, was shown to imply $\sup_{0\leqq t<\infty} -x(t) < \infty$—may, under the assumption

$$\int_0^\infty |f(\tau) - F|\,d\tau < \infty,$$

be weakened to read

$$\left[\lim_{x\to-\infty} \sup g(x)\right] \int_0^\infty |b(\tau)|\,d\tau - F - \alpha \int_0^\infty d(\tau)\,d\tau < 0.$$

As we now have $F \geqq 0$; $\alpha d(t) = \alpha e^{-t} > 0$; $\lim_{x\to-\infty} g(x) = 0$; $g'(x) \geqq 0$, $|x| < \infty$, and $b(t) \in L_1[0, \infty)$; then $\sup_{0\leqq t<\infty} |x(t)| < \infty$ follows. The lemma is proved.

LEMMA 2. *Let $h_i(x) \in C(-\infty, \infty)$, $|x| < \infty$, and let $h_i(x)$ be monotone non-decreasing for $|x| < \infty$; $i = 1, 2$. Also let, for some $\eta > 0$,*

$$\text{(2.14)} \qquad h_1(x_2) - h_1(x_1) \leqq \frac{1}{\eta}[h_2(x_2) - h_2(x_1)], \qquad x_1 \leqq x_2, \quad |x_1|, |x_2| < \infty;$$

and let

$$\text{(2.15)} \qquad z(t) \in C[0, \infty), \qquad \sup_{0\leqq t<\infty} |z(t)| < \infty,$$

$$\text{(2.16)} \qquad a(t), ta(t) \in L_1[0, \infty) \cap C[0, \infty), \quad a(t) \geqq 0, \quad 0 \leqq t < \infty.$$

Then, if $h_i(z(t)) \geqq 0$, $0 \leqq t < \infty$, one has

$$\int_0^t \int_0^\tau a(\tau - s)[h_1(z(\tau))h_2(z(\tau)) - h_1(z(s))h_2(z(\tau))]\,ds\,d\tau$$
$$> \eta \int_0^t \int_0^\tau a(\tau - s)[h_1^2(z(\tau)) - h_1(z(s))h_1(z(\tau))]\,ds\,d\tau - K_1, \qquad 0 \leqq t < \infty,$$

for some constant $K_1 < \infty$, independent of t.

Proof.

$$\int_0^t\int_0^\tau a(\tau - s)[h_1(z(\tau))h_2(z(\tau)) - h_1(z(s))h_2(z(\tau))]\,ds\,d\tau$$

$$= \int_0^t a(s)\left\{\int_0^t [h_1(\tau)h_2(\tau) - \bar{h}_1(\tau, s)h_2(\tau)]\,d\tau\right\} ds$$

$$- \int_0^t a(s)\left\{\int_0^s [h_1(\tau)h_2(\tau) - \bar{h}_1(\tau, s)h_2(\tau)]\,d\tau\right\} ds,$$

where we write $h_i(\tau) = h_i(z(\tau))$ and define

$$\bar{h}_1(\tau, s) = \begin{cases} h_1(z(\tau - s)), & s \leqq \tau \leqq t, \\ h_1(z(t - \tau)), & 0 \leqq \tau < s. \end{cases} \tag{2.17}$$

We assert that for any s, $0 \leqq s \leqq t$,

$$\int_0^t [h_1(\tau)h_2(\tau) - \bar{h}_1(\tau, s)h_2(\tau)]\,d\tau \geqq \eta \int_0^t [h_1^2(\tau) - \bar{h}_1(\tau, s)h_1(\tau)]\,d\tau.$$

To realize that the assertion holds we make the following observation. Suppose we have $2n$ nonnegative real numbers, x_i, y_i, $i = 1, \cdots, n$, such that $x_1 \leqq x_2 \leqq \cdots \leqq x_n$; $y_1 \leqq y_2 \leqq \cdots \leqq y_n$, and $x_i - x_j \leqq (1/\eta)[y_i - y_j]$, for any i, j such that $i \geqq j$. Then

$$\sum_{i=1}^n [x_i y_i - x_{q_i} y_i] \geqq \eta \sum_{i=1}^n [x_i^2 - x_i x_{q_i}], \tag{2.18}$$

where $q_1, q_2, \cdots, q_n$ represents an arbitrary ordering of the integers $1, \cdots, n$.

To see that (2.18) is true one only notices that any ordering $q_1, q_2, \cdots, q_n$ of the integers $1, \cdots, n$ may be obtained by a certain number of interchanges of the following type:

$$(r_1, r_2, \cdots, r_n) \to (s_1, s_2, \cdots, s_n),$$

where, for some i_0,

$$r_i = s_i, \qquad i = 1, \cdots, i_0 - 1, i_0 + 2, \cdots, n,$$

$$r_{i_0} = s_{i_0+1}, \quad r_{i_0+1} = s_{i_0}, \quad r_{i_0} < r_{i_0+1}.$$

At each such step needed to obtain $q_1, \cdots, q_n$ from $1, \cdots, n$ we get the following nonnegative contribution to the left side in (2.18):

$$\sum_{i=1}^n y_i[x_{r_i} - x_{s_i}] = y_{i_0}[x_{r_{i_0}} - x_{s_{i_0}}] + y_{i_0+1}[x_{r_{i_0+1}} - x_{s_{i_0+1}}]$$

$$= [y_{i_0+1} - y_{i_0}][x_{r_{i_0+1}} - x_{r_{i_0}}] \geqq \eta[x_{i_0+1} - x_{i_0}][x_{r_{i_0+1}} - x_{r_{i_0}}]$$

$$= \eta x_{i_0}[x_{r_{i_0}} - x_{s_{i_0}}] + \eta x_{i_0+1}[x_{r_{i_0+1}} - x_{s_{i_0+1}}] = \eta \sum_{i=1}^n x_i[x_{r_i} - x_{s_i}].$$

Invoking now the monotonicity of $h_i(x)$, (2.14), (2.15) and (2.17), one has (after taking the limit of a discrete version of the assertion) that the assertion holds. Thus

$$\begin{aligned}
&\int_0^t \int_0^\tau a(\tau - s)[h_1(\tau)h_2(\tau) - h_1(s)h_2(\tau)]\,ds\,d\tau \\
&\qquad \geqq \eta \int_0^t a(s)\left\{\int_s^t [h_1^2(\tau) - \bar{h}_1(\tau, s)h_1(\tau)]\,d\tau\right\} ds \\
&\qquad\quad + \eta \int_0^t a(s)\left\{\int_0^s [h_1^2(\tau) - \bar{h}_1(\tau, s)h_1(\tau)]\,d\tau\right\} ds \\
&\qquad\quad - \int_0^t a(s)\left\{\int_0^s [h_1(\tau)h_2(\tau) - \bar{h}_1(\tau, s)h_2(\tau)]\,d\tau\right\} ds \\
&\qquad \geqq \eta \int_0^t \int_0^\tau a(\tau - s)[h_1^2(\tau) - h_1(s)h_1(\tau)]\,ds\,d\tau - K_1,
\end{aligned}$$

where we have used (2.15) and (2.16). The lemma is proved.

Using now Lemma 2, with $g(x) = h_1(x)$, $G(x) = h_2(x)$, $\gamma = \eta$, $x(t) = z(t)$ and $-b''(t) = a(t)$ gives, by (2.6), Lemma 1 and the hypothesis (note that (1.7) and (1.8) together imply $tb''(t) \in L_1[0, \infty)$ and $b''(t) \leqq 0$, $0 \leqq t < \infty$; and also that we can of course, by Lemma 1, choose the x_0 value defining $G(x)$ such that $G(x(t)) \geqq 0$, $0 \leqq t < \infty$)

$$\begin{aligned}
&-F^{-1} \int_0^t x'(\tau)g(x(\tau)) \int_0^\tau b'(\tau - s)g(x(s))\,ds\,d\tau \\
&\qquad \geqq -F^{-1}G(x(t)) \int_0^t b'(t - \tau)g(x(\tau))\,d\tau \\
&\qquad\quad - \gamma\frac{1}{2F} \int_0^t \int_0^\tau b''(\tau - s)[g(x(s)) - g(x(\tau))]^2\,ds\,d\tau \qquad (2.19)\\
&\qquad\quad + \frac{\gamma}{2F} \int_0^t b'(t - \tau)g^2(x(\tau))\,d\tau - \frac{\gamma}{2F} \int_0^t b'(\tau)g^2(x(\tau))\,d\tau \\
&\qquad\quad + F^{-1} \int_0^t b'(\tau)g(x(\tau))G(x(\tau))\,d\tau - k_1,
\end{aligned}$$

for some constant k_1 and where we have also used

$$\begin{aligned}
&\int_0^t \int_0^\tau b''(\tau - s)[g(x(s))g(x(\tau)) - g^2(x(\tau))]\,ds\,d\tau \\
&\qquad = -\tfrac{1}{2} \int_0^t \int_0^\tau b''(\tau - s)[g(x(s)) - g(x(\tau))]^2\,ds\,d\tau \qquad (2.20)\\
&\qquad\quad + \tfrac{1}{2} \int_0^t b'(t - \tau)g^2(x(\tau))\,d\tau - \tfrac{1}{2} \int_0^t b'(\tau)g^2(x(\tau))\,d\tau.
\end{aligned}$$

Substituting $g(x(t))$ as given by (1.1b) into the second term on the right side of (2.4) yields

$$
\begin{aligned}
F^{-1} & \int_0^t y(\tau) \int_0^\tau b'(\tau - s) g(x(s))\, ds\, d\tau \\
&= \frac{\beta}{\alpha F} \int_0^t \int_0^\tau b'(\tau - s) y(\tau) y(s)\, ds\, d\tau + \frac{1}{\alpha F} \int_0^t \int_0^\tau b'(\tau - s) y(\tau) y'(s)\, ds\, d\tau \\
&= F^{-1} \int_0^t \int_0^\tau \left[\frac{\beta}{\alpha} b'(\tau - s) + \frac{1}{\alpha} b''(\tau - s) \right] y(\tau) y(s)\, ds\, d\tau \\
&\quad + \frac{\beta b(0)}{\alpha F} \int_0^t y^2(\tau)\, d\tau - \frac{\beta b(0)}{\alpha F} \int_0^t y^2(\tau)\, d\tau \\
&\quad + \frac{b'(0)}{\alpha F} \int_0^t y^2(\tau)\, d\tau - \frac{y(0)}{\alpha F} \int_0^t b'(\tau) y(\tau)\, d\tau \\
&= -\frac{1}{2F} \int_0^t \int_0^\tau \left[\frac{\beta}{\alpha} b'(\tau - s) + \frac{1}{\alpha} b''(\tau - s) \right] [y(s) - y(\tau)]^2\, ds\, d\tau \\
&\quad + \frac{1}{2F} \int_0^t \left[\frac{\beta}{\alpha} b(t - \tau) + \frac{1}{\alpha} b'(t - \tau) \right] y^2(\tau)\, d\tau \\
&\quad + \frac{1}{2F} \int_0^t \left[\frac{\beta}{\alpha} b(\tau) + \frac{1}{\alpha} b'(\tau) \right] y^2(\tau)\, d\tau - \frac{\beta b(0)}{\alpha F} \int_0^t y^2(\tau)\, d\tau \\
&\quad - \frac{y(0)}{\alpha F} \int_0^t b'(\tau) y(\tau)\, d\tau,
\end{aligned}
$$

where we have integrated the term involving $y'(s)$ by parts and applied the same reasoning that gives (2.20).

Analyzing next the third term on the right side of (2.4) one has, by (1.1a),

$$
\begin{aligned}
F^{-1} & \int_0^t g(x(\tau)) \left[\int_0^\tau b(\tau - s) g(x(s))\, ds \right] \left[\int_0^\tau b'(\tau - s) g(x(s))\, ds \right] d\tau \\
&= F^{-1} \int_0^t g(x(\tau)) \left[\int_0^\tau b(\tau - s) g(x(s))\, ds \right] \frac{d}{d\tau} \left[\int_0^\tau b(\tau - s) g(x(s))\, ds \right] d\tau \\
&\quad - \frac{b(0)}{F} \int_0^t g^2(x(\tau)) \int_0^\tau b(\tau - s) g(x(s))\, ds\, d\tau \\
&= \frac{1}{2F} \int_0^t g(x(\tau)) \frac{d}{d\tau} \left[\int_0^\tau b(\tau - s) g(x(s))\, ds \right]^2 d\tau \\
&\quad - \frac{b(0)}{F} \int_0^t g^2(x(\tau)) [x'(\tau) - y(\tau)[g(x(\tau))]^{-1} - f(\tau)]\, d\tau \\
&= \frac{g(x(t))}{2F} \left[\int_0^t b(t - \tau) g(x(\tau))\, d\tau \right]^2 - \frac{1}{2F} \int_0^t \frac{d}{d\tau}[g(x(\tau))] \left[\int_0^\tau b(\tau - s) g(x(s))\, ds \right]^2 d\tau
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{b(0)}{F}\int_0^t \frac{d}{d\tau}\int_{x(0)}^{x(\tau)} g^2(u)\,du\,d\tau + \frac{b(0)}{F}\int_0^t y(\tau)\left[\frac{1}{\alpha}y'(\tau) + \frac{\beta}{\alpha}y(\tau)\right]d\tau \\
&+ \frac{b(0)}{F}\int_0^t g^2(x(\tau))[f(\tau) - F]\,d\tau + b(0)\int_0^t g^2(x(\tau))\,d\tau \\
= {}& \frac{g(x(t))}{2F}\left[\int_0^t b(t-\tau)g(x(\tau))\,d\tau\right]^2 \\
&- \frac{1}{2F}\int_0^t g'(x(\tau))x'(\tau)\left[\int_0^\tau b(\tau - s)g(x(s))\,ds\right]^2 d\tau \\
&- \frac{b(0)}{F}\int_{x(0)}^{x(t)} g^2(u)\,du + \frac{b(0)}{2\alpha F}[y^2(t) - y^2(0)] + \frac{\beta b(0)}{\alpha F}\int_0^t y^2(\tau)\,d\tau \\
&+ \frac{b(0)}{F}\int_0^t g^2(x(\tau))[f(\tau) - F]\,d\tau + b(0)\int_0^t g^2(x(\tau))\,d\tau.
\end{aligned}
$$

Consider now the term

$$
-\frac{1}{2F}\int_0^t g'(x(\tau))x'(\tau)\left[\int_0^\tau b(\tau - s)g(x(s))\,ds\right]^2 d\tau \tag{2.21}
$$

above. Solving (1.1a) for $\int_0^\tau b(\tau - s)g(x(s))\,ds$ and substituting into (2.21) gives

$$
\begin{aligned}
(2.21) = {}& -\frac{1}{2F}\int_0^t g'(x(\tau))x'^2(\tau)[x'(\tau) - 2f(\tau) - 2y(\tau)[g(x(\tau))]^{-1}]\,d\tau \\
&- \frac{1}{2F}\int_0^t g'(x(\tau))x'(\tau)y^2(\tau)[g(x(\tau))]^{-2}\,d\tau \\
&- F^{-1}\int_0^t g'(x(\tau))x'(\tau)y(\tau)f(\tau)[g(x(\tau))]^{-1}\,d\tau \\
&- \frac{1}{2F}\int_0^t g'(x(\tau))x'(\tau)f^2(\tau)\,d\tau \\
= {}& -\frac{1}{2F}\int_0^t g'(x(\tau))x'^2(\tau)\left[\int_0^\tau b(\tau - s)g(x(s))\,ds - F - y(\tau)[g(x(\tau))]^{-1}\right]d\tau \\
&- \frac{1}{2F}\int_0^t g'(x(\tau))x'^2(\tau)[F - f(\tau)]\,d\tau \\
&- \frac{1}{2F}\int_0^t g'(x(\tau))x'(\tau)y^2(\tau)[g(x(\tau))]^{-2}\,d\tau \\
&- \int_0^t g'(x(\tau))x'(\tau)y(\tau)[g(x(\tau))]^{-1}\,d\tau \\
&- F^{-1}\int_0^t g'(x(\tau))x'(\tau)[f(\tau) - F]y(\tau)[g(x(\tau))]^{-1}\,d\tau \\
&- \frac{F}{2}[g(x(t)) - g(x(0))] - \frac{1}{2F}\int_0^t g'(x(\tau))x'(\tau)[f^2(\tau) - F^2]\,d\tau.
\end{aligned}
$$

Observe that the integrand in the first term after the last equality sign above is $\leqq 0$, $0 \leqq \tau < \infty$. The term

$$-\frac{1}{2F}\int_0^t g'(x(\tau))x'(\tau)y^2(\tau)[g(x(\tau))]^{-2}\,d\tau \tag{2.22}$$

gives, after integrating by parts and using (1.1),

$$(2.22) \;=\; +\frac{1}{2F}\left[\frac{y^2(t)}{g(x(t))} - \frac{y^2(0)}{g(x(0))}\right] - \frac{\alpha}{\beta F}[y(t) - y(0)] + \frac{1}{\beta F}\int_0^t y'^2(\tau)[g(x(\tau))]^{-1}\,d\tau.$$

Integrating next

$$-\int_0^t g'(x(\tau))x'(\tau)y(\tau)[g(x(\tau))]^{-1}\,d\tau \tag{2.23}$$

by parts we obtain

$$(2.23) = -y(t)\ln\{g(x(t))\} + y(0)\ln\{g(x(0))\} + \int_0^t y'(\tau)\ln\{g(x(\tau))\}\,d\tau.$$

Let

$$I_{1t} = \{\tau | g(x(\tau)) \geqq (\beta/\alpha)y(\tau),\ 0 \leqq \tau \leqq t\},$$

$$I_{2t} = \{\tau | g(x(\tau)) < (\beta/\alpha)y(\tau),\ 0 \leqq \tau \leqq t\}.$$

Then, as $y'(\tau) \geqq 0$ on I_{1t} and $y'(\tau) < 0$ on I_{2t},

$$\int_0^t y'(\tau)\ln\{g(x(\tau))\}\,d\tau \geqq \int_0^t y'(\tau)\ln\{y(\tau)\}\,d\tau + \ln\left\{\frac{\beta}{\alpha}\right\}[y(t) - y(0)]$$

$$= y(t)\ln\{y(t)\} - y(0)\ln\{y(0)\} - [y(t) - y(0)] + \ln\left\{\frac{\beta}{\alpha}\right\}[y(t) - y(0)] \geqq -k_2.$$

Collecting terms and substituting into (2.23) one has that there exists $k_3 < \infty$ such that (2.23) $\geqq -k_3$. Also, by Lemma 1,

$$(2.22) \geqq -k_4 + \frac{1}{\beta F}\int_0^t y'^2(\tau)[g(x(\tau))]^{-1}\,d\tau.$$

Thus, invoking (1.6), one has that there exists $k_5 < \infty$ such that

$$(2.21) \geqq -k_5 + \frac{1}{\beta F}\int_0^t y'^2(\tau)[g(x(\tau))]^{-1}\,d\tau.$$

Then, by Lemma 1 and (1.6),

$$F^{-1}\int_0^t g(x(\tau))\left[\int_0^\tau b(\tau - s)g(x(s))\,ds\right]\left[\int_0^\tau b'(\tau - s)g(x(s))\,ds\right]d\tau$$

$$\geqq -k_6 + \frac{\beta b(0)}{\alpha F}\int_0^t y^2(\tau)\,d\tau + b(0)\int_0^t g^2(x(\tau))\,d\tau$$

$$+ \frac{1}{\beta F}\int_0^t y'^2(\tau)[g(x(\tau))]^{-1}\,d\tau.$$

Collecting terms and substituting into (2.1) gives, as $b^{(k)}(t) \in L_1[0, \infty)$, $k = 0, 1$,

$$\frac{1}{\beta F}\int_0^t y'^2(\tau)[g(x(\tau))]^{-1}\,d\tau$$
$$-\frac{1}{2}\int_0^t\int_0^\tau \left[b'(\tau - s) + \frac{\gamma}{F}b''(\tau - s)\right][g(x(s)) - g(x(\tau))]^2\,ds\,d\tau$$
$$-\frac{1}{2F}\int_0^t\int_0^\tau \left[\frac{\beta}{\alpha}b'(\tau - s) + \frac{1}{\alpha}b''(\tau - s)\right][y(s) - y(\tau)]^2\,ds\,d\tau \leqq k_7,$$

and, remembering (1.8), and also that the hypothesis implies $b''(t) \leqq 0, 0 \leqq t < \infty$, we have

$$\frac{1}{\beta F}\int_0^t y'^2(\tau)[g(x(\tau))]^{-1}\,d\tau \leqq k_7, \qquad 0 \leqq t < \infty, \tag{2.24}$$

for some $k_7 < \infty$. By the hypothesis and Lemma 1, $y''(t)$ exists on $0 \leqq t < \infty$ and satisfies $\sup_{0 \leqq t < \infty} |y''(t)| < \infty$. Thus we conclude from (2.24) that $\lim_{t \to \infty} y'(t)$ exists and is equal to zero, or by (1.1b), that $\lim_{t \to \infty} [\alpha g(x(t)) - \beta y(t)] = 0$. Suppose $\lim_{t \to \infty} g(x(t))$ either does not exist or (if it exists) does not satisfy (1.9). Then there clearly exist intervals $[t_n, t_n + T_n]$ such that $\lim_{n \to \infty} t_n = \lim_{n \to \infty} T_n = \infty$ and such that, e.g.,

$$x(t_n + T_n) - x(t_n) \geqq -\delta_n, \qquad \lim_{n \to \infty} \delta_n = 0, \tag{2.25}$$

$\alpha g(x(t)) - \beta y(t)$ sufficiently small,

$$g(x(t)) \geqq \left[\frac{\alpha}{\beta} + F\right]\left[\int_0^\infty |b(\tau)|\,d\tau\right]^{-1} + \delta, \qquad t_n \leqq t \leqq t_n + T_n,$$

for some $\delta > 0$. Integrating (1.1a) over these intervals, remembering (1.6) and that $b(t) \in L_1[0, \infty)$, readily gives a contradiction to (2.25). Thus $\lim_{t \to \infty} g(x(t))$, $\lim_{t \to \infty} y(t)$ exist and satisfy (1.9).

Of course, if in addition to the hypothesis, $g'(x) > 0$ for x such that $g(x) = g(x(\infty))$, then $\lim_{t \to \infty} x(t)$ exists.

It is obvious that the arguments above remain much the same if $F = 0$. One may then begin by considering (2.2) and replace $g(x(t))$ in this term by

$$x'(t)g(x(t)) - y(t) - g(x(t))\int_0^t b(t - \tau)g(x(\tau))\,d\tau - f(t)g(x(t))$$

(which is identically zero) and then proceed as above.

This completes the proof.

3. Concluding remarks. In this paper we have investigated the solutions of a certain nonlinear Volterra integrodifferential equation and given a sufficient hypothesis under which the nonlinear function $g(x(t))$ of the solution $x(t)$ tends to a limiting value as $t \to \infty$. The equation considered may be viewed as a particular case of the more general equation

$$x(t) = \int_0^t B(t - \tau)g(x(\tau))\,d\tau + \int_0^t \int_0^\tau \frac{d(\tau - s)g(x(s))}{g(x(\tau))}\,ds\,d\tau + F(t),$$

on which extensions on the present work might be formulated.

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SIAM J. MATH. ANAL.
Vol. 2, No. 3, August 1971

A CONSTRUCTIVE EXISTENCE THEOREM FOR A NONLINEAR ELLIPTIC EQUATION*

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Abstract. A constructive existence theorem for the homogeneous Dirichlet problem in $\mathscr{R}$ (in two or three dimensions) for the equation

$$P(u) = \Delta u + \alpha_{ij} u_{x_i} u_{x_j} + f = 0$$

is obtained by placing P in the role of an operator mapping $W_{2,0}^2(\mathscr{R})$ into $L_2(\mathscr{R})$ and proving convergence of a Newton sequence for P. The theorem is "local" in the sense that the L_2-norm of f must satisfy a bound which becomes infinite as the diameter of $\mathscr{R}$ shrinks to zero. An essential feature of the proof is an application of Sobolev's lemma to show that $P(u)$ is an element of $L_2(\mathscr{R})$ when $u \in W_{2,0}^2(\mathscr{R})$, and that P satisfies the hypotheses of a theorem of Kantorovich.

Introduction. The purpose of this note is to present a constructive proof of the existence of a solution of the homogeneous Dirichlet problem for the equation

$$\Delta u + \alpha_{ij} u_{x_i} u_{x_j} + f = 0 \tag{1}$$

in a bounded domain $\mathscr{R}$ in either two or three dimensions. (The summation convention on repeated indices is employed here. We assume this unless stated otherwise.) The coefficients α_{ij} are bounded and measurable, the function f is square summable, and the solution sought is a "generalized" solution with square-summable second derivatives. The proof is accomplished by showing that the left-hand side of the equation defines an operator which satisfies the hypotheses of a theorem of Kantorovich on convergence of Newton's method. In order to do this it is necessary to impose a bound on the L_2-norm of the function f. This bound, which is explicitly given below, depends only on the size and geometry of the region $\mathscr{R}$. In order to facilitate obtaining this explicit value we place two restrictions on the region $\mathscr{R}$. The first is that the boundary of $\mathscr{R}$ be smooth and that its mean curvature be everywhere nonnegative (where the direction of the outward pointing normal is taken as positive). Secondly, we assume that $\mathscr{R}$ can be written as the union of a finite number of convex regions which overlap in sets of positive measure.

1. Preliminary lemmas. The solution of our problem is sought in the Hilbert space $W_{2,0}^2(\mathscr{R})$, which is defined to be the completion of the vector space of functions in $C_2(\mathscr{R})$ which vanish on the boundary of $\mathscr{R}$ in the inner product

$$\langle u, v \rangle = \int_{\mathscr{R}} (uv + u_{x_i} v_{x_i} + u_{x_i x_j} v_{x_i x_j}).$$

* Received by the editors March 26, 1970, and in revised form October 7, 1970.

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We adopt the notation

$$\|u\|_2 = (\langle u, u\rangle)^{1/2}$$

for the norm derived from this inner product.

We shall give several results needed in the proof of our main theorem in a series of lemmas.

LEMMA 1. *For* $u \in W_{2,0}^2(\mathscr{R})$,

$$\int_{\mathscr{R}} (u_{x_ix_j}u_{x_ix_j}) = \int_{\mathscr{R}} (\Delta u)^2 - (n-1)\int_{\partial\mathscr{R}} H\left(\frac{\partial u}{\partial n}\right)^2,$$

where H *is the mean curvature of the boundary of* $\mathscr{R}$ (*and* n *is the dimension of* $\mathscr{R}$).

The proof is essentially given in [1, p. 171]. The important thing for us to notice is that our first assumption about $\mathscr{R}$ implies that

$$\int_{\mathscr{R}} (u_{x_ix_j}u_{x_ix_j}) \leqq \int_{\mathscr{R}} (\Delta u)^2 \tag{2}$$

for u in $W_{2,0}^2(\mathscr{R})$.

In what follows we need the existence of

$$\lambda = \inf\left(\int_{\mathscr{R}} |\nabla u|^2 \Big/ \int_{\mathscr{R}} u^2\right),$$

where the infimum is taken over smooth functions that vanish on the boundary of $\mathscr{R}$. (See [2] for example.) We use the notation

$$\|u\|_0 = \left(\int_{\mathscr{R}} u^2\right)^{1/2}$$

for the norm in the Hilbert space $L_2(\mathscr{R})$.

LEMMA 2. *The linear operator* Δ *maps* $W_{2,0}^2(\mathscr{R})$ *onto* $L_2(\mathscr{R})$, *is invertible, and*

$$\|\Delta^{-1}\| \leqq \left(1 + \frac{1}{\lambda} + \frac{1}{\lambda^2}\right)^{1/2}.$$

Proof. We begin by establishing the validity of the inequality

$$(\|u\|_2)^2 \leqq \left(1 + \frac{1}{\lambda} + \frac{1}{\lambda^2}\right)(\|\Delta u\|_0)^2$$

for smooth functions vanishing on the boundary of $\mathscr{R}$. It then follows for u in $W_{2,0}^2(\mathscr{R})$ by taking limits.

First note that, from (2), we have

$$(\|u\|_2)^2 \leqq \int_{\mathscr{R}} u^2 + \int_{\mathscr{R}} |\nabla u|^2 + \int_{\mathscr{R}} (\Delta u)^2.$$

Then, using the inequalities

$$\int_{\mathscr{R}} u^2 \leqq \frac{1}{\lambda^2}\int_{\mathscr{R}} (\Delta u)^2$$

[3, p. 555], and

$$\int_{\mathscr{R}} |\nabla u|^2 \leqq \frac{\varepsilon}{2}\int_{\mathscr{R}} u^2 + \frac{1}{2\varepsilon}\int_{\mathscr{R}} (\Delta u)^2, \qquad \varepsilon > 0,$$

we have

$$(\|u\|_2)^2 \leqq \left(1 + \frac{\varepsilon}{2\lambda^2} + \frac{1}{2\varepsilon} + \frac{1}{\lambda^2}\right)\int_{\mathscr{R}} (\Delta u)^2.$$

The required inequality then follows by minimizing the constant with respect to ε.

Now, since Δ maps $W^2_{2,0}(\mathscr{R})$ into $L_2(\mathscr{R})$ and is bounded below, the lemma follows once it is known that the range of Δ is dense in $L_2(\mathscr{R})$. That this is the case follows from well-known results on linear elliptic equations. (See [4, p. 114], for example.)

In the next lemma we shall use the notation

$$M(u) = \frac{1}{m(\mathscr{R})}\int_{\mathscr{R}} u$$

for the mean value of a function integrable over $\mathscr{R}$.

LEMMA 3. *Suppose that $\mathscr{R}$ is a convex n-dimensional region of diameter δ, and that*

$$p \leqq q, \qquad q < \frac{np}{n-p}.$$

Then

$$\|u - M(u)\|_{L_q} \leqq K\| |\nabla u| \|_{L_p}, \qquad u \in W^1_p(\mathscr{R}),$$

where

$$K = \frac{\delta^{1+n/r}}{nm(\mathscr{R})}\left[\frac{\omega_n}{n-(n-1)r}\right]^{1/r}, \qquad r = \frac{pq}{(p-1)q+p},$$

and ω_n is the volume of the unit n-sphere.

This result is proved in [5, Chap. 10]. The following extension of Lemma 3 is also proved there.

LEMMA 3′. *Suppose $\mathscr{R}$ can be written as the union of two convex regions $\mathscr{R}_1$ and $\mathscr{R}_2$ which overlap in a set of positive measure $\mathscr{R}_3$. Then*

$$\|u - M(u)\|_{L_q} \leqq K'\| |\nabla u| \|_{L_p}, \qquad u \in W^1_p(\mathscr{R}), \tag{3}$$

where

$$\begin{aligned} K' = {} & K_1\{1 + m(\mathscr{R}_3)^{-1/q}(m(\mathscr{R}_1)^{1/q} + 2m(\mathscr{R}_2)^{1/q})\} \\ & + K_2\{1 + m(\mathscr{R}_3)^{-1/q}(m(\mathscr{R}_1)^{1/q} + 2m(\mathscr{R}_2)^{1/q}) \\ & \qquad + m(\mathscr{R}_2 \cap \mathscr{R}_1')(m(\mathscr{R}_1)^{1/q} + m(\mathscr{R}_2)^{1/q})\} \end{aligned}$$

and K_1, K_2 are the constants obtained in Lemma 3 for $\mathscr{R}_1$ and $\mathscr{R}_2$, respectively.

This result has an obvious inductive extension. Henceforth, we shall assume that $\mathscr{R}$ is such that (3) holds for some constant K. The explicit value can easily be computed from Lemma 3 and Lemma 3′ for the type of region we are considering.

For our purposes we choose $q = 4$, $p = 2$ and obtain

$$\|u - M(u)\|_{L_4} \leqq K\| |\nabla u| \|_{L_2}$$

provided $n < 4$. We have then the following lemma.

LEMMA 4. *For u in $W_2^1(\mathscr{R})$,*

$$\|u\|_{L_4} \leqq \sqrt{\tfrac{3}{2}} \max \{K, m(\mathscr{R})^{-1/4}\} \|u\|_{W_2^1} \equiv K_0 \|u\|_{W_2^1}.$$

Proof. First

$$\|u\|_{L_4} \leqq \|u - M(u)\|_{L_4} + \|M(u)\|_{L_4} \leqq K\| |\nabla u| \|_{L_2} + |M(u)| m(\mathscr{R})^{1/4}.$$

The result then follows from an application of Hölder's inequality to the last term.

The following result, which is a specialization of a theorem of Kantorovich (proved in [5]), is the vehicle for proving existence mentioned in the Introduction.

LEMMA 5. *Suppose that P is a twice continuously F-differentiable mapping of a Banach space X into a Banach space Y such that $P'(0)$ is invertible, and that*

$$\|[P'(0)]^{-1}\| \leqq B,$$

$$\|P''(u)\| \leqq C_0 \quad \text{for each } u \text{ in } X.$$

Then, if

$$\|P(0)\| \leqq \eta,$$

where

$$\eta < 1/(2B^2 C_0),$$

the Newton sequence

$$u_{n+1} = u_n - [P'(u_n)]^{-1} P(u_n), \qquad u_0 = 0,$$

is well-defined and converges to a solution u^* of the equation

$$P(u) = 0.$$

Error bounds and an a posteriori bound on the solution also can be obtained, but these need not concern us here.

2. The proof of the existence theorem. Suppose that the nonlinear operator P is defined by

$$P(u) = \Delta u + \alpha_{ij} u_{x_i} u_{x_j} + f$$

for u in $W_{2,0}^2(\mathscr{R})$. Then, since by Lemma 3′,

$$\int_{\mathscr{R}} (u_{x_i}^2 u_{x_j}^2) \leqq \text{const.} (\|u\|_2)^2, \qquad u \in W_{2,0}^2(\mathscr{R}),$$

P maps $W_{2,0}^2(\mathscr{R})$ into $L_2(\mathscr{R})$ and our problem is equivalent to showing that the equation

$$P(u) = 0$$

has a solution in $W^2_{2,0}(\mathscr{R})$. We do this by showing that P satisfies the hypotheses of Lemma 5 with $W^2_{2,0}(\mathscr{R})$ and $L_2(\mathscr{R})$ in the roles of X and Y, respectively.

We define

$$A = \max\{\|\alpha_{ij}\|_\infty\}.$$

LEMMA 6. *P is continuous.*

Proof. Let $u \in X$. Then, for $v \in X$,

$$P(u) - P(v) = \Delta(u - v) + \alpha_{ij}u_{x_i}(u_{x_j} - v_{x_j}) + \alpha_{ij}v_{x_j}(u_{x_i} - v_{x_i}),$$

and

$$\|P(u) - P(v)\|_0^2 \leqq (\|u - v\|_2)^2 + A^2[\|u_{x_i}^2\|_0 + \|v_{x_i}^2\|_0]\|(u_{x_i} - v_{x_i})^2\|_0,$$

and, by Lemma 3′,

$$\|P(u) - P(v)\|_0^2 \leqq [1 + c(\|u\|_2 + \|v\|_2)]\,\|u - v\|_2,$$

where c is a constant independent of u and v. Continuity of P at u then follows routinely.

We proceed by noting that

$$P(u + h) = P(u) + L_u(h) + N(h, h), \qquad u, h \in X,$$

where

$$L_u(h) = \Delta h + \alpha_{ij}u_{x_i}h_{x_j} + \alpha_{ij}u_{x_j}h_{x_i}$$

and

$$N(h, k) = \alpha_{ij}h_{x_i}k_{x_j}.$$

Hence if L_u and N are continuous linear and bilinear operators, respectively, it follows that P has two continuous Fréchet derivatives, and that

$$P'(u) = L_u, \qquad P''(u) = 2N.$$

LEMMA 7. *L_u is a continuous linear operator.*

Proof. An application of Lemma 3′ yields

$$\|L_u(h)\|_0 \leqq (1 + c\|u\|_2)\|h\|_2,$$

where c is a constant independent of u and h.

LEMMA 8. *N is a continuous bilinear operator, and*

$$\|N\| \leqq AK_0^2.$$

Proof. We have

$$\|N(h, k)\|_0 \leqq A\|h_ik_i\|_0 \leqq A(\|h_{x_i}^2\|_0)^{1/2}(\|k_{x_i}^2\|_0)^{1/2},$$

so that, again using Lemma 3′,

$$\|N(h, k)\|_0 \leqq AK_0^2\|h\|_2\|k\|_2.$$

Our preliminary work is finished once we note that $P'(0) = \Delta$, so that $P'(0)$ is invertible and

$$\|[P'(0)]^{-1}\| \leqq \left(1 + \frac{1}{\lambda} + \frac{1}{\lambda^2}\right)^{1/2}.$$

Our principal result is stated in the following theorem.

THEOREM. *The equation* (1) *has a solution in* $W^2_{2,0}(\mathscr{R})$ *provided that*

$$\|f\|_0 < \frac{\lambda^2}{2AK_0^2(\lambda^2 + \lambda + 1)}.$$

Proof. We have shown that P satisfies all of the hypotheses of Lemma 5 except the required bound on the norm of $P(0) = f$. This is precisely the bound that is imposed in the statement of the theorem.

Remarks. The theorem has the character of a "local" existence theorem in the sense that the required bound on $\|f\|_0$ goes to infinity as the diameter of $\mathscr{R}$ shrinks to zero. This follows from examination of K and an application of the Faber–Krahn inequality [6, p. 462] to λ.

In regard to our assumption on the mean curvature H of the boundary of $\mathscr{R}$ it should be remarked that this can be relaxed to the requirement that H be bounded. This could be done by utilizing an inequality of the form

$$\int_{\partial\mathscr{R}} \left(\frac{\partial u}{\partial n}\right)^2 \leqq \text{const.} \int_{\mathscr{R}} (\Delta u)^2.$$

(See, for example, [3].)

On the other hand, the restriction to two- or three-dimensional regions is necessary in order that Lemma 3 hold. It appears to be a difficult matter to extend the method of proof used above to higher dimensions.

Finally, we remark that the Kantorovich theorem provides error bounds for the convergence of the Newton sequence, so that these hold in particular for the convergence of the sequence of solutions of linear elliptic equations that converge to our solution.

3. Conclusion. We have proved a "local" existence theorem for (1). In particular if we assume f to be given and allow the region to vary, a solution will always exist if the diameter of the region is sufficiently small. It follows from the Sobolev imbedding theorem [8, Chap. 3] that this solution is uniformly continuous in $\mathscr{R}$. In light of this it is interesting to take note of the (two-dimensional) example

$$\Delta u + (2 - x^2 - y^2)(u_x^2 + u_y^2) + c = 0,$$

where c is a constant. It was pointed out in [7] that, when $c \geqq 4$, the Dirichlet problem for this equation on the unit disc has no bounded solution. Therefore, a complete "global" result for equations of the form (1) is not possible, and the truth must lie somewhere between such a result and one of the type presented here.

Acknowledgment. The author is indebted to the referee for helpful suggestions.

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SIAM J. MATH. ANAL.
Vol. 2, No. 3, August 1971

AN ASYMPTOTIC ANALYSIS OF A CERTAIN HYPERBOLIC CAUCHY PROBLEM*

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Abstract. An asymptotic expansion valid for small values of the positive parameter ε is given for the solution of the Cauchy problem (1.1), (1.2). The expansion is shown to be asymptotically correct (as $\varepsilon \to 0$) uniformly on compact sets in the upper half-plane $t \geqq 0$. In the case that the reduced linear algebraic system obtained by setting $\varepsilon = 0$ in (1.1) has infinitely many solutions along a certain subcharacteristic, it is shown that a particular one of these infinitely many solutions is distinguished as the unique limit of the solution of the Cauchy problem (1.1), (1.2) as ε vanishes for fixed $t > 0$. If the reduced system has no solution along the subcharacteristic, it is shown that the solution of (1.1), (1.2) becomes unbounded like ε^{-1} as $\varepsilon \to 0$. Finally, if the reduced system is nonsingular, then the solution of (1.1), (1.2) tends to the unique solution of the reduced system as $\varepsilon \to 0$.

1. Introduction. We consider the behavior of solutions of the hyperbolic system

$$\begin{aligned} &\varepsilon\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)u + au + bv = f(x, t), \\ &\varepsilon\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)v + cu + dv = g(x, t) \end{aligned} \tag{1.1}$$

as the positive parameter ε tends toward zero. The coefficients a, b, c and d are fixed constants while the forcing terms f and g are specified functions defined and smooth for $t \geqq 0$. We specify also the values of u and v along $t = 0$:

$$u = u_0(x), \qquad v = v_0(x) \quad \text{for} \quad t = 0. \tag{1.2}$$

The data a, b, c, d, f, g, u_0 and v_0 are independent of ε.

We assume throughout that the coefficient matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has nonnegative diagonal elements and positive trace,

$$a \geqq 0, \qquad d \geqq 0, \qquad a + d > 0. \tag{1.3}$$

Moreover we assume that the product of the off-diagonal elements is nonnegative and dominated by the product of the diagonal elements

$$0 \leqq bc \leqq ad. \tag{1.4}$$

Actually we need only assume $|bc| \leqq ad$ instead of (1.4), but (1.4) makes certain estimates simpler by eliminating certain complex-valued expressions (see, for example, (2.11), (2.18), (3.7), etc.). The conditions (1.3) and (1.4) are stability conditions, and will be seen to ensure that solutions of (1.1), (1.2) do *not* become exponentially unbounded as $\varepsilon \to 0$. Indeed, in the case when all the data are independent of x, so that (1.1), (1.2) reduce to an initial value problem for a system of ordinary differential equations, then (1.3) and (1.4) are known to ensure stability.

* Received by the editors June 30, 1970, and in revised form December 7, 1970.

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In fact, it is well known in that case that (1.4) can be replaced with the weaker condition $bc \leqq ad$.

One expects that the solution of the Cauchy problem (1.1), (1.2) will exhibit some sort of singular behavior as $\varepsilon \to 0$. The problem is linear with constant coefficients and may, of course, be solved in closed form in terms of the Riemann function (which may be given in this case in terms of a modified Bessel function), and the resulting closed form solution may be studied as $\varepsilon \to 0$. Alternately one may attempt somehow to use the system (1.1), (1.2) directly to obtain a suitable approximation to the solution for small ε. Each of these approaches has been used successfully by Whitham [1] to obtain appropriate first approximations to the solutions of a class of similar problems for various linear and nonlinear hyperbolic equations. (The author is indebted to a referee for bringing [1] to his attention.) In the present paper we use a standard version of the latter approach (see, for example, [2, pp. 449–451] and the references given there) to obtain a complete asymptotic expansion for the solution of (1.1), (1.2), and we show that the resulting expansion is asymptotically correct (as $\varepsilon \to 0$) uniformly on compact sets in the upper half-plane $t \geqq 0$. The proof of the asymptotic correctness of the expansion is based on a study of the consequences of the inequalities (see (3.14) below)

$$(1.5)\qquad \begin{aligned} e^{ta}R(t) &\leqq |b| \int_0^t e^{sa}S(s)\,ds + U(t), \\ e^{td}S(t) &\leqq |c| \int_0^t e^{sd}R(s)\,ds + V(t) \end{aligned}$$

for nonnegative functions R and S, where U and V are known functions.

Of particular interest for (1.1), (1.2) is the situation where the reduced system

$$(1.6)\qquad \begin{aligned} au + bv &= f(x, t), \\ cu + dv &= g(x, t), \end{aligned}$$

obtained by setting $\varepsilon = 0$ in (1.1), has infinitely many solutions (i.e., the coefficient matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is singular and the forcing vector $\begin{pmatrix} f \\ g \end{pmatrix}$ is in the column space of that matrix). The special example

$$(1.7)\qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}, \qquad f(x, t) = g(x, t)$$

falls into this category, and arises in a study of the telegraph equation. This example has been studied in [3], where Gronwall's inequality is used to prove the asymptotic correctness of the appropriate expansion. Indeed, if $|b|$, $|c| \leqq a = d$ as in (1.7), then the inequalities of (1.5) can be added and Gronwall's inequality suffices for the result. (The author has learned recently of work by Fife [4] in which a significant and large first step is made towards a general theory for such problems whose reduced forms have many solutions.)

For positive ε the values at (x_0, t_0) of the solution functions $u = u(x_0, t_0; \varepsilon)$ and $v = v(x_0, t_0; \varepsilon)$ of (1.1), (1.2) depend on the data restricted to the fixed triangular region

$$\{(x,t): |x - x_0| \leqq x_0 + t_0 - t, \text{all } 0 \leqq t \leqq t_0\},$$

which is independent of ε, as shown in Fig. 1.

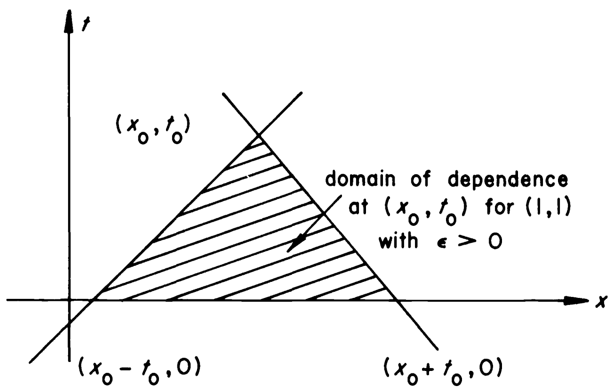

FIG. 1

As shown by Whitham [1], however, one expects that *the limiting behavior of* $u(x_0, t_0; \varepsilon)$ *and* $v(x_0, t; \varepsilon)$ *as* $\varepsilon \to 0$ *is governed entirely by the data restricted to the line segment*

$$\left\{(x,t): x = x_0 + \frac{a-d}{a+d}(t_0 - t), \text{all } 0 \leqq t \leqq t_0\right\} \tag{1.8}$$

as shown in Fig. 2. Indeed if we eliminate v

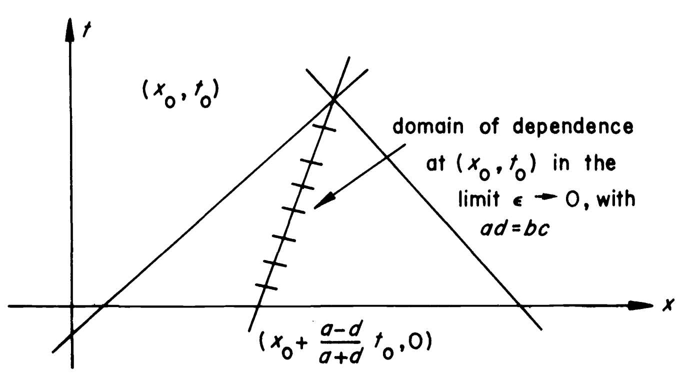

FIG. 2

from the system (1.1), (1.2), we find for u the second order equation

$$\begin{aligned}\varepsilon\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)u + \left[(a+d)\frac{\partial}{\partial t} + (-a+d)\frac{\partial}{\partial x}\right]u + \left(\frac{ad-bc}{\varepsilon}\right)u \\ = \frac{df - bg}{\varepsilon} + \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)f\end{aligned} \tag{1.9}$$

with initial data

$$u = u_0(x),$$
$$\frac{\partial u}{\partial t} = -u_0'(x) + \frac{f(x,0) - au_0(x) - bv_0(x)}{\varepsilon} \quad \text{for} \quad t = 0.$$

The line segment (1.8) is then recognized as the subcharacteristic of (1.9) as $\varepsilon \to 0$.

The *asymmetry* of the limiting domain of dependence as shown in Fig. 2 reflects a certain *struggle* which occurs between the two equations of (1.1) as $\varepsilon \to 0$. Indeed, these equations may be rewritten as (see (1.10) below)

$$a^{-1}\varepsilon\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)u + u = -a^{-1}bv + a^{-1}f(x,t),$$
$$d^{-1}\varepsilon\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)v + v = -d^{-1}cu + d^{-1}g(x,t),$$

where the first equation carries up data from below and primarily from the *left* of the point (x_0, t_0), while the second equation carries up data primarily from the right. In the limit $\varepsilon \to 0$, *that equation loses the struggle which has the smaller coefficient multiplying its derivative term*, so that in the limit $\varepsilon \to 0$ the data is transmitted along the line segment given by (1.8) issuing up from the left if $a < d$ and from the right if $d < a$. In the special case $a = d$, the limiting behavior at (x_0, t_0) is governed entirely by the time history of the data at $x = x_0$.

Assuming always that (1.3) and (1.4) hold, *we show that if the reduced system* (1.6) *has infinitely many solutions at the points on the line segment* (1.8) (i.e., if the coefficient matrix is singular and the forcing vector is in the column space of the coefficient matrix along (1.8)), then the values of the solution functions $u(x_0, t_0; \varepsilon)$, $v(x_0, t_0; \varepsilon)$ of (1.1), (1.2) have well-defined (finite) limits as $\varepsilon \to 0$ at any fixed point (x_0, t_0) with $t_0 > 0$, and these limiting values satisfy the reduced system (1.6). We exhibit in this case which of the infinitely many solutions of (1.6) is selected by the Cauchy problem (1.1), (1.2) as ε vanishes. (See (4.4) below.) On the other hand, *if the reduced system* (1.6) *has no solution along the line segment* (1.8) (i.e. if the coefficient matrix is singular and the forcing vector is not in the column space of the coefficient matrix along the line segment (1.8)), then we show that the values $u(x_0, t_0; \varepsilon)$, $v(x_0, t_0; \varepsilon)$ of the solution functions of (1.1), (1.2) become unbounded like ε^{-1} as $\varepsilon \to 0$ for fixed $t_0 > 0$. *Finally, if the reduced system* (1.6) *is nonsingular*, then the values of the solution functions u, v of (1.1), (1.2) tend to the unique solution of (1.6) as $\varepsilon \to 0$.

In all cases the solution functions $u(x, t; \varepsilon)$, $v(x, t; \varepsilon)$ of (1.1), (1.2) exhibit a boundary layer behavior near $t = 0$ as ε vanishes, and we obtain uniformly valid asymptotic expansions in all cases.

We remark finally that we may without essential loss assume (in addition to (1.3)) that *both diagonal elements are positive*:

$$a > 0, \qquad d > 0. \tag{1.10}$$

Indeed, if either a or d is zero, then (1.4) implies also that either b or c vanishes, and the resulting four special cases can be handled explicitly; i.e., in these special cases the Cauchy problem (1.1), (1.2) can be solved in closed form up to simple quadratures of the data. One then finds directly by integration by parts that the stated results of this paper hold in those special cases. Hence we shall assume (1.10) whenever convenient.

2. Formal construction of the asymptotic expansions. We seek asymptotic expansions for the solution functions u, v of (1.1), (1.2) in the form

$$
\begin{aligned}
u(x,t;\varepsilon) &\sim \sum_{\nu=-1}^{\infty} \{u_\nu^1(x,t) + e^{-\alpha(x,t)/\varepsilon}u_\nu^2(x,t) + e^{-\beta(x,t)/\varepsilon}u_\nu^3(x,t)\}\cdot\varepsilon^\nu, \\
v(x,t;\varepsilon) &\sim \sum_{\nu=-1}^{\infty} \{v_\nu^1(x,t) + e^{-\alpha(x,t)/\varepsilon}v_\nu^2(x,t) + e^{-\beta(x,t)/\varepsilon}v_\nu^3(x,t)\}\cdot\varepsilon^\nu,
\end{aligned}
\tag{2.1}
$$

where the functions u_ν^1, u_ν^2, u_ν^3, v_ν^1, v_ν^2, v_ν^3, α and β are to be independent of ε. The terms in (2.1) with $\nu = -1$ are included to allow for the possibility that u and v may become unbounded as $\varepsilon \to 0$ (for example, if the reduced system (1.6) has no solution). We might include other negative powers of ε, but it turns out that (2.1) suffices (see also [4] in this regard).

Note that we have anticipated in (2.1) that the coupling of the system (1.1) will likely lead to the same type boundary layer terms $e^{-\alpha/\varepsilon}$ and $e^{-\beta/\varepsilon}$ in both u and v; this is indeed the case. We are naturally led to allow for two different exponential terms in (2.1) since the nonlinear equation (2.8) below has in general two distinct solutions satisfying given initial data. As usual we require that $\alpha(x,t)$ and $\beta(x,t)$ vanish initially:

$$
\alpha(x,0) = 0, \quad \beta(x,0) = 0. \tag{2.2}
$$

Using now the notation $D_\pm = \dfrac{\partial}{\partial t} \pm \dfrac{\partial}{\partial x}$, and inserting (2.1) into (1.1) we obtain

$$
\begin{aligned}
&\sum_{\nu=-1}^{\infty} [D_+u_{\nu-1}^1 + au_\nu^1 + bv_\nu^1]\cdot\varepsilon^\nu \\
&\quad + e^{-\alpha/\varepsilon}\sum_{\nu=-1}^{\infty} [D_+u_{\nu-1}^2 + (a - D_+\alpha)u_\nu^2 + bv_\nu^2]\cdot\varepsilon^\nu \\
&\quad + e^{-\beta/\varepsilon}\sum_{\nu=-1}^{\infty} [D_+u_{\nu-1}^3 + (a - D_+\beta)u_\nu^3 + bv_\nu^3]\cdot\varepsilon^\nu = f(x,t), \\
&\sum_{\nu=-1}^{\infty} [D_-v_{\nu-1}^1 + cu_\nu^1 + dv_\nu^1]\cdot\varepsilon^\nu \\
&\quad + e^{-\alpha/\varepsilon}\sum_{\nu=-1}^{\infty} [D_-v_{\nu-1}^2 + cu_\nu^2 + (d - D_-\alpha)v_\nu^2]\cdot\varepsilon^\nu \\
&\quad + e^{-\beta/\varepsilon}\sum_{\nu=-1}^{\infty} [D_-v_{\nu-1}^3 + cu_\nu^3 + (d - D_-\beta)v_\nu^3]\cdot\varepsilon^\nu = g(x,t),
\end{aligned}
\tag{2.3}
$$

where we have set

$$u^i_{-2} = v^i_{-2} = 0 \quad \text{for} \quad i = 1, 2, 3. \tag{2.4}$$

We now set to zero the expressions multiplying the different exponentials in (2.3), and then equate coefficients of like powers of ε in the resulting equations. In this way we obtain the conditions:

$$\begin{aligned} D_+u^1_{\nu-1} + au^1_\nu + bv^1_\nu &= \begin{cases} f(x,t) & \text{for} \quad \nu = 0, \\ 0 & \text{otherwise,} \end{cases} \\ D_-v^1_{\nu-1} + cu^1_\nu + dv^1_\nu &= \begin{cases} g(x,t) & \text{for} \quad \nu = 0, \\ 0 & \text{otherwise,} \end{cases} \end{aligned} \tag{2.5}$$

$$\begin{aligned} &D_+u^2_{\nu-1} + (a - D_+\alpha)u^2_\nu + bv^2_\nu = 0, \\ &D_-v^2_{\nu-1} + cu^2_\nu + (d - D_-\alpha)v^2_\nu = 0 \quad \text{for all } \nu, \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} &D_+u^3_{\nu-1} + (a - D_+\beta)u^3_\nu + bv^3_\nu = 0, \\ &D_-v^3_{\nu-1} + cu^3_\nu + (d - D_-\beta)v^3_\nu = 0 \quad \text{for all } \nu. \end{aligned} \tag{2.7}$$

We now require $\alpha(x, t)$ and $\beta(x, t)$ to be such that the matrix

$$\begin{pmatrix} a - D_+\gamma & b \\ c & d - D_-\gamma \end{pmatrix}$$

occurring in (2.6) and (2.7) is *singular* for $\gamma = \alpha$ and $\gamma = \beta$ since otherwise (2.4) along with (2.6) and (2.7) would require

$$u^2_\nu \equiv 0, \qquad v^2_\nu \equiv 0 \quad \text{and} \quad u^3_\nu \equiv 0, \qquad v^3_\nu \equiv 0$$

for all ν. Hence α and β must satisfy the nonlinear first order equation

$$(D_+\gamma)(D_-\gamma) - (aD_-\gamma + dD_+\gamma) + ad - bc = 0 \tag{2.8}$$

for $\gamma = \alpha$ and $\gamma = \beta$, subject to the initial condition (see (2.2))

$$\gamma(x, 0) = 0. \tag{2.9}$$

One easily checks that the initial value problem (2.8), (2.9) has precisely two solutions given as

$$\begin{aligned} \gamma_+(x,t) &= \frac{a + d + \lambda}{2} \cdot t, \\ \gamma_-(x,t) &= \frac{a + d - \lambda}{2} \cdot t, \end{aligned} \tag{2.10}$$

where

$$\lambda = \sqrt{(a + d)^2 - 4(ad - bc)} = \sqrt{(a - d)^2 + 4bc}. \tag{2.11}$$

We set

$$\alpha = \gamma_+, \quad \beta = \gamma_- \tag{2.12}$$

so that in particular (2.6) and (2.7) become

$$\text{(2.13)}\qquad \begin{aligned} &D_+u^2_{\nu-1} + \frac{a-d-\lambda}{2}u^2_\nu + bv^2_\nu = 0,\\ &D_-v^2_{\nu-1} + cu^2_\nu + \frac{-a+d-\lambda}{2}v^2_\nu = 0 \quad \text{for all } \nu, \end{aligned}$$

and

$$\text{(2.14)}\qquad \begin{aligned} &D_+u^3_{\nu-1} + \frac{a-d+\lambda}{2}u^3_\nu + bv^3_\nu = 0,\\ &D_-v^3_{\nu-1} + cu^3_\nu + \frac{-a+d+\lambda}{2}v^3_\nu = 0 \quad \text{for all } \nu. \end{aligned}$$

As initial conditions we find with (1.2), (2.1), (2.10), (2.11) and (2.12) the requirements

$$\sum_{\nu=-1}^{\infty}[u^1_\nu(x,0) + u^2_\nu(x,0) + u^3_\nu(x,0)]\cdot\varepsilon^\nu = u_0(x),$$

$$\sum_{\nu=-1}^{\infty}[v^1_\nu(x,0) + v^2_\nu(x,0) + v^3_\nu(x,0)]\cdot\varepsilon^\nu = v_0(x),$$

which will hold if

$$\text{(2.15)}\qquad \begin{aligned} &u^1_\nu(x,0) + u^2_\nu(x,0) + u^3_\nu(x,0) = \begin{cases} u_0(x) & \text{if } \nu = 0,\\ 0 & \text{otherwise,}\end{cases}\\ &v^1_\nu(x,0) + v^2_\nu(x,0) + v^3_\nu(x,0) = \begin{cases} v_0(x) & \text{if } \nu = 0,\\ 0 & \text{otherwise.}\end{cases} \end{aligned}$$

We find it convenient at this point to distinguish between the two cases (see (1.4))

$$\text{(2.16)}\qquad bc < ad$$

and

$$\text{(2.17)}\qquad bc = ad.$$

In the former case of (2.16) both γ_+ and γ_- (hence also α and β) are *positive* (and distinct—see (1.3) and (1.4)), while in the latter case of (2.17) γ_- (and hence β) *vanishes identically.*

Considering first the case (2.16), we show that the relations (2.4), (2.5), (2.13), (2.14) and (2.15) can be used recursively to determine unique values for the functions $u^i_\nu(x,t)$ and $v^i_\nu(x,t)$ for $i = 1,2,3$, for $\nu \geqq -1$, and for all $t \geqq 0$, all x.

Since the linear systems (2.13) and (2.14) are *singular*, they will have solutions for u^2_ν, v^2_ν and u^3_ν, v^3_ν respectively only if the "forcing vectors" involving $D_+u^2_{\nu-1}$, $D_-v^2_{\nu-1}$ and $D_+u^3_{\nu-1}$, $D_-v^3_{\nu-1}$ respectively are in the column spaces of the corresponding coefficient matrices. Hence we have the following necessary compatibility conditions:

$$\frac{-a+d-\lambda}{2} D_+ u_\nu^2 = b D_- v_\nu^2 \tag{2.18}$$

and

$$\frac{-a+d+\lambda}{2} D_+ u_\nu^3 = b D_- v_\nu^3 \tag{2.19}$$

for all $\nu \geqq -1$ and for all (x, t).

Now (2.4) and (2.5) imply with (2.16) the result

$$u_{-1}^1(x,t) = 0, \quad v_{-1}^1(x,t) = 0 \quad \text{for all } (x,t). \tag{2.20}$$

Then (2.15) and (2.20) give initially

$$u_{-1}^2(x,0) + u_{-1}^3(x,0) = 0,$$
$$v_{-1}^2(x,0) + v_{-1}^3(x,0) = 0 \quad \text{for all } x,$$

which with (2.4), (2.13) and (2.14) with $\nu = -1$ imply necessarily

$$u_{-1}^2(x,0) = u_{-1}^3(x,0) = v_{-1}^2(x,0) = v_{-1}^3(x,0) = 0. \tag{2.21}$$

But now the differential equations (2.18) and (2.19) for $\nu = -1$ can be integrated using (2.13) and (2.14) with $\nu = -1$ to give with (2.21) the result

$$\begin{gathered} u_{-1}^2(x,t) = 0, \qquad u_{-1}^3(x,t) = 0, \qquad v_{-1}^2(x,t) = 0, \\ v_{-1}^3(x,t) = 0 \quad \text{for all } (x,t). \end{gathered} \tag{2.22}$$

Having obtained u_{-1}^i and v_{-1}^i for $i = 1, 2, 3$, in the case (2.16), we show now that the relations (2.5), (2.13), (2.14), (2.15), (2.18) and (2.19) determine unique values for the functions u_ν^i and v_ν^i, $i = 1, 2, 3$, *if we have already functions* $u_{\nu-1}^i$ *and* $v_{\nu-1}^i$ *satisfying those relations.* In view of (2.16) *it is clear that* (2.5) *determines unique values for* $u_\nu^1(x, t)$ *and* $v_\nu^1(x, t)$ for all (x, t) in terms of the expressions $u_{\nu-1}^1$ and $v_{\nu-1}^1$ and the data. For example, in the case $\nu = 0$, (2.5) gives (with (2.20))

$$\begin{aligned} u_0^1(x,t) &= \frac{df(x,t) - bg(x,t)}{ad - bc}, \\ v_0^1(x,t) &= \frac{-cf(x,t) + ag(x,t)}{ad - bc}. \end{aligned} \tag{2.23}$$

Using now the values from (2.5) for u_ν^1 and v_ν^1 in (2.15) we find initially,

$$\begin{aligned} u_\nu^2(x,0) + u_\nu^3(x,0) &= \text{known function of } x, \\ v_\nu^2(x,0) + v_\nu^3(x,0) &= \text{known function of } x \end{aligned} \tag{2.24}$$

for all x. On the other hand, (2.13) and (2.14), along with the fact that $u_{\nu-1}^2$, $v_{\nu-1}^2$, $u_{\nu-1}^3$ and $v_{\nu-1}^3$ are assumed already to satisfy (2.18) and (2.19), imply that we need only impose *one* of the two relations *each* of (2.13) and (2.14), since the other two relations will then automatically hold (recall that the systems (2.13) and (2.14) are *singular* linear systems). For definiteness we impose the first relation from each of (2.13) and (2.14), where $D_+u^2_{\nu-1}$ and $D_+u^3_{\nu-1}$ are known. In particular, we obtain from those relations upon evaluating them at $t = 0$,

$$\begin{aligned} &\frac{a-d-\lambda}{2}\,u^2_\nu(x,0) + bv^2_\nu(x,0) = \text{known function},\\ &\frac{a-d+\lambda}{2}\,u^3_\nu(x,0) + bv^3_\nu(x,0) = \text{known function} \end{aligned} \tag{2.25}$$

for all x. One easily checks now that the system of four relations in $u^2_\nu(x,0)$, $u^3_\nu(x,0)$, $v^2_\nu(x,0)$ and $v^3_\nu(x,0)$ given by (2.24) and (2.25) is *nonsingular*. (The coefficient matrix is nonsingular; moreover it is independent of ν.) Hence (2.24) and (2.25) give unique initial values for the functions $u^2_\nu(x,0)$, $u^3_\nu(x,0)$, $v^2_\nu(x,0)$ and $v^3_\nu(x,0)$. For example, in the case $v = 0$ we find

$$\begin{aligned} u^2_0(x,0) &= \frac{a-d+\lambda}{2\lambda}\left[u_0(x) - \frac{df(x,0)-bg(x,0)}{ad-bc}\right]\\ &\quad + \frac{b}{\lambda}\left[v_0(x) + \frac{cf(x,0)-ag(x,0)}{ad-bc}\right],\\ u^3_0(x,0) &= \frac{-a+d+\lambda}{2\lambda}\left[u_0(x) - \frac{df(x,0)-bg(x,0)}{ad-bc}\right]\\ &\quad - \frac{b}{\lambda}\left[v_0(x) + \frac{cf(x,0)-ag(x,0)}{ad-bc}\right],\\ v^2_0(x,0) &= \frac{-a+d+\lambda}{2\lambda}\left[v_0(x) + \frac{cf(x,0)-ag(x,0)}{ad-bc}\right]\\ &\quad + \frac{c}{\lambda}\left[u_0(x) - \frac{df(x,0)-bg(x,0)}{ad-bc}\right],\\ v^3_0(x,0) &= \frac{a-d+\lambda}{2\lambda}\left[v_0(x) + \frac{cf(x,0)-ag(x,0)}{ad-bc}\right]\\ &\quad - \frac{c}{\lambda}\left[u_0(x) - \frac{df(x,0)-bg(x,0)}{ad-bc}\right]. \end{aligned} \tag{2.26}$$

It is now possible to integrate the differential equations (2.18) and (2.19) along with the first relation each from (2.13) and (2.14) in terms of the known initial values determined by (2.24) and (2.25). In fact, (2.18), (2.19) and (2.13), (2.14) specify the values of the following directional derivatives;

$$\begin{aligned} &\left(\lambda\frac{\partial}{\partial t} + (a-d)\frac{\partial}{\partial x}\right)v^2_\nu(x,t) = \text{specified (known) function},\\ &\left(\lambda\frac{\partial}{\partial t} - (a-d)\frac{\partial}{\partial x}\right)v^3_\nu(x,t) = \text{specified (known) function} \end{aligned} \tag{2.27}$$

for all (x, t). The equations (2.27) can then be integrated using the initial values from (2.24) and (2.25) to get values for v^2_ν and v^3_ν everywhere for $t \geqq 0$. Finally the first relations from each of (2.13) and (2.14) give the values for u_ν^2 and u_ν^3. For example, in the case $\nu = 0$ this procedure gives with (2.26),

$$\begin{aligned}
u_0^2(x,t) &= \frac{a-d+\lambda}{2\lambda} \\
&\quad\cdot\left[u_0\left(x-\frac{a-d}{\lambda}t\right)-\frac{df(x-t(a-d)/\lambda,0)-bg(x-t(a-d)/\lambda,0)}{ad-bc}\right] \\
&\quad+\frac{b}{\lambda}\left[v_0\left(x-\frac{a-d}{\lambda}t\right)+\frac{cf(x-t(a-d)/\lambda,0)-ag(x-t(a-d)/\lambda,0)}{ad-bc}\right], \\
v_0^2(x,t) &= \frac{-a+d+\lambda}{2\lambda} \\
&\quad\cdot\left[v_0\left(x-\frac{a-d}{\lambda}t\right)+\frac{cf(x-t(a-d)/\lambda,0)-ag(x-t(a-d)/\lambda,0)}{ad-bc}\right] \\
&\quad+\frac{c}{\lambda}\left[u_0\left(x-\frac{a-d}{\lambda}t\right)-\frac{df(x-t(a-d)/\lambda,0)-bg(x-t(a-d)/\lambda,0)}{ad-bc}\right], \\
u_0^3(x,t) &= \frac{-a+d+\lambda}{2\lambda} \\
&\quad\cdot\left[u_0\left(x+\frac{a-d}{\lambda}t\right)-\frac{df(x+t(a-d)/\lambda,0)-bg(x+t(a-d)/\lambda,0)}{ad-bc}\right] \\
&\quad-\frac{b}{\lambda}\left[v_0\left(x+\frac{a-d}{\lambda}t\right)+\frac{cf(x+t(a-d)/\lambda,0)-ag(x+t(a-d)/\lambda,0)}{ad-bc}\right], \\
v_0^3(x,t) &= \frac{a-d+\lambda}{2\lambda} \\
&\quad\cdot\left[v_0\left(x+\frac{a-d}{\lambda}t\right)+\frac{cf(x+t(a-d)/\lambda,0)-ag(x+t(a-d)/\lambda,0)}{ad-bc}\right] \\
&\quad-\frac{c}{\lambda}\left[u_0\left(x+\frac{a-d}{\lambda}t\right)-\frac{df(x+t(a-d)/\lambda,0)-bg(x+t(a-d)/\lambda,0)}{ad-bc}\right].
\end{aligned} \tag{2.28}$$

Hence we see that in the case (2.16) the above procedure can be used recursively to determine uniquely the functions u_ν^i and v_ν^i for $i = 1, 2, 3$, for all $\nu = -1, 0, 1, \cdots$ and for all (x, t) with $t \geqq 0$, such that the relations (2.4), (2.5), (2.13), (2.14) and (2.15) hold, provided of course that the data $u_0(x)$, $v_0(x)$, $f(x, t)$ and $g(x, t)$ are sufficiently smooth.

We now indicate briefly the situation in the remaining case (2.17). In this case $\lambda = a + d$ (see (2.11)) so that $\gamma_- = \beta = 0$ for all $t \geqq 0$. Hence the expressions $u_\nu^1(x, t) + u_\nu^3(x, t)$ and $v_\nu^1(x, t) + v_\nu^3(x, t)$ appearing in the expansions of (2.1) can be consolidated, and the simplest way to do this is formally to set

$$u_\nu^3(x,t) = 0, \quad v_\nu^3(x,t) = 0 \tag{2.29}$$

for all (x, t) and for all ν. The functions $u_\nu^1(x, t)$, $v_\nu^1(x, t)$, $u_\nu^2(x, t)$ and $v_\nu^2(x, t)$ are determined as before using (2.4), (2.5), (2.13) and (2.15) along with (2.29). We again have (2.18) as a compatibility condition obtained from (2.13). Now, however, (2.5) no longer automatically determines u_ν^1 and v_ν^1 since the coefficient matrix there is singular. But then we have the added compatibility condition obtained from (2.5);

$$dD_+u_{\nu-1}^1(x,t) - bD_-v_{\nu-1}^1(x,t) = \begin{cases} df(x,t) - bg(x,t) & \text{if } \nu = 0, \\ 0 & \text{otherwise} \end{cases} \tag{2.30}$$

for all (x, t). Now it is clear that the procedure goes just as before, using now (2.18) and (2.30) along with (2.4), (2.5), (2.13), (2.15) and (2.29). In particular we find for $\nu = -1$ the results (note that $\lambda = a + d$):

$$\begin{aligned} u_{-1}^1(x,t) &= \frac{1}{a+d}\int_0^t \left[df\left(x + \frac{a-d}{a+d}(t-s), s\right) - bg\left(x + \frac{a-d}{a+d}(t-s), s\right)\right] ds, \\ v_{-1}^1(x,t) &= \frac{1}{a+d}\int_0^t \left[-cf\left(x + \frac{a-d}{a+d}(t-s), s\right) + ag\left(x + \frac{a-d}{a+d}(t-s), s\right)\right] ds, \\ u_{-1}^2(x,t) &= 0, \quad v_{-1}^2(x,t) = 0 \end{aligned} \tag{2.31}$$

for any (x, t) with $t \geqq 0$. Similarly for $\nu = 0$ we obtain at any point (x_0, t_0) with $t_0 \geqq 0$,

$$\begin{aligned} u_0^1(x_0,t_0) &= \frac{du_0(x_0 + t_0(a-d)/(a+d)) - bv_0(x_0 + t_0(a-d)/(a+d))}{a+d} \\ &\quad + \frac{af(x_0 + t_0(a-d)/(a+d), 0) + bg(x_0 + t_0(a-d)/(a+d), 0)}{(a+d)^2} \\ &\quad + \frac{1}{a+d}\int_0^{t_0} \Big[df(x,t) - bg(x,t) \\ &\quad + D_-f(x,t) - D_-D_+u_{-1}^1(x,t)\Big]\Big|_{\substack{x = x_0 + (t_0 - s)(a-d)/(a+d) \\ t = s}} ds, \\ u_0^2(x_0,t_0) &= \frac{au_0(x_0 - t_0(a-d)/(a+d)) + bv_0(x_0 - t_0(a-d)/(a+d))}{a+d} \\ &\quad - \frac{af(x_0 - t_0(a-d)/(a+d), 0) + bg(x_0 - t_0(a-d)/(a+d), 0)}{(a+d)^2} \end{aligned} \tag{2.32}$$

and associated values for $v_0^1(x_0, t_0)$ and $v_0^2(x_0, t_0)$ obtained directly from (2.5) and (2.32).

Hence, also in the case (2.17) the above procedure can be used recursively to determine uniquely the functions u_ν^i and v_ν^i for all i, all ν and all (x, t), such that the relations (2.4), (2.5), (2.13), (2.15) and (2.29) hold.

We turn next to a proof of the asymptotic correctness of the resulting expansions (2.1) constructed using this procedure.

3. Proof of the asymptotic correctness of the expansions. Let $u(x, t; \varepsilon)$ and $v(x, t; \varepsilon)$ solve the initial value problem (1.1), (1.2) for $t \geqq 0$ and for any fixed $\varepsilon > 0$, and assume that the initial functions u_0 and v_0 of (1.2) are $2N + 2$ times continuously differentiable. Assume that the forcing terms $f = f(x, t)$ and $g = g(x, t)$ have $2N + 3$ continuous x-derivatives and $N + 1$ continuous t-derivatives. Then one easily checks (by induction) that the functions $u_\nu^i(x, t)$ and $v_\nu^i(x, t)$ constructed in §2 exist and are continuous on $t \geqq 0$ for $i = 1, 2, 3$ and for all $\nu = -1, 0, 1, \cdots, N, N + 1$, with u_N^i and v_N^i being continuously differentiable. Here N is any fixed nonnegative integer.

Let now functions $R_{N+1}(x, t; \varepsilon)$ and $S_{N+1}(x, t; \varepsilon)$ be defined for $t \geqq 0$, all x, and $\varepsilon > 0$ by the relations:

$$
\begin{aligned}
u(x, t; \varepsilon) &= \sum_{\nu=-1}^{N} [u_\nu^1(x, t) + e^{-\alpha(x,t)/\varepsilon} u_\nu^2(x, t) + e^{-\beta(x,t)/\varepsilon} u_\nu^3(x, t)] \cdot \varepsilon^\nu \\
&\quad + R_{N+1}(x, t; \varepsilon) \cdot \varepsilon^{N+1}, \\
v(x, t; \varepsilon) &= \sum_{\nu=-1}^{N} [v_\nu^1(x, t) + e^{-\alpha(x,t)/\varepsilon} v_\nu^2(x, t) + e^{-\beta(x,t)/\varepsilon} v_\nu^3(x, t)] \cdot \varepsilon^\nu \\
&\quad + S_{N+1}(x, t; \varepsilon) \cdot \varepsilon^{N+1},
\end{aligned} \tag{3.1}
$$

where α and β are given by (2.10), (2.11) and (2.12). We have then the following result.

THEOREM. *The functions R_{N+1} and S_{N+1} defined by* (3.1) *are uniformly bounded on compact subsets of $-\infty < x < +\infty$, $t \geqq 0$ as $\varepsilon \to 0$.*

Proof. Inserting (3.1) into the system (1.1) and using (2.5), (2.13) and (2.14), we find that R_{N+1} and S_{N+1} satisfy the system

$$
\begin{aligned}
\varepsilon D_+ R_{N+1} + aR_{N+1} + bS_{N+1} + D_+ u_N^1 + e^{-\alpha/\varepsilon} D_+ u_N^2 + e^{-\beta/\varepsilon} D_+ u_N^3 &= 0, \\
\varepsilon D_- S_{N+1} + cR_{N+1} + dS_{N+1} + D_- v_N^1 + e^{-\alpha/\varepsilon} D_- v_N^2 + e^{-\beta/\varepsilon} D_- v_N^3 &= 0,
\end{aligned} \tag{3.2}
$$

where as before $D_\pm = \partial/\partial t \pm \partial/\partial x$. In order to prove that R_{N+1} and S_{N+1} remain bounded as $\varepsilon \to 0$, we replace $D_+ u_N^1$ and $D_- v_N^1$ in (3.2) with their equivalents in terms of u_{N+1}^1 and v_{N+1}^1 obtained from (2.5) with $\nu = N + 1$, giving

$$
\begin{aligned}
&\varepsilon D_+(R_{N+1} - u_{N+1}^1) + a(R_{N+1} - u_{N+1}^1) + b(S_{N+1} - v_{N+1}^1) \\
&\qquad + \varepsilon D_+ u_{N+1}^1 + e^{-\alpha/\varepsilon} D_+ u_N^2 + e^{-\beta/\varepsilon} D_+ u_N^3 = 0, \\
&\varepsilon D_-(S_{N+1} - v_{N+1}^1) + c(R_{N+1} - u_{N+1}^1) + d(S_{N+1} - v_{N+1}^1) \\
&\qquad + \varepsilon D_- v_{N+1}^1 + e^{-\alpha/\varepsilon} D_- v_N^2 + e^{-\beta/\varepsilon} D_- v_N^3 = 0,
\end{aligned}
$$

or, defining $\mathscr{R}(x, t)$ and $\mathscr{S}(x, t)$ by the formulas

$$
\begin{aligned}
\mathscr{R}(x, t) &= R_{N+1}(x, t; \varepsilon) - u_{N+1}^1(x, t), \\
\mathscr{S}(x, t) &= S_{N+1}(x, t; \varepsilon) - v_{N+1}^1(x, t),
\end{aligned} \tag{3.3}
$$

where we have suppressed the dependency of $\mathscr{R}$ and $\mathscr{S}$ on N and ε, we have

$$
\begin{aligned}
\varepsilon D_+ \mathscr{R} + a\mathscr{R} + b\mathscr{S} + \varepsilon D_+ u_{N+1}^1 + e^{-\alpha/\varepsilon} D_+ u_N^2 + e^{-\beta/\varepsilon} D_+ u_N^3 &= 0, \\
\varepsilon D_- \mathscr{S} + c\mathscr{R} + d\mathscr{S} + \varepsilon D_- v_{N+1}^1 + e^{-\alpha/\varepsilon} D_- v_N^2 + e^{-\beta/\varepsilon} D_- v_N^3 &= 0.
\end{aligned} \tag{3.4}
$$

Setting $t = 0$ in (3.1) and using (1.2), (2.15) and (3.3), we find as initial conditions for $\mathscr{R}$ and $\mathscr{S}$ the conditions

$$\mathscr{R}(x,0) = -u_{N+1}^1(x,0), \qquad \mathscr{S}(x,0) = -v_{N+1}^1(x,0), \tag{3.5}$$

independent of $\varepsilon > 0$. (Note that $N \geqq 0$.)

We now integrate separately the equations of (3.4) along the appropriate characteristics to find with (3.5) the equivalent system of integral equations:

$$\begin{aligned}
&e^{ta/\varepsilon}\mathscr{R}(x,t) + u_{N+1}^1(x-t,0)\\
&\qquad + \frac{b}{\varepsilon}\int_0^t e^{sa/\varepsilon}\mathscr{S}(x-t+s,s)\,ds + \int_0^t e^{sa/\varepsilon}\frac{du_{N+1}^1(x-t+s,s)}{ds}\,ds\\
&\qquad + \frac{1}{\varepsilon}\int_0^t \Bigg[e^{s(a-d-\lambda)/2\varepsilon}\frac{du_N^2(x-t+s,s)}{ds}\\
&\qquad\qquad + e^{s(a-d+\lambda)/2\varepsilon}\frac{du_N^3(x-t+s,s)}{ds}\Bigg]\,ds = 0,\\
&e^{td/\varepsilon}\mathscr{S}(x,t) + v_{N+1}^1(x+t,0)\\
&\qquad + \frac{c}{\varepsilon}\int_0^t e^{sd/\varepsilon}\mathscr{R}(x+t-s,s)\,ds + \int_0^t e^{sd/\varepsilon}\frac{dv_{N+1}^1(x+t-s,s)}{ds}\,ds\\
&\qquad + \frac{1}{\varepsilon}\int_0^t \Bigg[e^{s(-a+d-\lambda)/2\varepsilon}\frac{dv_N^2(x+t-s,s)}{ds}\\
&\qquad\qquad + e^{s(-a+d+\lambda)/2\varepsilon}\frac{dv_N^3(s+t-s,s)}{ds}\Bigg]\,ds = 0,
\end{aligned} \tag{3.6}$$

where we have used (2.10), (2.11) and (2.12). We can now use (3.6) to prove that $\mathscr{R}$ and $\mathscr{S}$ remain uniformly bounded on compact sets as $\varepsilon \to 0$; the same result will then follow for R_{N+1} and S_{N+1} from (3.3).

It suffices for this purpose to consider an arbitrary set J of the form (see Fig. 3)

$$J = \{(x,t) : |x - \bar{x}| \leqq L + T - t, \text{ all } 0 \leqq t \leqq T\} \tag{3.7}$$

for arbitrary positive numbers L and T and

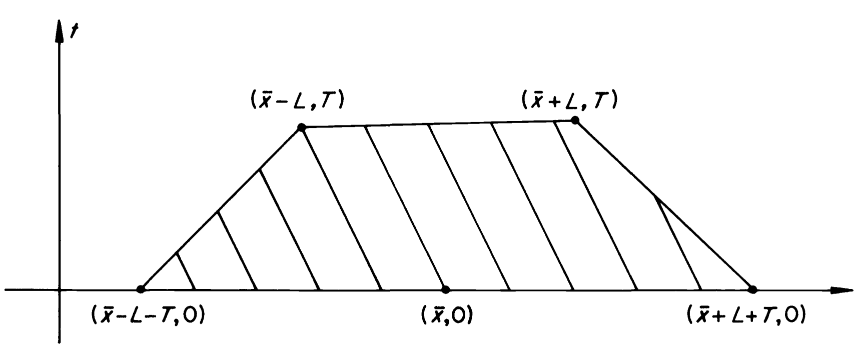

FIG. 3

for any number $\bar{x}$. For such a set J the assumed smoothness of the data implies the existence of a constant K such that the expressions $u_{N+1}^1(x - t, 0)$ and $v_{N+1}^1(x + t, 0)$ and all derivatives of u_N^i, v_N^i, u_{N+1}^1 and v_{N+1}^1 appearing in (3.6) *are bounded in magnitude by* K, uniformly on J.

Considering first the case (2.16), and increasing the magnitude of K, we obtain easily from (3.6) the estimates

$$\begin{aligned} e^{ta/\varepsilon}|\mathscr{R}(x,t)| &\leqq \frac{|b|}{\varepsilon}\int_0^t e^{sa/\varepsilon}|\mathscr{S}(x - t + s, s)|\, ds \\ &\quad + K\cdot\left(1 + \varepsilon e^{ta/\varepsilon} + \frac{1 - e^{-t(\lambda + d - a)/2\varepsilon}}{\lambda + d - a} + \frac{e^{t(\lambda - d + a)/2\varepsilon} - 1}{\lambda - d + a}\right), \\ e^{td/\varepsilon}|\mathscr{S}(x,t)| &\leqq \frac{|c|}{\varepsilon}\int_0^t e^{sd/\varepsilon}|\mathscr{R}(x + t - s, s)|\, ds \\ &\quad + K\cdot\left(1 + \varepsilon e^{td/\varepsilon} + \frac{1 - e^{-t(\lambda - d + a)/2\varepsilon}}{\lambda - d + a} + \frac{e^{t(\lambda + d - a)/2\varepsilon} - 1}{\lambda + d - a}\right), \end{aligned} \tag{3.8}$$

where (1.4) and (2.11) imply

$$|a - d| \leqq \lambda, \tag{3.9}$$

and where (2.16) implies

$$\lambda < a + d. \tag{3.10}$$

It is understood in (3.8) that the expression $(e^x - 1)/x$ takes the value 1 at $x = 0$.

The following calculation is somewhat simplified if we first assume (see (3.9))

$$|a - d| < \lambda, \tag{3.11}$$

or, what is the same thing, $bc > 0$. In this case we can replace (3.8) with the following inequalities by increasing the magnitude of K:

$$\begin{aligned} e^{ta/\varepsilon}|\mathscr{R}(x,t)| &\leqq \frac{|b|}{\varepsilon}\int_0^t e^{sa/\varepsilon}|\mathscr{S}(x - t + s, s)|\, ds \\ &\quad + K\cdot(1 + \varepsilon e^{ta/\varepsilon} + e^{t(\lambda - d + a)/2\varepsilon}), \\ e^{td/\varepsilon}|\mathscr{S}(x,t)| &\leqq \frac{|c|}{\varepsilon}\int_0^t e^{sd/\varepsilon}|\mathscr{R}(x + t - s, s)|\, ds \\ &\quad + K\cdot(1 + \varepsilon e^{td/\varepsilon} + e^{t(\lambda + d - a)/2\varepsilon}). \end{aligned} \tag{3.12}$$

Defining functions $R(t)$ and $S(t)$ for $0 \leqq t \leqq T$ (see (3.7)) as

$$\begin{aligned} R(t) &= \max_{\text{all } (x,t) \text{ in } J} |\mathscr{R}(x,t)|, \\ S(t) &= \max_{\text{all } (x,t) \text{ in } J} |\mathscr{S}(x,t)|, \end{aligned} \tag{3.13}$$

we then find from (3.12) and (3.13) the estimates

$$
\begin{aligned}
e^{ta/\varepsilon}R(t) &\leqq \frac{|b|}{\varepsilon}\int_0^t e^{sa/\varepsilon}S(s)\,ds \\
&\qquad + K\cdot(1 + \varepsilon e^{ta/\varepsilon} + e^{t(\lambda-d+a)/2\varepsilon}), \\
e^{td/\varepsilon}S(t) &\leqq \frac{|c|}{\varepsilon}\int_0^t e^{sd/\varepsilon}R(s)\,ds \\
&\qquad + K\cdot(1 + \varepsilon e^{td/\varepsilon} + e^{t(\lambda+d-a)/2\varepsilon}).
\end{aligned}
\tag{3.14}
$$

Using the second inequality of (3.14) in the right side of the first inequality of (3.14), we obtain with (1.4) and (3.11) (again increasing the size of K),

$$
\begin{aligned}
e^{ta/\varepsilon}R(t) &\leqq \frac{ad}{\varepsilon^2}\int_0^t e^{s(a-d)/\varepsilon}\int_0^s e^{\sigma d/\varepsilon}R(\sigma)\,d\sigma\,ds \\
&\qquad + K\left\{1 + \varepsilon e^{ta/\varepsilon} + e^{t(\lambda-d+a)/2\varepsilon} + \frac{e^{t(a-d)/\varepsilon}-1}{a-d}\right\}.
\end{aligned}
\tag{3.15}
$$

Hence, if we define a function $P(t)$ by the formula (again suppressing the dependency on $\varepsilon > 0$)

$$
P(t) = \int_0^t e^{s(a-d)/\varepsilon}\int_0^s e^{\sigma d/\varepsilon}R(\sigma)\,d\sigma\,ds \tag{3.16}
$$

for $0 \leqq t \leqq T$, with

$$
P'' = e^{ta/\varepsilon}R + \frac{a-d}{\varepsilon}P',
$$

we then find from (3.15) and (3.16) the differential inequality

$$
P''(t) \leqq \frac{ad}{\varepsilon^2}P(t) + \frac{a-d}{\varepsilon}P'(t) + K\left\{1 + \varepsilon e^{ta/\varepsilon} + e^{t(\lambda-d+a)/2\varepsilon} + \frac{e^{t(a-d)/\varepsilon}-1}{a-d}\right\}. \tag{3.17}
$$

On the other hand we find directly by integrations by parts the identity

$$
P(t) = \frac{\varepsilon}{a+d}\int_0^t (e^{(t-s)a/\varepsilon} - e^{-(t-s)d/\varepsilon})\left(P''(s) - \frac{a-d}{\varepsilon}P'(s) - \frac{ad}{\varepsilon^2}P(s)\right)ds \tag{3.18}
$$

since $P(0) = P'(0) = 0$ as follows from (3.16). Since $e^{(t-s)a/\varepsilon} - e^{-(t-s)d/\varepsilon}$ is non-negative for $0 \leqq s \leqq t$, there follows from (3.17) and (3.18) the estimate

$$
\begin{aligned}
P(t) &\leqq \frac{K\cdot\varepsilon}{a+d}\int_0^t (e^{(t-s)a/\varepsilon} - e^{-(t-s)d/\varepsilon}) \\
&\qquad \cdot\left(1 + \varepsilon e^{sa/\varepsilon} + e^{s(\lambda-d+a)/2\varepsilon} + \frac{e^{s(a-d)/\varepsilon}-1}{a-d}\right)ds.
\end{aligned}
\tag{3.19}
$$

If $a \neq d$, there follows easily from (3.19), (1.3) and (3.10), the estimate

$$
P(t) \leqq \text{const.}\ \varepsilon^2(1 + t)e^{ta/\varepsilon},
$$

which with (3.15) and (3.16) gives

$$R(t) \leqq \text{const.}\,(1 + T) \tag{3.20}$$

for all $0 \leqq t \leqq T$, with the constant independent of $\varepsilon > 0$ as $\varepsilon \to 0$. On the other hand, if $a = d$, then (3.19) leads to the estimate

$$P(t) \leqq \text{const.}\,\left\{\varepsilon^2(1 + t)e^{ta/\varepsilon} + \varepsilon t\right\}, \tag{3.21}$$

which with (3.15) and (3.16) gives

$$R(t) \leqq \text{const.}\,\left\{1 + T + \frac{ta}{\varepsilon}e^{-ta/\varepsilon}\right\} \tag{3.22}$$

for all $0 \leqq t \leqq T$ and all small $\varepsilon > 0$. This last result (3.22) again gives a uniform bound on $R(t)$ as $\varepsilon \to 0$.

Hence if (3.11) holds, we find in every case a bound

$$R(t) \leqq \text{const.} \tag{3.23}$$

for all $0 \leqq t \leqq T$ and for all small $\varepsilon \to 0$.

The second inequality of (3.14) then gives a similar uniform bound for $S(t)$,

$$S(t) \leqq \text{const.}, \tag{3.24}$$

where we again used (3.10).

To finish the case (2.16), we only need consider the remaining special case $\lambda = |a - d|$ (cf. (3.9) and (3.11)). In this case the same type calculation beginning with (3.8) gives

$$e^{ta/\varepsilon}R(t) \leqq \frac{ad}{\varepsilon^2}\int_0^t e^{s(a-d)/\varepsilon}\int_0^s e^{\sigma d/\varepsilon}R(\sigma)\,d\sigma\,ds + K(1 + t/\varepsilon + \varepsilon e^{ta/\varepsilon} + e^{t(a-d)/\varepsilon}),$$

which as before leads in this case to the estimate (3.21) for $P(t)$ (see (3.16)), giving again the bounds (3.23) and (3.24).

Hence we finally need only consider the case (2.17). Since (2.29) holds in this case, we obtain from (3.6) and (2.11) the estimates

$$e^{ta/\varepsilon}|\mathscr{R}(x, t)| \leqq \frac{|b|}{\varepsilon}\int_0^t e^{sa/\varepsilon}|\mathscr{S}(x - t + s, s)|\,ds + K\cdot(1 + \varepsilon e^{ta/\varepsilon}),$$

$$e^{td/\varepsilon}|\mathscr{S}(x, t)| \leqq \frac{|c|}{\varepsilon}\int_0^t e^{sd/\varepsilon}|\mathscr{R}(x + t - s, s)|\,ds + K\cdot(1 + \varepsilon e^{td/\varepsilon})$$

uniformly for all (x, t) in J and for a fixed constant K independent of $\varepsilon > 0$. Again, the same type calculation used above (only simpler in this case) leads to uniform bounds of the type (3.23) and (3.24) for the functions $R(t)$ and $S(t)$ defined as before by (3.13).

Hence in every case the uniform estimates (3.23) and (3.24) hold, giving with (3.13) analogous uniform estimates for $|\mathscr{R}(x, t)|$ and $|\mathscr{S}(x, t)|$ for all (x, t) in J and uniformly for all small $\varepsilon > 0$. The theorem then follows directly from (3.3) and the boundedness (on compact sets) of the functions u_{N+1}^1 and v_{N+1}^1.

4. Discussion of results. The construction in § 2 of the formal asymptotic expansions along with the theorem of § 3 show in particular that the solution functions u and v of the Cauchy problem (1.1), (1.2) (subject to (1.3), (1.4)) satisfy

$$\begin{aligned} u(x, t; \varepsilon) &= \frac{1}{\varepsilon} u_{-1}^1(x, t) + u_0^1(x, t) + e^{-\alpha/\varepsilon} u_0^2(x, t) + e^{-\beta/\varepsilon} u_0^3(x, t) + O(\varepsilon), \\ v(x, t; \varepsilon) &= \frac{1}{\varepsilon} v_{-1}^1(x, t) + v_0^1(x, t) + e^{-\alpha/\varepsilon} v_0^2(x, t) + e^{-\beta/\varepsilon} v_0^3(x, t) + O(\varepsilon), \end{aligned} \tag{4.1}$$

where

$$\alpha = \frac{a + d + \sqrt{(a - d)^2 + 4bc}}{2} t,$$

$$\beta = \frac{a + d - \sqrt{(a - d)^2 + 4bc}}{2} t$$

and where the remainder terms in (4.1) are $O(\varepsilon)$ uniformly on compact sets in $-\infty < x < +\infty, t \geqq 0$ as $\varepsilon \to 0$.

If the coefficient matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is *nonsingular*, so that (2.16) holds, then α and β are positive for each fixed $t > 0$, while u_{-1}^1 and v_{-1}^1 vanish identically (see (2.20)). Since u_0^1 and v_0^1 are given in this case by (2.23), we find from (4.1) for any fixed $t > 0$ the result

$$\begin{aligned} \lim_{\varepsilon \to 0} u(x, t; \varepsilon) &= \frac{df(x, t) - bg(x, t)}{ad - bc}, \\ \lim_{\varepsilon \to 0} v(x, t; \varepsilon) &= \frac{-cf(x, t) + ag(x, t)}{ad - bc}, \end{aligned} \tag{4.2}$$

so that u and v tend in this case to the unique solution of the reduced system (1.6).

If the coefficient matrix is *singular*, so that (2.17) holds, then $\alpha = (a + d)t$ and $\beta \equiv 0$, while (2.29) gives $u_\nu^3 \equiv 0$ and $v_\nu^3 \equiv 0$. Hence in this case it follows that *if the reduced system of linear equations* (1.6) *has no solution for points* (x, t) *on the line segment* (1.8) *for a given point* (x_0, t_0), *then the values* $u(x_0, t_0; \varepsilon)$ *and* $v(x_0, t_0; \varepsilon)$ *become unbounded like* ε^{-1} *as* $\varepsilon \to 0$ for fixed $t_0 > 0$. In fact, (4.1) and (2.31) show in this case that

$$\begin{aligned} &\lim_{\varepsilon \to 0} \varepsilon \cdot u(x_0, t_0; \varepsilon) \\ &\quad = \frac{1}{a + d} \int_0^{t_0} \left[df\left(x_0 + \frac{a - d}{a + d}(t_0 - s), s\right) - bg\left(x_0 + \frac{a - d}{a + d}(t_0 - s), s\right) \right] ds, \\ &\lim_{\varepsilon \to 0} \varepsilon \cdot v(x_0, t_0; \varepsilon) \\ &\quad = \frac{1}{a + d} \int_0^{t_0} \left[-cf\left(x_0 + \frac{a - d}{a + d}(t_0 - s), s\right) + ag\left(x_0 + \frac{a - d}{a + d}(t_0 - s), s\right) \right] ds, \end{aligned} \tag{4.3}$$

where these limits will in general be *nonzero.*

Finally if the coefficient matrix is singular and *if the reduced system* (1.6) *has solutions for all points* (x, t) *on the line segment* (1.8) (hence infinitely many), *then the limiting expressions in* (4.3) *vanish, and* (4.1), (2.32) *and* (2.5) *give in this case* (after some calculation, using $cf = ag$, $df = bg$),

$$\begin{aligned}
&\lim_{\varepsilon\to 0} u(x_0, t_0; \varepsilon) \\
&\quad = \frac{f(x_0, t_0) + du_0(x_0 + t_0(a-d)/(a+d)) - bv_0(x_0 + t_0(a-d)/(a+d))}{a+d} \\
&\qquad - \frac{2d}{(a+d)^2}\int_0^{t_0} f_x\left(x_0 + \frac{a-d}{a+d}(t_0 - s), s\right) ds, \\
&\lim_{\varepsilon\to 0} v(x_0, t_0; \varepsilon) \\
&\quad = \frac{g(x_0, t_0) - cu_0(x_0 + t_0(a-d)/(a+d)) + av_0(x_0 + t_0(a-d)/(a+d))}{a+d} \\
&\qquad + \frac{2a}{(a+d)^2}\int_0^{t_0} g_x\left(x_0 + \frac{a-d}{a+d}(t_0 - s), s\right) ds
\end{aligned} \tag{4.4}$$

for any fixed $t_0 > 0$, where $f_x = \partial f/\partial x$ and $g_x = \partial g/\partial x$. The limit functions in (4.4) are easily seen to satisfy the reduced system (1.6) in this case. Hence among the infinitely many solutions of (1.6) the Cauchy problem (1.1), (1.2) distinguishes in this case the particular solution given by (4.4) as $\varepsilon \to 0$. Note that the limiting solutions given by (4.4) depend on the data restricted to *any neighborhood* of the line segment (1.8) in view of the spatial derivatives f_x and g_x in (4.4).

Acknowledgment. The author wishes to express his appreciation to a referee for several valuable comments and suggestions concerning an earlier version of this paper.

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SIAM J. MATH. ANAL.
Vol. 2, No. 3, August 1971

SINGULAR POINTS OF STURM-LIOUVILLE SERIES*

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Abstract. The results of Gilbert and Howard on the location of singularities of Sturm-Liouville series $\sum b_n u_n(x)$ are extended to the cases when the coefficients satisfy: (i) $b_n = O(n^p)$, p an integer, and (ii) $\limsup |b_n|^{1/n} > 1$. In the first case the series converges to a distribution and in the second to an ultra-distribution. The analytic representations $\hat{u}_n^{\pm}(z)$ of the u_n are defined as appropriated second solutions to the differential equation. The series $\sum b_n \hat{u}_n^{\pm}(z)$ is then shown to converge to an analytic function which is the analytic representation of $\sum b_n u_n(x)$. The singular points of this analytic function are compared to those of the function given by $\sum b_n t^n$.

1. Introduction. In a 1956 paper, which has had many imitators (including this one), Nehari [1] devised a method for locating the singular points of an analytic function given by a Legendre series. Nehari's method was in turn an adaptation of a method used earlier by Hadamard for power series and Watson for Neumann series. This method has been exploited extensively by Gilbert in a series of papers dealing with the analytic properties of certain partial differential equations (see [2] for details and bibliography). Recently Gilbert and Howard [3] used the method for a function given by a Sturm-Liouville series arising from a differential operator with holomorphic functions as coefficients. This paper presents a procedure for doing the same when the function to which the series converges is not analytic (in fact, is not even a function). The procedure is a modification of Nehari's which uses the "multiplication of singularities" artifice of Hadamard, in which integral operators are found which map the given function into a function given by an associated power series and inversely.

The Sturm-Liouville series we consider are the series of eigenfunctions of the system

$$\begin{aligned} &\frac{d^2u}{dx^2} + \lambda^2 u = q(x)u, \qquad x \in (0, \pi), \\ &u'(0) - hu(0) = u'(\pi) + Hu(\pi) = 0, \end{aligned} \tag{1}$$

where $q(x)$ is holomorphic on the entire plane and is real on the real axis.

In the work cited above, Gilbert and Howard studied the location of singularities of functions given by series

$$\sum a_n u_n,$$

where $\limsup |a_n|^{1/n} < 1$ and the u_n are the eigenfunctions of the system. We shall study two additional cases, depending on the behavior of the coefficients:

(i) $b_n = O(n^p)$, p an integer,
(ii) $\limsup |c_n|^{1/n} > 1$.

In neither case does the series ($\sum b_n u_n$ or $\sum c_n u_n$) converge to a holomorphic function. Thus it is impossible to talk of singular points of functions given by

* Received by the editors April 21, 1970, and in revised form October 5, 1970.
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the series. However we shall see that an associated series $\sum b_n \hat{u}_n$ or $\sum c_n \hat{u}_n$, in which the $\hat{u}_n$ are other solutions to the equation corresponding to the same eigenvalues, does converge to a holomorphic function in part of the plane. It is the corresponding analytic function whose singularities we locate. Moreover, this analytic function will turn out to be the analytic representation of the function (or generalized function) given by the Sturm–Liouville series.

The normalized eigenfunctions $u_n(x)$ have the asymptotic expression:

$$u_n(x) = \sqrt{\frac{2}{\pi}} \cos nx \, \{1 + O(n^{-2})\} + \sin nx \, \{\beta(x) n^{-1} + O(n^{-2})\}, \qquad n = 1, 2, \cdots,$$

where $\beta(x)$ is continuous on $[0, \pi]$ (see Ince [4, p. 273]). We make the additional assumption that $u_n(z)$ satisfies:

(i) $|u_n(z)| \leqq C_1 e^{\alpha_1 \lambda_n^{\frac{1}{2}} |\mathrm{Im}\, z|}$ for some constants C_1 and α_1 greater than 0, and that for $|\mathrm{Im}\, z|$ sufficiently large, $|u_n(z)| \geqq C_2 e^{\alpha_2 |\mathrm{Im}\, z| \lambda_n^{\frac{1}{2}}}$ for other constants $C_2 > 0$ and $2\alpha_2 > \alpha_1$.

(ii) $\sum u_n(z) t^n$ has a singular point at most at whichever of the points $t = e^{\pm iz}$ is on its circle of convergence.

Both (i) and (ii) are valid (trivially) for systems with constants q. They may also be shown to be true for a large class of Sturm–Liouville systems (see Gilbert and Howard [3] for more details).

2. Properties of a second eigenfunction. We define $\hat{u}_n^+$ (respectively $\hat{u}_n^-$) to be an eigenfunction of the differential operator vanishing at $i\infty$ (respectively $-i\infty$). Since a second solution to the differential equation (1) is given in terms of the first by

$$\hat{u}(z) = u(z) \int_a^z \frac{c}{u^2(\xi)} \, d\xi,$$

we see that

$$\hat{u}_n^+(z) = u_n(z) \int_{i\infty}^z \frac{d_n}{u_n^2(\xi)} \, d\xi \quad \text{and} \quad \hat{u}_n^-(z) = u_n(z) \int_{-i\infty}^z \frac{d_n}{u_n^2(\xi)} \, d\xi,$$

where d_n is a constant to be determined.

Clearly because of the assumption (i) these integrals exist and are independent of path for Im z sufficiently large in the first integral and sufficiently small in the second. They may be extended to any complex z which is not a zero of u_n by extending the integration along a contour which avoids those zeros. The value will be the same for all such contours since the residue of $1/u_n^2$ at such a zero is zero.

LEMMA. *Let*

$$d_n^{-1} = \int_{i\infty}^{-i\infty} \frac{1}{u_n^2(\xi)} \, d\xi.$$

Then d_n is a constant which satisfies $|d_n| \leqq d\lambda_n$ for all n, some $d > 0$, and

(i) $$\hat{u}_n^+(x) - \hat{u}_n^-(x) = u_n(x),$$

(ii) $$\hat{u}_n^{\pm}(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{u_n(x)}{x - z}\,dx \quad \begin{bmatrix} \operatorname{Im} z > 0 \\ \operatorname{Im} z < 0 \end{bmatrix}.$$

To prove that d_n is a constant, we observe that $\hat{u}_n^+ - \hat{u}_n^-$ is a solution whose Wronskian with u_n is 0. Thus $\hat{u}_n^+ - \hat{u}_n^-$ is a multiple of u_n. But

$$\hat{u}_n^+(x) - \hat{u}_n^-(x) = u_n(x)\int_{i\infty}^{-i\infty} \frac{d_n}{u_n^2(\xi)}\,d\xi,$$

where the path of integration crosses the real axis at x. Since the integral $\int_{i\infty}^{-i\infty} (d_n/u_n^2(\xi))\,d\xi$ is constant in intervals containing no zeros of u_n, it must equal the constant factor of the multiple of u_n. Thus it is constant everywhere and equal to 1.

It may be observed that d_n is the Wronskian of $\hat{u}_n$ with u_n by differentiating $\hat{u}_n/u_n = \int_{i\infty}^{z}(d_n/u_n^2(\xi))\,d\xi$. Now assuming that part (ii) has been proved, we use it to get an estimate for d_n, namely,

$$|d_n| = \left|\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{u_n(x)u_n'(z) - u_n(z)u_n'(x)}{x - z}\,dx\right| \leqq \lambda_n d$$

for some constant d.

To prove that (ii) holds we observe first that for $\operatorname{Im} z > 0$,

$$\hat{u}_n^+(z) = \frac{1}{2\pi i}\left\{\int_{-M}^{M} + \int_{R_M^+}\right\} \frac{\hat{u}_n^+(x)}{x - z}\,dx,$$

where R_M^+ is a semicircle of radius M and center 0 which, together with $[-M, M]$, encloses z. On the other hand we have

$$0 = \frac{1}{2\pi i}\left\{\int_{-M}^{M} + \int_{R_M^-}\right\} \frac{\hat{u}_n^-(x)}{x - z}\,dx,$$

where R_M^- is a semicircle in the lower half-plane having the same radius and center. Now let $M \to \infty$. Then

$$\left|\int_{R_M^{\pm}} \frac{\hat{u}_n^{\pm}(x)}{x - z}\,dx\right| < M\int_0^{\pi} \frac{e^{-\alpha_1\lambda_n^2 M\sin\theta}}{|Me^{i\theta} - z|}\,d\theta$$

$$\leqq (\text{const.})\int_0^{\pi/2} e^{-2\alpha_1\lambda_n^2\theta M/\pi}\,d\theta \to 0.$$

Thus we conclude that

$$\hat{u}_n^+(z) - 0 = \frac{1}{2\pi i}\left\{\int_{-\infty}^{\infty} \frac{\hat{u}_n^+(x)}{x - z}\,dx - \int_{-\infty}^{\infty} \frac{\hat{u}_n^-(x)}{x - z}\,dx\right\}$$

$$= \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{u_n(x)}{x - z}\,dx.$$

Similarly, we reach the same conclusion for $\hat{u}_n^-$.

3. Singularity theorems. We have two theorems concerning the singular points of Sturm–Liouville series, one for each of the cases mentioned in the introduction. In both theorems $\{u_n\}$ denotes the orthonormal system of eigenfunctions discussed in the introduction and $\{\hat{u}_n^{\pm}\}$ the second solution to the differential equation discussed in §2.

THEOREM 1. *Let $\{b_n\}$ be a sequence of complex numbers such that $b_n = O(n^p)$ for some integer p; furthermore, let $\varphi(\xi) = \sum b_n \xi^n, |\xi| < 1$. Then:*

(i) *the Sturm–Liouville series $\sum b_n u_n$ converges to a distribution f on R^1;*

(ii) *the series $\sum b_n \hat{u}_n^{\pm}(z)$ converges to a function $\hat{f}^{\pm}(z)$, holomorphic for $\operatorname{Im} z > 0\,(+)$ or $\operatorname{Im} z < 0\,(-)$ which is the analytic representation of f;*

(iii) *the function $\varphi(\xi)$ is singular at $\xi = \alpha, |\alpha| = 1, \alpha \neq \pm 1$ if and only if either $\hat{f}^{+}(z)$ or $\hat{f}^{-}(z)$ has a singular point in $(0, \pi)$ at $z = \beta$, where $\cos \beta = \frac{1}{2}(\alpha + 1/\alpha)$.*

THEOREM 2. *Let $\{c_n\}$ be a sequence of complex numbers such that $\lim_{n\to\infty} \sup |c_n|^{1/n} = 1/\rho,\ \rho < 1$; furthermore, let $\psi(\xi) = \sum c_n \xi^n,\ |\xi| < \rho$. Then:*

(i) *the Sturm–Liouville series $\sum c_n u_n$ converges to the ultradistribution g (in the sense of Z') and moreover g is an analytic functional on Z;*

(ii) *the series $\sum c_n \hat{u}_n^{\pm}$ converges to a function $\hat{g}^{\pm}(z)$ holomorphic for $\operatorname{Im} z > c$, a constant $(+)$, or $\operatorname{Im} z < c\ (-)$ which corresponds to the analytic functional g;*

(iii) *the function $\psi(\xi)$ is singular at $\xi = \alpha,\ |\alpha| = \rho$ $(\arg \alpha \neq 0, \pi)$, if and only if either $\hat{g}^{+}(z)$ or $\hat{g}^{-}(z)$ has a singular point at $z = \beta$, where $\cos \beta = \frac{1}{2}(\alpha + 1/\alpha)$.*

4. Proof of theorems. The statement (i) of Theorem 1 is known (and easy to prove). The proof of Theorem 2 is more difficult. In order to show that the series converges, we shall show that $\int_{-\infty}^{\infty} u_n \varphi \, dx = 0$ for each test function φ in Z for n sufficiently large (see [8] for properties of this space).

The test functions φ in Z are entire functions satisfying

$$|z|^p |\varphi(z)| < M_p e^{a|\operatorname{Im} z|}$$

for some positive constants a and M_p and all integers $p \geqq 0$. We may write, for n sufficiently large (to be specified more precisely later),

$$\int_{-\infty}^{\infty} u_n \, dx = \int_{-\infty}^{\infty} (\hat{u}_n^+ - \hat{u}_n^-) \, dx = \int_{-\infty}^{\infty} \hat{u}_n^+ \, dx - \int_{-\infty}^{\infty} \hat{u}_n^- \, dx$$
$$= \int_{C^+} \hat{u}_n^+ \, d\xi - \int_{C^-} \hat{u}_n^- \, d\xi,$$

where C^+ and C^- are contours above and below the x-axis respectively and parallel to it. We choose C^+ such that

$$|u_n(z)| \geqq C_2 e^{\alpha_2 \lambda_n^2 |\operatorname{Im} z|}$$

for any z lying on C^+. Then $\hat{u}_n^+(z)$ for z on C^+ satisfies

$$|\hat{u}_n^+(z)| = \left| u_n(z) \int_{i\infty}^{z} \frac{d_n}{u_n^2(\xi)} \, d\xi \right|$$
$$\leqq C_1 e^{\alpha_1 \lambda_n^2 |\operatorname{Im} z|} \, d\lambda_n \int_{i\infty}^{z} C_2^2 e^{-2\alpha_2 \lambda_n^2 |\operatorname{Im} \xi|} \, d\xi$$
$$\leqq C e^{(\alpha_1 - 2\alpha_2) \lambda_n^2 |\operatorname{Im} z|}$$

for some constant C independent of z and n.

Similarly $\varphi(z)$ satisfies

$$|\varphi(z)| \leqq M\frac{e^{a|\mathrm{Im}\, z|}}{1 + |z|^2}.$$

Thus whenever $(2\alpha_2 - \alpha_1)\lambda_n^2 > a$ we obtain

$$\left|\int_{C^+} \hat{u}_n^+ \varphi \, d\xi\right| \leqq Ke^{\{(\alpha_1 - 2\alpha_2)\lambda_n^2 + a\}|\mathrm{Im}\, z|},$$

which vanishes when $|\mathrm{Im}\, z| \to \infty$. Since the integrand is holomorphic it follows that $\int_{C^+} \hat{u}_n^+ \varphi \, d\xi = 0$ for all n sufficiently large. The same holds for $\int_{C^-} \hat{u}_n^- \varphi \, d\xi$. Thus $\int u_n \varphi \, dx = 0$ for each $\varphi \in Z$ and all but a finite set of values of n; whence it follows that $\sum_{n=1}^\infty c_n u_n$ converges weakly in Z'. Since Z' is weakly complete, it must converge to an element g in Z'.

We shall show later that $\sum_{n=1}^\infty c_n \hat{u}_n^\pm(z)$ converges for $|\mathrm{Im}\, z|$ sufficiently large. Assuming this to be true for the moment, we have

$$\begin{aligned}
\langle g, \varphi\rangle &= \langle \textstyle\sum c_n u_n, \varphi\rangle \\
&= \textstyle\sum c_n \langle u_n, \varphi\rangle \\
&= \textstyle\sum c_n \displaystyle\int_{-\infty}^{\infty} u_n \varphi \, dx \\
&= \textstyle\sum c_n \left\{\displaystyle\int_{-\infty}^{\infty} \hat{u}_n^+ \varphi \, dx - \int_{-\infty}^{\infty} \hat{u}_n^- \varphi \, dx\right\} \qquad (4.1)\\
&= \textstyle\sum c_n \left\{\displaystyle\int_{-\infty+ai}^{\infty+ai} \hat{u}_n^+ \varphi \, d\xi - \int_{-\infty-ai}^{+\infty-ai} \hat{u}_n^- \varphi \, d\xi\right\} \\
&= \int_{-\infty+ai}^{\infty+ai} (\textstyle\sum c_n \hat{u}_n^+)\varphi \, d\xi - \displaystyle\int_{-\infty-ai}^{\infty-ai} (\textstyle\sum c_n \hat{u}_n^-)\varphi \, d\xi \\
&= \int_{-\infty+ai}^{\infty+ai} \hat{g}^+ \varphi \, d\xi - \int_{-\infty-ai}^{\infty-ai} \hat{g}^- \varphi \, d\xi,
\end{aligned}$$

which may be expressed as a single integral. (The real number a is taken sufficiently large to ensure that $\sum c_n \hat{u}_n^+(ai)$ converges.) Thus g is an analytic functional.

In order to prove the statement (ii) of Theorem 1 we merely integrate by parts repeatedly:

$$\begin{aligned}
\frac{d\hat{u}_n^\pm(z)}{dz} &= \frac{-1}{2\pi i}\int_{-\infty}^{\infty} \frac{u_n(x)}{(x-z)^2}\, dx \\
&= \frac{-1}{\lambda_n^2}\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{u_n(x)q(x) - u_n''(x)}{(x-z)^2}\, dx \\
&= \frac{1}{\lambda_n^2}\frac{1}{2\pi i}\left\{\int_{-\infty}^{\infty} \frac{u_n(x)q(x)}{(x-z)^2}\, dx - \int_{-\infty}^{\infty} \frac{6u_n(x)}{(x-z)^4}\, dx\right\} \qquad \text{(cont.)}
\end{aligned}$$

$$= -\frac{1}{\lambda_n^2}\frac{1}{2\pi i}\left\{\int_{-\infty}^{\infty}\frac{u_n(x)q^2(x) - u_n''(x)q(x)}{(x-z)^2}dx - 6\int_{-\infty}^{\infty}\frac{u_n(x)q(x) - u_n''(x)}{(x-z)^4}dx\right\}$$

$$= -\frac{1}{\lambda_n^4}\frac{1}{2\pi i}\left\{\int_{-\infty}^{\infty}\frac{u_n(x)(q^2(x) - q''(x))}{(x-z)^2}dx + 2\int_{-\infty}^{\infty}\frac{u_n(x)q'(x)}{(x-z)^3}dx\right.$$

$$\left. -12\int_{-\infty}^{\infty}\frac{u_n(x)q(x)}{(x-z)^4}dx + 120\int_{-\infty}^{\infty}\frac{u_n(x)}{(x-z)^6}dx\right\}$$

$$= -\frac{1}{\lambda_n^{2r}}\frac{1}{2\pi i}\left\{\int_{-\infty}^{\infty}\frac{u_n(x)q^r(x)}{(x-z)^2}dx - \cdots \pm \int_{-\infty}^{\infty}\frac{(2r+1)!\,u_n(x)}{(x-z)^{2r+2}}dx\right\},$$

where r is any integer such that $2r - 2 \geqq p + 1$. From this equality it follows that

$$\left|\frac{d\hat{u}_n^{\pm}}{dz}(z)\right| \leqq \frac{C(z)}{z^2}|\lambda_n|^{-2r}, \qquad \operatorname{Im} z \neq 0, \quad n = 1, 2, \cdots,$$

where $C(z)$ is a bounded function for $|\operatorname{Im} z| \geqq \varepsilon > 0$. Since $\hat{u}_n^{\pm}(z)$ has a zero at $z = \pm i\infty$, we deduce that

$$|\hat{u}_n^{\pm}(z)| \leqq \tilde{C}(z)|\lambda_n|^{-2r}, \qquad \operatorname{Im} z \neq 0, \quad n = 1, 2, \cdots, \quad r \geqq (p+3)/2,$$

and therefore, using the fact that $\lambda_n^{-2} = O(n^{-2})$, that $\sum b_n\hat{u}_n^{\pm}(z)$ converges uniformly for z in compact sets avoiding the real axis.

To prove statement (ii) of Theorem 2 and to justify the calculation made in (4.1), we first observe that, for $|\operatorname{Im} z|$ sufficiently large,

$$|\hat{u}_n^{\pm}(z)| = \left|u_n(z)\int_{\pm i\infty}^{z}\frac{d_n}{u_n^2(\xi)}d\xi\right|$$

$$\leqq C_1 e^{\alpha_1|\operatorname{Im} z|\lambda_n^2}\int_{\pm i\infty}^{z}\lambda_n\, dC_2^2 e^{-2\alpha_2\lambda_n^2|\operatorname{Im}\xi|}\,d\xi$$

$$\leqq Ce^{-|\operatorname{Im} z|\lambda_n^2(2\alpha_2 - \alpha_1)}$$

for a constant C independent of n and z, whence it follows that

$$\sum C_n\hat{u}_n^{\pm}(z) = \hat{g}^{\pm}(z)$$

for $|\operatorname{Im} z|$ sufficiently large. This is the same $\hat{g}$ appearing in (4.1). Therefore this $\hat{g}$ must be the one corresponding to the analytic functional g.

The statement (iii) is proved using a variant of the Hadamard argument mentioned in § 1. To do this we need to find an integral operator relating $\varphi(\xi)$ and $\hat{f}^{\pm}(z)$ in the case of Theorem 1, and $\psi(\xi)$ and $\hat{g}^{\pm}(z)$ in Theorem 2. In one direction this is easy:

$$\hat{f}^{\pm}(z) = \sum_n b_n\hat{u}_n^{\pm}(z) = \frac{1}{2\pi i}\int_{\Gamma}\sum_n b_n t^n \sum_k \hat{u}_k^{\pm}(z)t^{-k}\frac{dt}{t}$$

$$= \frac{1}{2\pi i}\int_{\Gamma}\varphi(t)\hat{K}^{\pm}(z, t^{-1})\frac{dt}{t}, \tag{4.2}$$

where $\hat{K}^{\pm}(z, t)$ is the function given by the series $\sum \hat{u}_n^{\pm}(z)t^n$ which converges for $|\mathrm{Im}\, z| > 0$, $|t| < 1$, and Γ is a circular contour inside the unit disk enclosing the origin. Similarly we have

$$g^{\pm}(z) = \frac{1}{2\pi i}\int_{\Gamma'} \psi(t)\hat{K}^{\pm}(z, t^{-1})\frac{dt}{t},$$

where Γ' is a contour inside the disk of radius ρ enclosing the origin.

On the other hand, we have

$$\begin{aligned}\varphi(t) = \sum b_n t^n &= \int_0^{\pi} \sum_n \frac{b_n u_n(x)}{\lambda_n^{2r}} \sum_k u_k(x)\lambda_k^{2r} t^k \, dx \\ &= \int_0^{\pi} F(x)(q(x) - D_x^2)^r K(x, t)\, dx,\end{aligned} \tag{4.3}$$

where r is chosen so large that $\sum b_n u_n(x)/\lambda_n^{2r}$ converges uniformly in $[0, \pi]$.

Each analytic representation $\hat{F}^{\pm}$ of F satisfies $\hat{F}^{+}(x + i\varepsilon) - \hat{F}^{-}(x - i\varepsilon) \to F(x)$, where convergence is uniform on $[0, \pi]$ as $\varepsilon \to 0$. Thus (4.3) may be written:

$$\begin{aligned}\varphi(t) &= \int_0^{\pi} \{\hat{F}^{+}(x + i0) - \hat{F}^{-}(x - i0)\}(q(x) - D_x^2)^r K(x, t)\, dx \\ &= \int_{\Gamma+} \hat{F}^{+}(z)(q(z) - D_z^2)^r K(z, t)\, dz - \int_{\Gamma-} \hat{F}^{-}(z)(q(z) - D_z^2)^r K(z, t)\, dz,\end{aligned}$$

where Γ^{+} is the path from 0 to $i\varepsilon$ to $\pi + i\varepsilon$ to π and Γ^{-} its reflection in the real axis. We may then integrate by parts on the segments from $i\varepsilon$ to $\pi + i\varepsilon$ and $-i\varepsilon$ to $\pi - i\varepsilon$.

For example, on $i\varepsilon$ to $\pi + i\varepsilon$ we obtain

$$\begin{aligned}&\int_0^{\pi} \hat{F}^{+}(x + i\varepsilon)(q(x + i\varepsilon) - D_x^2)^r K(x + i\varepsilon, t)\, dx \\ &\qquad = \int_0^{\pi} (q(x + i\varepsilon) - D_x^2)^r \hat{F}^{+}(x + i\varepsilon)K(x + i\varepsilon, t)\, dx + \varphi_{\varepsilon}(t),\end{aligned}$$

where $\varphi_{\varepsilon}(t)$ represents the integrated terms. Since an analytic representation of F is given by $\hat{F}^{\pm}(z) = \sum b_n \hat{u}_n^{\pm}(z)/\lambda_n^{2r}$ we may replace $(q(x + i\varepsilon) - D_x^2)^r \hat{F}^{+}(x + i\varepsilon)$ by $\sum b_n \hat{u}_n^{+}(z) = \hat{f}^{+}(z)$. Then denoting by $\varphi_1(t)$ the contribution from the integrated terms as well as the integral over the segments from $-i\varepsilon$ to $i\varepsilon$ and $\pi + i\varepsilon$ to $\pi - i\varepsilon$, we obtain

$$\varphi(t) = \varphi_1(t) + \int_{i\varepsilon}^{\pi + i\varepsilon} \hat{f}^{+}(z)K(z, t)\, dz - \int_{-i\varepsilon}^{\pi - i\varepsilon} \hat{f}^{-}(z)K(z, t)\, dz, \tag{4.4}$$

where $\varphi_1(t)$ is not affected by singular points of $\hat{f}^{\pm}$ in the interior of $(0, \pi)$.

In the other case, we obtain almost the same result:

$$\begin{aligned}\psi(t) &= \sum_n c_n \int_0^\pi u_n(x) \sum_k u_k(x) t^k \, dx \\ &= \sum_n c_n \left[\left\{ \int_0^{ai} + \int_{ai}^{\pi + ai} - \int_{a+ai}^{\pi} \right\} \hat{u}_n^+(x) K(x,t)\, dx \right. \\ &\qquad \left. - \left\{ \int_0^{-ai} + \int_{-ai}^{\pi - ai} - \int_{\pi - ai}^{\pi} \right\} \hat{u}_n^-(x) K(x,t)\, dx \right] \\ &= \sum_n c_n \left\{ \int_{ai}^{\pi+ai} \hat{u}_n^+(x) K(x,t)\, dx - \int_{-ai}^{\pi - ai} \hat{u}_n^-(x) K(x,t)\, dx \right\} + \psi_1(t) \qquad (4.5) \\ &= \int_{ai}^{\pi+ai} \hat{g}^+(x) K(x,t)\, dx - \int_{-ai}^{\pi-ai} \hat{g}^-(x) K(x,t)\, dx + \psi_1(t)\end{aligned}$$

if a is sufficiently large.

We now have the basic tool necessary to prove part (iii) of each lemma. We recall that the only possible singularities of the kernel $K(x, t)$ are at $t = e^{\pm ix}$. Thus the only possible singularity of the kernel

$$\hat{K}^{\pm}(z,t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{K(x,t)}{x - z}\, dx$$

is the point $t = e^{+iz}$ for $\hat{K}^+(z,t)$ and $t = e^{-iz}$ for $\hat{K}^-(z,t)$ by the original "Hadamard argument" (see [1], [2], [7] for details).

We use the same argument to locate a singular point of $\hat{f}^+(z)$. By (4.2),

$$\hat{f}^+(z) = \frac{1}{2\pi i} \int_\Gamma \varphi(t) \hat{K}^+(z, t^{-1}) \frac{dt}{t}.$$

Let us suppose that φ has an isolated singularity at α; then by deforming the contour we can extend $\hat{f}^+(z)$ to any point z on the real axis such that the singular points of φ and $\hat{K}^+$ do not coincide. They do coincide if $\alpha^{-1} = e^{iz}$ or $\alpha = e^{-iz}$, which is therefore the only possible singular point of $\hat{f}^+(z)$. Similarly the only possible singular point of $\hat{f}^-(z)$ is given by $e^{iz} = \alpha$.

In the same way, if $\psi(\xi)$ has an isolated singularity at α, then $\hat{g}^{\pm}(z)$ has a possible singularity only when $\alpha = e^{\pm iz}$.

To go in the other direction, we assume that either $\hat{f}^+(z)$ or $\hat{f}^-(z)$ has an isolated singular point at $z = \beta$ in $(0, \pi)$. Then either the first or the second integral in (4.4) gives a function whose only possible singularity is at $t = e^{\pm i\beta}$, since $\varphi_1(t)$ will not be affected by such a singularity.

Furthermore, we may deduce that $\hat{f}^+(z)$ or $\hat{f}^-(z)$ is indeed singular at $z = \beta$, where $\cos\beta = \frac{1}{2}(\alpha + 1/\alpha)$, provided $\varphi(t)$ is singular at $t = \alpha$. For if neither were singular at β, then $\varphi(t)$ could not be singular at $e^{i\beta}$ or $e^{-i\beta}$ (since $\cos\beta = \frac{1}{2}(e^{-i\beta} + e^{-i\beta})$. Furthermore, it follows that if $\hat{f}^+(z)$ or $\hat{f}^-(z)$ is singular at β, then φ is singular at α.

In the case of Theorem 2, the same arguments hold, the only difference being that the singular points in the z-plane are no longer on the real axis.

5. Extensions. A number of fairly obvious extensions of this theory suggest themselves. For example, extension to orthogonal systems coming from singular problems is possible in some cases (see [5], for example). In other cases such as Hermite series, it is not. Furthermore, it would be interesting to delineate exactly which Sturm–Liouville systems have the property that the corresponding kernel has singular points only at the two points indicated. Perhaps it is true for all systems, or perhaps there is an example with an infinite number of singular points.

Another extension is to systems in which q is not necessarily holomorphic. A different notion of "singular" has to be used in this case, making it more difficult to establish necessary and sufficient conditions.

Finally we observe that the proof that $\sum c_n u_n$ converges in Z' is valid for all sequences of coefficients $\{c_n\}$, no matter how fast they might grow. This could be used to develop a theory which completely removes the distinction between Sturm–Liouville series and expansions. This has already been done by the author in the case of trigonometric series in [6].

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SIAM J. MATH. ANAL.
Vol. 2, No. 3, August 1971

HYPERBOLIC EQUATIONS WITH MULTIPLE CHARACTERISTICS AND TIME DEPENDENT COEFFICIENTS*

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Abstract. This paper considers hyperbolic operators of order n in the $m + 1$ variables $(t, x_1, \cdots, x_m)$ with multiple characteristics and coefficients depending on the time variable t. The roots of the principal part of the characteristic polynomial are real. The form of the lower order terms of the operator and the dependence of the coefficients on t are subject to restrictions which depend on the multiplicity of the characteristic roots. Energy inequalities are developed for these operators. These inequalities are then used to show that the corresponding Cauchy problem is well-posed and also to derive the differentiability of the solution.

1. Preliminaries. We consider linear hyperbolic partial differential operators of order n in $m + 1$ variables, with coefficients depending on the time variable, that is, the variable with respect to which the operator is defined to be hyperbolic. The sheets of the normal characteristic cone may coincide, intersect or coalesce. Certain restrictions, depending on the multiplicity of the characteristics, are imposed on the lower order terms and on the time dependent coefficients. We develop energy integral estimates for these hyperbolic operators and apply these estimates to show that the corresponding Cauchy problem is well-posed in the space of square integrable functions. We also use these estimates to derive the differentiability of the solution of the Cauchy problem.

For strictly hyperbolic operators with coefficients depending on all the variables, the solution of the Cauchy problem, with the aid of energy integrals, was given by Gårding [3]. In [2] Gårding treated hyperbolic operators with constant coefficients and multiple characteristics (that is, not necessarily strictly hyperbolic), using methods other than energy inequalities. The present author in [5] developed energy inequalities for a special class of such operators.

We introduce the operator $L(\partial/\partial t, \partial/\partial x) \equiv L(\partial/\partial t, \partial/\partial x_1, \cdots, \partial/\partial x_m)$ in the $m + 1$ variables $(t, x) \equiv (t, x_1, \cdots, x_m)$, defined by

$$L\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right) = \left(\frac{\partial}{\partial t}\right)^n + \sum_{\substack{|\alpha| \leqq n \\ \alpha_0 < n}} a_\alpha(t)\left(\frac{\partial}{\partial t}\right)^{\alpha_0}\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_m}\right)^{\alpha_m}, \tag{1}$$

where $\alpha = (\alpha_0, \alpha_1, \cdots, \alpha_m)$, $|\alpha| = \alpha_0 + \alpha_1 + \cdots + \alpha_m$. Corresponding to L we have the polynomial in τ with parameters $\xi \equiv (\xi_1, \cdots, \xi_m)$,

$$L(\tau, \xi) = \tau^n + \sum_{\substack{|\alpha| \leqq n \\ \alpha_0 < n}} a_\alpha(t)\tau^{\alpha_0}\xi^{\alpha'}, \tag{2}$$

where $\alpha' \equiv (\alpha_1, \cdots, \alpha_m)$. We define the pseudo-differential operator $\tilde{L}$ by

$$\tilde{L} \equiv L\left(\frac{\partial}{\partial t}, i\xi\right) = \left(\frac{\partial}{\partial t}\right)^n + \sum_{\substack{|\alpha| \leqq n \\ \alpha_0 < n}} a_\alpha(t)\left(\frac{\partial}{\partial t}\right)^{\alpha_0}(i\xi)^{\alpha'}. \tag{3}$$

* Received by the editors April 30, 1970, and in revised form October 22, 1970.

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We impose several conditions on the polynomial $L(\tau, \xi)$, under which the operator $L(\partial/\partial t, \partial/\partial x)$ is defined to be hyperbolic. The domains under consideration will be the dual domains

$$V: 0 \leqq t \leqq 1, \quad -\infty < x_i < \infty, \qquad i = 1, \cdots, m,$$

and

$$\tilde{V}: 0 \leqq t \leqq 1, \quad -\infty < \xi_i < \infty.$$

$H(\tau, \xi)$ will denote the principal part of $L(\tau, \xi)$:

$$H(\tau, \xi) = \tau^n + \sum_{\substack{|\alpha| = n \\ \alpha_0 < n}} a_\alpha(t) \tau^{\alpha_0} \xi^{\alpha'} \tag{4}$$
$$= (\tau - \lambda_1(t, \xi)) \cdots (\tau - \lambda_n(t, \xi)).$$

Assumption A. The roots $\lambda_i(t, \xi)$ are real for all $(t, \xi) \in \tilde{V}$.

We number the roots, for any fixed t, according to increasing order: $\lambda_1(t, \xi) \leqq \cdots \leqq \lambda_n(t, \xi)$. From classical considerations it follows that the roots are continuous functions of ξ. Regarding the dependence of the roots on t we make the following assumption.

Assumption B. The roots $\lambda_i(t, \xi)$ are $n - 1$ times continuously differentiable with respect to t. Furthermore, the derivatives up to order $n - 2$ of

$$[\partial(\lambda_i - \lambda_j)/\partial t]/(\lambda_i - \lambda_j),$$

for all $i \neq j$, are uniformly bounded in $\tilde{V}$. More precisely, it is assumed that

$$\frac{\partial}{\partial t}(\lambda_i - \lambda_j) = C_{ij}(\lambda_i - \lambda_j), \tag{5}$$

where the functions $C_{ij}(t, \xi)$ are $n - 2$ times differentiable with respect to t and these derivatives are continuous and uniformly bounded in $\tilde{V}$.

It is readily seen that Assumption B implies that if ξ is fixed and two roots coincide for some value of t, then they are equal for all values of t.

The lower order terms of $L(\partial/\partial t, \partial/\partial x)$ are subject to certain restrictions, reflecting the multiplicity of the sheets of the normal characteristic cone of $H(\partial/\partial t, \partial/\partial x)$.

We define the jth characteristic factor by

$$\partial_j \equiv \frac{\partial}{\partial t} - i\lambda_j(t, \xi). \tag{6}$$

Assumption C. The pseudo-differential operator $\tilde{L}$ can be written in the form

$$\tilde{L} = \partial_1 \partial_2 \cdots \partial_n + \sum_{k \leqq n-1} \gamma_{i_1 \cdots i_k} \partial_{i_1} \cdots \partial_{i_k}, \tag{7}$$

where the summation extends over k characteristic factors, $k = 0, \cdots, n - 1$, in arbitrary order, with coefficients $\gamma_{i_1 \cdots i_k}(t, \xi)$, continuous and uniformly bounded in $\tilde{V}$.

It is clear that the pseudo-differential operator $\partial_1 \cdots \partial_n$ depends on the order of the characteristic factors. However, we show next that any change in the order of the characteristic factors introduces additional pseudo-operators of lower order, of the type occurring in Assumption C.

LEMMA 1. *If* $\partial_{i_1} \cdots \partial_{i_n}$ *is a rearrangement of* $\partial_1 \cdots \partial_n$, *then*

$$\partial_{i_1} \cdots \partial_{i_n} - \partial_1 \cdots \partial_n = \sum_{k \leqq n-1} \delta_{j_1 \cdots j_k} \partial_{j_1} \cdots \partial_{j_k} \tag{8}$$

with coefficients $\delta_{j_1 \cdots j_k}(t, \xi)$ *continuous and uniformly bounded in* $\tilde{V}$.

Proof. We employ a method similar to the one used by A. Lax [4]. Consider first the case when only two adjacent characteristic factors of $\partial_1 \cdots \partial_n$ are interchanged. It follows from Assumption B that

$$\begin{aligned}
&\partial_1 \cdots \partial_i \partial_{i+1} \cdots \partial_n - \partial_1 \cdots \partial_{i+1} \partial_i \cdots \partial_n \\
&= \partial_1 \cdots \partial_{i-1} (\partial_i \partial_{i+1} - \partial_{i+1} \partial_i) \partial_{i+2} \cdots \partial_n \\
&= \partial_1 \cdots \partial_{i-1} \left(i \frac{\partial \lambda_i}{\partial t} - i \frac{\partial \lambda_{i+1}}{\partial t} \right) \partial_{i+2} \cdots \partial_n \\
&= \partial_1 \cdots \partial_{i-1} C_{i,i+1} (i\lambda_i - i\lambda_{i+1}) \partial_{i+2} \cdots \partial_n \\
&= \partial_1 \cdots \partial_{i-1} C_{i,i+1} (\partial_{i+1} - \partial_i) \partial_{i+2} \cdots \partial_n \\
&= \partial_1 \cdots \partial_{i-1} C_{i,i+1} \partial_{i+1} \partial_{i+2} \cdots \partial_n - \partial_1 \cdots \partial_{i-1} C_{i,i+1} \partial_i \partial_{i+2} \cdots \partial_n .
\end{aligned} \tag{9}$$

Now consider the first term on the right of (9):

$$\begin{aligned}
\partial_1 \cdots \partial_{i-1} C_{i,i+1} \partial_{i+1} \partial_{i+2} \cdots \partial_n = {}& \partial_1 \cdots \partial_{i-2} C_{i,i+1} \partial_{i-1} \partial_{i+1} \partial_{i+2} \cdots \partial_n \\
&+ \partial_1 \cdots \partial_{i-2} C'_{i,i+1} \partial_{i+1} \partial_{i+2} \cdots \partial_n ,
\end{aligned} \tag{10}$$

where

$$C'_{i,i+1} = \frac{\partial}{\partial t} C_{i,i+1} .$$

The second term in (9) can be broken up in a similar way. We continue this process with each term in (10), successively moving the factors $C_{i,i+1}, C'_{i,i+1}, \cdots$ one place to the left. It follows that after a number of steps, involving t-derivatives of $C_{i,i+1}$ at most up to order $n-2$, each term has the required form.

Returning to $\partial_{i_1} \cdots \partial_{i_n}$, we permute the characteristic factors one at a time, a process which has just been shown to have the effect of adding lower order terms of the required form, with coefficients involving the t-derivatives of C_{ij} of order $\leqq n-2$. This completes the proof of the lemma.

2. Energy inequalities. We shall make repeated use of a well-known classical lemma. For the sake of completeness we shall include a simple proof.

LEMMA 2. *If* $g(y)$ *is a continuous function and* $h(y)$ *is a nondecreasing function in the interval* $a \leqq y \leqq b$, *satisfying the inequality* $g(y) \leqq k \int_a^y g(s)\, ds + h(y)$, $k > 0$, *then* $g(y) \leqq e^{k(y-a)} h(y)$.

Proof. Since $h(y)$ is nondecreasing, it follows that for fixed y and $a \leqq \eta \leqq y$,

$$g(\eta) \leqq k \int_a^\eta g(s)\, ds + h(y). \tag{11}$$

We denote the right-hand side of (11) by $W(\eta)$. It follows that

$$\frac{d}{d\eta} W(\eta) \leqq k W(\eta).$$

Hence,

$$\frac{d}{d\eta} e^{-k\eta} W(\eta) \leqq 0.$$

Integration with respect to η from a to y implies that $e^{-ky}W(y) - e^{-ka}W(a) \leqq 0$. Since $W(a) = h(y)$, the lemma follows.

We denote by $\mathscr{H}$ the space of complex-valued functions $w(t, x)$, which are square integrable in V (V: $0 \leqq t \leqq 1$, $-\infty < x_i < \infty$). The inner product and norm are defined as usual by

$$(12) \qquad \begin{aligned} (w, v) &= \iint_V w\bar{v}\, dt\, dx, \qquad dx \equiv dx_1\, dx_2 \cdots dx_m, \\ \|w\|^2 &= (w, w). \end{aligned}$$

Similarly $\tilde{\mathscr{H}}$ denotes the space of complex-valued functions $\tilde{w}(t, \xi)$, which are square integrable in $\tilde{V}$ ($\tilde{V}$: $0 \leqq t \leqq 1$, $-\infty < \xi_i < \infty$), with inner product $(\tilde{w}, \tilde{v})$ and norm $\|\tilde{w}\|$.

DEFINITION. $w(t, x)$ is defined to be *smooth in V* if it has continuous derivatives up to and including order n in the t-variable, and has continuous derivatives of all orders in the x-variables (that is, if it has the continuous derivatives of the form $(\partial/\partial t)^{\alpha_0}(\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_m)^{\alpha_m}$ with $\alpha_0 \leqq n$ and all $(\alpha_1, \cdots, \alpha_m)$), and such that each derivative is $O(|x|^{-N})$, as $|x| \equiv (x_1^2 + \cdots + x_m^2)^{1/2} \to \infty$, uniformly in t, for each nonnegative integer N.

Smooth functions in $\tilde{V}$ are defined in the same way.

We shall use the following elementary facts concerning Fourier transforms. Let $\tilde{w}(t, \xi)$ be the Fourier transform of $w(t, x)$ with respect to the x-variables (we shall refer to it as the transform of w), given by

$$\tilde{w}(t, \xi) = \frac{1}{(2\pi)^{m/2}} \int_{-\infty}^{\infty} e^{-ix\cdot\xi} w(t, x)\, dx,$$

$$x \cdot \xi = x_1\xi_1 + \cdots + x_m\xi_m.$$

$w \in \mathscr{H}$ implies that $\tilde{w} \in \tilde{\mathscr{H}}$. Furthermore, if w is smooth, then also $\tilde{w}$ is smooth.

Smooth functions $w(t, x)$ and $\tilde{w}(t, \xi)$ will be said to satisfy homogeneous data on $t = 0$ if, respectively,

$$(13) \qquad \begin{aligned} &\left(\frac{\partial}{\partial t}\right)^s \tilde{w}(0, x) = 0 \quad \text{and} \\ &\left(\frac{\partial}{\partial t}\right)^s \tilde{w}(0, \xi) = 0 \quad \text{for } s = 0, \cdots, n - 1. \end{aligned}$$

S_T will denote the surface $t = T$, $-\infty < x_i < \infty$. V_T denotes the slab $0 \leqq t \leqq T$, $-\infty < x_i < \infty$. Similarly $\tilde{S}_T$ and $\tilde{V}_T$ denote the corresponding surface and slab in the (t, ξ)-space.

Throughout the paper K will denote an unspecified positive constant depending only on the coefficients and the order of the given operator and on the number of independent variables.

LEMMA 3. *If $\tilde{w}(t, \xi)$ is smooth with homogeneous data on $t = 0$, and if we denote*

$$\begin{aligned} \partial_{i_1}\partial_{i_2} \cdots \partial_{i_k} &= \tilde{P}_1, \\ \partial_{i_2} \cdots \partial_{i_k} &= \tilde{P}_2, \end{aligned} \qquad k \leqq n,$$

then for $0 \leqq T \leqq 1$,

$$\int_{\tilde{S}_T} |\tilde{P}_2 \tilde{w}|^2 \, d\xi \leqq K \iint_{\tilde{V}_T} |\tilde{P}_1 \tilde{w}|^2 \, dt \, d\xi. \tag{14}$$

Proof.

$$\operatorname{Re}\{\tilde{P}_1 \tilde{w} \cdot \overline{\tilde{P}_2 \tilde{w}}\} = \operatorname{Re}\left\{\left(\frac{\partial}{\partial t} - i\lambda_{i_1}\right)\tilde{P}_2 \tilde{w} \cdot \overline{\tilde{P}_2 \tilde{w}}\right\} = \frac{1}{2}\frac{\partial}{\partial t}|P_2 w|^2. \tag{15}$$

Integration of (15) over $\tilde{V}_T$ implies that

$$\frac{1}{2}\int_{\tilde{S}_T} |\tilde{P}_2 \tilde{w}|^2 \, d\xi = \iint_{\tilde{V}_T} \operatorname{Re}\{\tilde{P}_1 \tilde{w} \cdot \overline{\tilde{P}_2 \tilde{w}}\} \, dt \, d\xi \leqq \frac{1}{2}\iint_{\tilde{V}_T} (|\tilde{P}_1 \tilde{w}|^2 + |\tilde{P}_2 \tilde{w}|^2) \, dt \, d\xi.$$

This can be rewritten in the following form:

$$\int_{\tilde{S}_T} |\tilde{P}_2 \tilde{w}|^2 \, d\xi \leqq \int_0^T \left[\int_{\tilde{S}_t} (|\tilde{P}_2 \tilde{w}|^2 + |\tilde{P}_1 \tilde{w}|^2) \, d\xi \right] dt.$$

We apply Lemma 2 to this inequality and (14) now follows. This completes the proof.

We consider next the two operators $\partial_{i_1} \cdots \partial_{i_k}$, $k \leqq n - 1$, and

$$\partial_{i_1} \cdots \partial_{i_k}\partial_{i_{k+1}} \cdots \partial_{i_n}.$$

Repeated application of (14) shows that if $\tilde{w}$ is smooth with homogeneous data on $t = 0$, then

$$\int_{\tilde{S}_T} |\partial_{i_1} \cdots \partial_{i_k} \tilde{w}|^2 \, d\xi \leqq K \iint_{\tilde{V}_T} |\partial_{i_1} \cdots \partial_{i_k}\partial_{i_{k+1}} \cdots \partial_{i_n} \tilde{w}|^2 \, dt \, d\xi. \tag{16}$$

We now derive the energy inequalities.

THEOREM 1. *If $\tilde{w}$ is smooth with homogeneous data on $t = 0$, then*

$$\int_{\tilde{S}_T} \sum_{k \leqq n-1} |\partial_{i_1} \cdots \partial_{i_k} \tilde{w}|^2 \, d\xi \leqq K \iint_{\tilde{V}_T} |\tilde{L}\tilde{w}|^2 \, dt \, d\xi. \tag{17}$$

Proof. From (7), (8) and (16) it follows that for $v \leqq n - 1$,

$$\int_{\tilde{S}_T} |\partial_{i_1} \cdots \partial_{i_v} \tilde{w}|^2 \, d\xi \leqq K \iint_{\tilde{V}_T} \left(|\tilde{L}\tilde{w}|^2 + \sum_{k \leqq n-1} |\partial_{i_1} \cdots \partial_{i_k} \tilde{w}|^2 \right) dt \, d\xi. \tag{18}$$

Summation of (18) with respect to v from 0 to $n - 1$ yields

$$\int_{\tilde{S}_T} \sum_{k \leqq n-1} |\partial_{i_1} \cdots \partial_{i_k} \tilde{w}|^2 \, d\xi \leqq K \iint_{\tilde{V}_T} \left(|\tilde{L}\tilde{w}|^2 + \sum_{k \leqq n-1} |\partial_{i_1} \cdots \partial_{i_k} \tilde{w}|^2 \right) dt \, d\xi.$$

From Lemma 2, (17) now follows. This completes the proof.

If $w(t, x)$ is smooth in V with homogeneous data on $t = 0$, then it follows from (17) that the transform $\tilde{w}(t, \xi)$ satisfies

$$\|\tilde{w}\|^2 \leqq K\|\tilde{L}\tilde{w}\|^2. \tag{19}$$

From Parseval's theorem it then follows that

$$\|w\|^2 \leqq K\|Lw\|^2. \tag{20}$$

We need another energy inequality, stronger than (20), which we proceed to derive. We multiply both sides of (17) by $e^{-\sigma T}$, where σ is a positive constant at our disposal. Integration with respect to T from 0 to 1, and use of integration by parts on the right results in the following theorem.

THEOREM 2. *If $\tilde{w}(t, \xi)$ is smooth with homogeneous data on $t = 0$, then for $\sigma > 0$,*

$$\sum_{k \leqq n-1} \iint_{\tilde{V}} |e^{-\sigma t/2}\partial_{i_1} \cdots \partial_{i_k}\tilde{w}|^2 \, dt \, d\xi \leqq (K/\sigma) \iint_{\tilde{V}} |e^{-\sigma t/2}\tilde{L}\tilde{w}|^2 \, dt \, d\xi. \tag{21}$$

Parseval's theorem, applied to both sides of (21), implies the following corollary.

COROLLARY. *If $w(t, x)$ is smooth in V with homogeneous data on $t = 0$, then*

$$\|e^{-\sigma t/2}w\|^2 \leqq (K/\sigma)\|e^{-\sigma t/2}Lw\|^2. \tag{22}$$

3. The Cauchy problem. We now consider the so-called strong solution of the Cauchy problem for the operator L with homogeneous data.

DEFINITION. The function $u \in \mathscr{H}$ is a *strong solution, in V, of*

$$Lu = g, \qquad g \in \mathscr{H}, \tag{23}$$

satisfying the homogeneous data

$$\left(\frac{\partial}{\partial t}\right)^s u(0, x) = 0, \qquad s = 0, \cdots, n-1, \tag{24}$$

in the strong sense, if there exists a sequence of smooth functions $u^{(r)}(t, x)$, satisfying homogeneous data on $t = 0$, such that

$$\|u^{(r)} - u\| + \|Lu^{(r)} - g\| \to 0 \quad \text{as} \quad r \to \infty. \tag{25}$$

THEOREM 3. *If $g \in \mathscr{H}$, then there exists a unique strong solution, in V, of $Lu = g$ satisfying the homogeneous data* (24) *in the strong sense. Furthermore, the strong solution u depends continuously, in the norm, on the given function g.*

Proof. Uniqueness and continuous dependence. From (20) it follows that $\|u^{(r)}\|^2 \leqq K\|Lu^{(r)}\|^2$. Hence in the limit $\|u\|^2 \leqq K\|g\|^2$. This implies the uniqueness of the strong solution and its continuous dependence, in the norm, on g.

Existence. From Parseval's theorem it follows that the proof of the existence of a function $u \in \mathscr{H}$ satisfying (25) is equivalent to showing that there exists a function $\tilde{u} \in \tilde{\mathscr{H}}$ and a sequence of smooth functions $\tilde{u}^{(r)}(t, \xi)$ with homogeneous data on $t = 0$, such that for $\tilde{g}$, the transform of g,

$$\|\tilde{u}^{(r)} - \tilde{u}\| + \|\tilde{L}\tilde{u}^{(r)} - \tilde{g}\| \to 0 \quad \text{as} \quad r \to \infty. \tag{26}$$

We introduce in $\tilde{\mathscr{H}}$ another inner product $(\tilde{w}, \tilde{v})_\sigma$ and norm $\|\tilde{w}\|_\sigma$:

$$\begin{aligned} (\tilde{w}, \tilde{v})_\sigma &= (e^{-\sigma t/2}\tilde{w}, e^{-\sigma t/2}\tilde{v}), \\ \|\tilde{w}\|_\sigma &= \|e^{-\sigma t/2}\tilde{w}\|. \end{aligned} \tag{27}$$

Let $\tilde{\mathscr{K}}$ be the subspace of $\tilde{\mathscr{H}}$ consisting of all functions $\tilde{f}$ for which there exists a function $\tilde{w} \in \tilde{\mathscr{H}}$ and a sequence of smooth functions $\tilde{w}^{(r)}(t, \xi)$ satisfying homogeneous data on $t = 0$, such that

$$\|\tilde{w}^{(r)} - \tilde{w}\|_\sigma + \|\tilde{L}\tilde{w}^{(r)} - \tilde{f}\|_\sigma \to 0. \tag{28}$$

It is our purpose to show that we can determine σ such that $\tilde{\mathscr{K}} = \tilde{\mathscr{H}}$. Since the norms $\|\tilde{w}\|_\sigma$ and $\|\tilde{w}\|$ are clearly equivalent in $\tilde{\mathscr{H}}$, the existence of the function $\tilde{u}$ satisfying (26) will then follow.

From (22) it follows that if $\tilde{f} \in \tilde{\mathscr{K}}$, and $\tilde{w} \in \tilde{\mathscr{H}}$ is the function corresponding to $\tilde{f}$ in (28), then $\|\tilde{w}\|_\sigma^2 \leqq (K/\sigma)\|\tilde{f}\|_\sigma^2$. This implies that $\tilde{\mathscr{K}}$ is a closed subspace of $\tilde{\mathscr{H}}$. Let $\tilde{v} \in \tilde{\mathscr{H}}$ be orthogonal to all of $\tilde{\mathscr{K}}$. Then

$$(\tilde{f}, \tilde{v})_\sigma = 0 \quad \text{for all} \quad \tilde{f} \in \tilde{\mathscr{K}}. \tag{29}$$

We shall show that $\tilde{v} = 0$, which will imply that $\tilde{\mathscr{K}} = \tilde{\mathscr{H}}$. From (29) it follows that $(\tilde{L}\tilde{w}, \tilde{v})_\sigma = 0$ for all smooth $\tilde{w}(t, \xi)$ with homogeneous data on $t = 0$. Let $\tilde{z}^{(r)}(t, \xi)$ be a sequence of smooth functions such that $\|\tilde{z}^{(r)} - \tilde{v}\|_\sigma \to 0$. We may assume that each $\tilde{z}^{(r)}$ has bounded support in the ξ-variables. For every $\tilde{z}^{(r)}$ we solve the pseudo-differential equation

$$\partial_1 \cdots \partial_n \tilde{q}^{(r)} = \tilde{z}^{(r)} \tag{30}$$

with $\tilde{q}^{(r)}$ satisfying homogeneous data on $t = 0$. We do this by solving the following system (the superscript r will be temporarily dropped):

$$\begin{aligned} &\frac{\partial}{\partial t}\tilde{q} - i\lambda_n\tilde{q} = \tilde{v}_{n-1}, \\ &\frac{\partial}{\partial t}\tilde{v}_{n-1} - i\lambda_{n-1}\tilde{v}_{n-1} = \tilde{v}_{n-2}, \\ &\qquad \vdots \\ &\frac{\partial}{\partial t}\tilde{v}_1 - i\lambda_1\tilde{v}_1 = \tilde{z} \end{aligned} \tag{31}$$

with data

$$\tilde{q}(0, \xi) = \tilde{v}_{n-1}(0, \xi) = \cdots = \tilde{v}_1(0, \xi) = 0. \tag{32}$$

Explicitly,

$$\tilde{v}_1 = \int_0^t \exp\left(i \int_\tau^t \lambda_1(\rho, \xi)\, d\rho\right) \tilde{z}(\tau, \xi)\, d\tau. \tag{33}$$

The functions $\tilde{v}_2, \cdots, \tilde{v}_{n-1}, \tilde{q}$, computed recursively, have expressions similar to (33). It follows that $\tilde{q}$ has $n - 1$ continuous t-derivatives in $\tilde{V}$. Since $\tilde{z}$ was assumed to have bounded support in the ξ-variables, the same holds for $\tilde{q}$. We show next that $\tilde{L}\tilde{q}$ belongs to the subspace $\tilde{\mathscr{K}}$. For this purpose we apply the Friedrichs mollifiers [1] to the function $\tilde{q}$ in the ξ-directions. Let $j(\xi)$ be an infinitely differentiable nonnegative function of ξ with support in the cube $-1 \leqq \xi_i \leqq 1$, and such that $\int_{-1}^{1} j(\xi)\, d\xi = 1$. For $0 < \varepsilon < 1$ we set

$$\tilde{q}_\varepsilon = \varepsilon^{-m} \int_{-\infty}^{\infty} j(\varepsilon^{-1}(\xi - \hat{\xi}))\tilde{q}(t, \hat{\xi})\, d\hat{\xi}. \tag{34}$$

The mollifiers satisfy the following simple property:

$$\|\tilde{q} - \tilde{q}_\varepsilon\|_\sigma \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{35}$$

Since

$$\left(\frac{\partial}{\partial t}\right)^s \tilde{q}_\varepsilon = \left(\left(\frac{\partial}{\partial t}\right)^s \tilde{q}\right)_\varepsilon,$$

it follows from (35) that also

$$\left\|\left(\frac{\partial}{\partial t}\right)^s \tilde{q} - \left(\frac{\partial}{\partial t}\right)^s \tilde{q}_\varepsilon\right\|_\sigma \to 0, \qquad s = 0, \cdots, n. \tag{36}$$

Now, $\tilde{q}$ has bounded support in the ξ-variables. Therefore the smooth functions $\tilde{q}_\varepsilon$ vanish outside a fixed bounded subdomain of $\tilde{V}$, independently of ε. This together with (36) implies that

$$\|\tilde{L}\tilde{q} - \tilde{L}\tilde{q}_\varepsilon\|_\sigma \to 0. \tag{37}$$

Hence $\tilde{L}\tilde{q}$ belongs to $\tilde{\mathscr{K}}$ and therefore

$$(\tilde{L}\tilde{q}, \tilde{v})_\sigma = 0. \tag{38}$$

Returning to the sequences $\tilde{q}^{(r)}$ and $\tilde{z}^{(r)}$, we have from (38) and (17) that

$$\begin{aligned} 0 &= |(\tilde{L}\tilde{q}^{(r)}, \tilde{v})_\sigma| \\ &\geqq |(\partial_1 \cdots \partial_n \tilde{q}^{(r)}, \tilde{v})_\sigma| - \left|\left(\sum_{k \leqq n-1} \gamma_{i_1 \cdots i_k} \partial_{i_1} \cdots \partial_{i_k} \tilde{q}^{(r)}, \tilde{v}\right)_\sigma\right| \\ &\geqq |(\tilde{z}^{(r)}, \tilde{v})_\sigma| - (K/\sigma)\|\partial_1 \cdots \partial_n \tilde{q}^{(r)}\|_\sigma \|\tilde{v}\|_\sigma \\ &= |(\tilde{z}^{(r)}, \tilde{v})_\sigma| - (K/\sigma)\|\tilde{z}^{(r)}\|_\sigma \|\tilde{v}\|_\sigma. \end{aligned} \tag{39}$$

Letting $r \to \infty$, it follows from the strong convergence of $\tilde{z}^{(r)}$ to $\tilde{v}$ that $(1 - K/\sigma)\|v\|_\sigma^2 \leqq 0$. Choosing $\sigma > K$ thus implies that $\|\tilde{v}\|_\sigma = 0$. This completes the existence proof.

4. Differentiability. The differentiability of the solution of the Cauchy problem depends on the differentiability of the right-hand side g of the given equation (23). In this section we derive the strong (square integrable) differentiability with respect to the x-variables. For this purpose we introduce another inner product and norm for smooth functions in V:

$$(\tilde{w}, \tilde{v})_{(p)} = \iint_{\tilde{V}} (1 + |\xi|^2)^p \tilde{w}\bar{\tilde{v}}\, dt\, d\xi, \tag{40}$$

$$\|\tilde{w}\|^2_{(p)} = (\tilde{w}, \tilde{w})_{(p)}, \qquad p = \text{positive integer}. \tag{41}$$

The space $\tilde{\mathscr{H}}_p$ is obtained by the completion of the space of smooth functions in $\tilde{V}$ under the norm (41). $\mathscr{H}_p$ will denote the space of the inverse transforms of the functions in $\tilde{\mathscr{H}}_p$. Hence $\mathscr{H}_p$ is the subspace of $\mathscr{H}$ consisting of all functions possessing strong x-derivatives of order $\leqq p$.

The factor $(1 + |\xi|^2)^{p/2}$ commutes with $\tilde{L}$ and with $\partial_{i_1} \cdots \partial_{i_k}$. Hence it follows from (17) that for smooth $\tilde{w}(t, \xi)$ with homogeneous data on $t = 0$,

$$\int_{\tilde{S}_T} \sum_{k \leqq n-1} (1 + |\xi|^2)^p |\partial_{i_1} \cdots \partial_{i_k} \tilde{w}|^2\, d\xi \leqq K \iint_{\tilde{V}_T} (1 + |\xi|^2)^p |\tilde{L}\tilde{w}|^2\, dt\, d\xi. \tag{42}$$

Consider the Cauchy problem (23) with homogeneous data (24). If g possesses all strong x-derivatives of order $\leqq p$, then $\tilde{g}$, the transform of g, belongs to $\tilde{\mathscr{H}}_p$. From (42) it follows in the same way as in the existence proof of Theorem 3, that there exists a function $\tilde{u} \in \tilde{\mathscr{H}}_p$ and a sequence of smooth functions $u^{(r)}(t, \xi)$ with homogeneous data on $t = 0$, such that

$$\|\tilde{u}^{(r)} - \tilde{u}\|_{(p)} + \|\tilde{L}\tilde{u}^{(r)} - \tilde{g}\|_{(p)} \to 0. \tag{43}$$

Since $\tilde{u} \in \tilde{\mathscr{H}}_p$ it follows that the inverse transform u, which is the strong solution of the Cauchy problem (23) with homogeneous data (24), possesses all strong x-derivatives of order $\leqq p$.

Next we consider strong derivatives of the solution, including the t-variable. For this purpose we now assume that $p = n$, that is, g has strong x-derivatives of order $\leqq n$ (the order of L). It follows from (42) that

$$\|\partial_1 \tilde{u}\|_{(n-1)} \leqq K \|\tilde{L}\tilde{u}\|_{(n-1)}.$$

Now, $\partial_1 = \partial/\partial t - i\lambda_1(t, \xi)$, where $\lambda_1(t, \xi) = |\xi|\lambda_1(t, \xi/|\xi|) = O(|\xi|)$. Therefore

$$\left\|\left(\frac{\partial}{\partial t}\right)\tilde{u}\right\|_{(n-1)} \leqq K(\|\tilde{L}\tilde{u}\|_{(n-1)} + \|\tilde{u}\|_{(n)}).$$

This implies that $(\partial/\partial t)\tilde{u} \in \tilde{\mathscr{H}}_{n-1}$. Similarly it follows from

$$\|\partial_1 \partial_2 \tilde{u}\|_{(n-2)} \leqq K \|\tilde{L}\tilde{u}\|_{(n-2)},$$

that

$$\left\|\left(\frac{\partial}{\partial t}\right)^2 \tilde{u}\right\|_{(n-2)} \leqq K\left(\|\tilde{L}\tilde{u}\|_{(n-2)} + \left\|\left(\frac{\partial}{\partial t}\right)\tilde{u}\right\|_{(n-1)} + \|\tilde{u}\|_{(n)}\right).$$

This implies that $(\partial/\partial t)^2\tilde{u} \in \tilde{\mathscr{H}}_{n-2}$. Continuing in this way we deduce that for $0 \leqq s \leqq n-1$, $(\partial/\partial t)^s\tilde{u}$ belongs to $\tilde{\mathscr{H}}_{n-s}$. This implies that all the functions

$$\left(\frac{\partial}{\partial t}\right)^{\alpha_0} \xi_1^{\alpha_1} \cdots \xi_m^{\alpha_m}\tilde{u}$$

with $\alpha_0 + \alpha_1 + \cdots + \alpha_m \leqq n$, $\alpha_0 \leqq n-1$, belong to $\tilde{\mathscr{H}}$. Therefore u possesses all strong derivatives of the form

$$\left(\frac{\partial}{\partial t}\right)^{\alpha_0}\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_m}\right)^{\alpha_m}.$$

Since $(\partial/\partial t)^n u = Lu$ + terms involving the t-derivatives of order $\leqq n-1$, it follows that u also possesses the strong nth order t-derivative.

Finally, consideration of the successive derivatives of $Lu \equiv (\partial/\partial t)^n u + \cdots = g$ results in the following theorem.

THEOREM 4. *If the coefficients of the hyperbolic operator* L (*in addition to the properties imposed by Assumptions* A, B *and* C) *have* p_0 *continuous t-derivatives and* g *possesses all strong derivatives*

$$\left(\frac{\partial}{\partial t}\right)^{\alpha_0}\left(\frac{\partial}{\partial x_i}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_m}\right)^{\alpha_m}$$

with $\alpha_0 + \alpha_1 + \cdots + \alpha_m \leqq n + p_0$, $\alpha_0 \leqq p_0$, *then the strong solution* u *of* $Lu = g$ *with homogeneous data* (24), *possesses all strong derivatives of order* $n + p_0$, *with respect to all variables.*

5. Summary. In § 1 we consider the hyperbolic operator $L(\partial/\partial t, \partial/\partial x)$ with time dependent coefficients. The corresponding pseudo-differential operator $\tilde{L} = L(\partial/\partial t, i\xi)$ has the form

$$\tilde{L} = \partial_1 \cdots \partial_n + \sum_{k \leqq n-1} \gamma_{i_1 \cdots i_k}(t)\partial_{i_1} \cdots \partial_{i_k},$$

where $\partial_i = (\partial/\partial t) - i\lambda_i(t, \xi)$. $\lambda_i(t, \xi)$ are the real roots of the principal characteristic polynomial, and their dependence on t is subject to restrictions related to the multiplicity of the characteristics.

In § 2 we derive energy inequalities for functions $\tilde{w}(t, \xi)$ in the Fourier transform space with homogeneous data on $t = 0$: $\|\partial_{i_1} \cdots \partial_{i_k}\tilde{w}\| \leqq K\|\tilde{L}\tilde{w}\|$. We use these inequalities in § 3 to derive the strong solution of the pseudo-differential equation $\tilde{L}\tilde{u} = \tilde{g}$ with homogeneous data. The inverse Fourier transform of $\tilde{u}$ is the strong solution of the Cauchy problem $Lu = g$ with homogeneous data.

Additional energy inequalities are derived in § 4, and these are used to show the strong differentiability of the solution of the Cauchy problem, provided the right-hand side g is sufficiently differentiable.

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SIAM J. MATH. ANAL.
Vol. 2, No. 3, August 1971

ON A DIFFERENTIAL EQUATION FOR THE EIGENVECTORS OF A REAL SYMMETRIC MATRIX*

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Abstract. For a given real symmetric matrix, a differential equation is established which has the property that eigenvectors of the matrix are asymptotically stable solutions for the differential equation. A Lyapunov function is constructed for this purpose, and subsequently, the regions of stability for the equation are characterized.

1. Introduction. We consider the question of finding a differential equation in order to obtain the eigenvectors of a real symmetric matrix A. The question arose from a problem concerning the estimation of eigenvalues of A. Block and Fuchs [1] obtained bounds for the eigenvalues. We use these bounds to obtain our differential equation. In particular, for each real unit vector x, we compute certain numbers $\mu = \mu(x)$ and $\sigma = \sigma(x)$, which have the property that $[\mu + \sigma, \mu - \sigma]$ contains some eigenvalue of A. We establish an autonomous differential equation for x which has the property that $\sigma^2(x)$ is nonincreasing along solutions of the differential equation. The main result (Theorem 6.1) is that the differential equation admits asymptotically stable solutions of the form, $x =$ eigenvector of A. That is, there is an open subset of the unit sphere (the region of attraction) such that any solution with initial value in this subset tends to some eigenvector of A.

The key to our results is the observation that the eigenvalue bound generates a Lyapunov function, $\sigma^2(x)$, for our choice of differential equation. (We note that our choice is certainly not unique. In the addendum, § 7, we indicate other possibilities.) The stationary set for the differential equation turns out to be precisely the set of critical points of $\sigma^2(x)$ on the unit sphere. This enables us to characterize the region of attraction.

In fact, the (nonzero) critical points of $\sigma^2(x)$ are those points on the unit sphere where the gradient of $\sigma^2(x)$ is normal to the sphere. These points constitute the boundary of the region of attraction and are shown to comprise a finite union of products of lower dimensional spheres.

2. An enclosure theorem for eigenvalues. Denote the real numbers by R and real n-dimensional Euclidean space by R^n. Let $\{e_1, e_2, \cdots, e_n\}$ be an orthonormal basis for R^n. Then each vector $x \in R^n$ may be uniquely written $x = \sum_{i=1}^n \xi_i e_i$ for some real scalars $\xi_1, \xi_2, \cdots, \xi_n$. For any other vector $y = \sum_{i=1}^n \zeta_i e_i$, we denote the usual inner product by $\langle x, y\rangle = \sum_{i=1}^n \xi_i \zeta_i$, noting that $\langle e_i, e_j\rangle = \delta_{ij}$, the Kronecker delta. Let $\|x\|$ denote the usual norm, $\langle x, x\rangle^{1/2}$, and let I designate the $n \times n$ identity matrix.

The following theorem by Block and Fuchs [1] provides the framework for our result. We include the proof, as the notation and computations therein will be used later on.

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* Received by the editors July 2, 1970, and in revised form October 29, 1970.

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THEOREM 2.1. *Let A be a real symmetric $n \times n$ matrix. For any unit vector $x \in R^n$, define*

$$\mu = \langle Ax, x \rangle,$$
$$K = A - \mu I,$$
$$\sigma = \|Kx\|.$$

Then there exists an eigenvalue of A in the interval $[\mu - \sigma, \mu + \sigma]$.

Proof. Since A is real symmetric, R^n has an orthonormal basis of eigenvectors for A. In particular, A has real eigenvalues $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ and a corresponding set of orthonormal eigenvectors $\{e_1, e_2, \cdots, e_n\}$ such that $Ae_i = \lambda_i e_i$, $1 \leqq i \leqq n$. Consider any vector $x = \sum_{i=1}^n \xi_i e_i$ with $\|x\|^2 = \sum_{i=1}^n \xi_i^2 = 1$. Then

$$\mu = \langle Ax, x \rangle = \left\langle \sum_{i=1}^n \xi_i \lambda_i e_i, \sum_{i=1}^n \xi_i e_i \right\rangle = \sum_{i=1}^n \lambda_i \xi_i^2,$$

and

$$\begin{aligned} \sigma^2 = \|Kx\|^2 &= \left\| \sum_{i=1}^n \xi_i \lambda_i e_i - \mu \sum_{i=1}^n \xi_i e_i \right\|^2 \\ &= \left\| \sum_{i=1}^n (\lambda_i - \mu) \xi_i e_i \right\|^2 = \sum_{i=1}^n (\lambda_i - \mu)^2 \xi_i^2. \end{aligned} \tag{2.1}$$

But

$$\sum_{i=1}^n (\lambda_i - \mu)^2 \xi_i^2 \geqq \min_i (\lambda_i - \mu)^2.$$

Hence there is some eigenvalue λ_i satisfying $(\lambda_i - \mu)^2 \leqq \sigma^2$. Consequently, there is some eigenvalue of A in the interval $[\mu - \sigma, \mu + \sigma]$.

3. A differential equation for the eigenvalues of A. We note that μ, K and σ^2 depend on x. In the event that x is a unit eigenvector of A, it follows that μ is the associated eigenvalue. Thus, if some initial choice of $x_0 \in R^n$ is subsequently made to vary as a function of time, $x(t)$, $t \in R$, $x(0) = x_0$, so as to reduce σ^2 continuously and monotonically to zero, then $x(t)$ will converge to an eigenvector of A. (In fact, we shall show that σ^2 can be chosen as a Lyapunov function which is strictly decreasing along the trajectories, $x(t)$.)

Specifically, let us represent x as a differentiable function of t, $x = x(t)$ with $x(t)$ lying on the unit sphere $S^{n-1} = \{x \in R^n : \|x\| = 1\}$ for all $t \in R$. To emphasize the dependence of σ^2 upon x, we write $\sigma^2 = \sigma^2(x)$. Then

$$\frac{d}{dt}\sigma^2(x) = \frac{d}{dt}\langle Kx, Kx \rangle = 2\langle K\dot{x} + \dot{K}x, Kx \rangle,$$

where $\cdot$ represents d/dt. Noting that $\dot{K} = -\dot{\mu} I$ and $\dot{\mu} = 2\langle Ax, \dot{x} \rangle$, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\sigma^2(x) &= \langle K\dot{x}, Kx \rangle - 2\langle Ax, \dot{x} \rangle \langle x, Kx \rangle \\ &= \langle K^2 x, \dot{x} \rangle \end{aligned}$$

since $\langle x, Kx \rangle = 0$ and K is symmetric.

To insure that $\sigma^2(x(t))$ is monotonically decreasing, we require $\langle K^2x, \dot{x}\rangle < 0$. Thus $\dot{x}$ must have a component in the direction of $-K^2x$. Since x is a unit vector, $\langle x, \dot{x}\rangle = 0$.

We set

$$\dot{x} = \alpha K^2 x + \beta x \tag{3.1}$$

for some $\alpha < 0$ and $\beta \neq 0$. Taking the inner product of x with each term in (3.1), we find that $\beta = -\alpha\langle K^2x, x\rangle = -\alpha\sigma^2$. Thus

$$\dot{x} = \alpha(K^2 - \sigma^2 I)x. \tag{3.2}$$

Furthermore,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\sigma^2(x) &= \langle K^2x, \dot{x}\rangle \\ &= \langle K^2x, \alpha K^2x - \alpha\sigma^2 x\rangle \\ &= \alpha(\|K^2x\|^2 - \sigma^2\langle K^2x, x\rangle) \\ &= \alpha(\|K^2x\|^2 - \|Kx\|^4).\end{aligned} \tag{3.3}$$

We observe that by the Cauchy–Schwarz inequality,

$$\begin{aligned}\|K^2x\|^2 - \|Kx\|^4 &= \|K^2x\|^2 - \langle K^2x, x\rangle^2 \\ &\geqq \|K^2x\|^2 - \|K^2x\|^2\|x\|^2 = 0.\end{aligned} \tag{3.4}$$

If we choose $\alpha = -1$ and recall that K depends upon x, then we obtain the autonomous differential equation (from (3.2)) on S^{n-1},

$$\frac{dx}{dt} = -(K^2x - \|Kx\|^2 x). \tag{3.5}$$

Now we demonstrate that the eigenvectors of A are solutions of (3.5). In fact, we have for $x \in S^{n-1}$, $Kx = 0$ if and only if x is an eigenvector of A. So let x be a unit eigenvector of A with eigenvalue λ. Then

$$Kx = Ax - \mu x = \lambda x - \langle Ax, x\rangle x = \lambda x - \lambda x = 0.$$

Conversely, if $Kx = 0$, then $Ax = \mu x$. Thus x is an eigenvector of A with eigenvalue μ. It follows that x satisfies (3.5).

We can even make a stronger statement when x is an eigenvector of A.

LEMMA 3.1. *Suppose $x \in S^{n-1}$. Then x is an eigenvector for A if and only if $K^2x = 0$.*

Proof. Clearly from the argument preceding this lemma, if x is an eigenvector for A, then $K^2x = K(Kx) = K(0) = 0$. Conversely, let $x \in S^{n-1}$ and $K^2x = 0$. Then $A(Kx) = \mu Kx$. Since $\langle Kx, x\rangle = \langle Ax - \mu x, x\rangle = \langle Ax, x\rangle - \mu\langle x, x\rangle = 0$, we obtain $\langle AKx, x\rangle = 0$. Expanding this we have

$$\begin{aligned}0 &= \langle A(A - \mu I)x, x\rangle \\ &= \langle A^2x, x\rangle - \mu\langle Ax, x\rangle \\ &= \|Ax\|^2 - \mu^2.\end{aligned}$$

But this is precisely σ^2, since

$$\sigma^2 = \|Kx\|^2 = \langle Ax - \mu x, Ax - \mu x\rangle$$
$$= \|Ax\|^2 - \mu^2.$$

Thus, $\sigma^2 = \|Kx\|^2 = 0$. Hence $Kx = 0$, so x is an eigenvector of A.

4. The stationary set for the differential equation. We now have that if $x \in S^{n-1}$ is an eigenvector of A, the right-hand side of (3.5) is zero. (Later we shall show that the eigenvector solutions are asymptotically stable.) We proceed to characterize all the points

$$M = \{x \in S^{n-1} : K^2x - \sigma^2 x = 0\}$$

for which the right-hand side of (3.5) vanishes, the *stationary set* for (3.5). Let the distinct eigenvalues of A be $\gamma_1, \gamma_2, \cdots, \gamma_d$ with multiplicities $m_1, m_2, \cdots, m_d$. Denote by E_k the eigenspace of γ_k, that is, $E_k = \{x \in R^n : Ax = \gamma_k x\}$. The dimension of E_k is m_k. Each of the E_k has a basis of vectors drawn from $\{e_i\}_{i=1}^n$; hence the subspaces $\{E_k\}_{k=1}^d$ are mutually orthogonal. For any $x \in R^n$, we have $x = \sum_{k=1}^d x_k$, $x_k \in E_k$, and $\langle x_j, x_k\rangle = 0$ whenever $j \neq k$. The following theorem characterizes the set M (see [3] for a related result).

THEOREM 4.1. *Let $x \in S^{n-1}$. Then $K^2x = \sigma^2 x$ if and only if exactly one of the following holds*:

(i) *x is an eigenvector of A,*

(ii) *$x = x_j + x_k$, $x_j \in E_j$, $x_k \in E_k$, $j \neq k$, and $\|x_j\|^2 = \|x_k\|^2 = \frac{1}{2}$. This representation is unique.*

Proof. Suppose $K^2x - \sigma^2x = 0$. Consider K^2x. We have for $x = \sum_{i=1}^n \xi_i e_i$, x not an eigenvector,

$$
\begin{aligned}
K^2x &= A^2x - 2\mu Ax + \mu^2 x \\
&= \sum_{i=1}^n \xi_i\lambda_i^2 e_i - 2\mu \sum_{i=1}^n \xi_i\lambda_i e_i + \mu^2 \sum_{i=1}^n \xi_i e_i \qquad (4.1)\\
&= \sum_{i=1}^n (\lambda_i - \mu)^2 \xi_i e_i.
\end{aligned}
$$

Then

$$K^2x - \sigma^2x = \sum_{i=1}^n \{(\lambda_i - \mu)^2 - \sigma^2\}\xi_i e_i = 0$$

implies that for all those nonzero ξ_i, $\sigma^2 = (\lambda_i - \mu)^2$. Denote by I those indices i such that $\xi_i \neq 0$. Set

$$I_+ = \{i \in I : \lambda_i = \mu + \sigma\},$$
$$I_- = \{i \in I : \lambda_i = \mu - \sigma\}.$$

In fact, let the number of indices in I_+ and I_- be m_+ and m_- respectively. Thus, x is such that $\mu + \sigma$ and $\mu - \sigma$ are eigenvalues of A with multiplicities m_+ and m_- respectively. However, for a given $x \in S^{n-1}$, at most, two distinct eigenvalues can satisfy $(\lambda - \mu)^2 = \sigma^2$.

Now $x \in M$ implies

$$x = \sum_{i \in I_+} \xi_i e_i + \sum_{i \in I_-} \xi_i e_i$$

uniquely. Letting x_+ and x_- denote the first and second sums respectively in the expression above for x, we have

$$\begin{aligned} Ax &= \sum_{i \in I_+} \lambda_i \xi_i e_i + \sum_{i \in I_-} \lambda_i \xi_i e_i \\ &= (\mu + \sigma)x_+ + (\mu - \sigma)x_- \\ &= \mu x + \sigma(x_+ - x_-). \end{aligned}$$

Moreover, taking the inner product $\langle Ax, x \rangle$ we find that

$$\begin{aligned} \langle Ax, x \rangle &= \mu + \sigma \langle x_+ - x_-, x_+ + x_- \rangle \\ &= \mu + \sigma(\|x_+\|^2 - \|x_-\|^2), \end{aligned}$$

since $x_+ \in E_+$, $x_- \in E_-$, the respective eigenspaces for $\mu + \sigma$ and $\mu - \sigma$. $\langle Ax, x \rangle$ is defined as μ so we must have $\|x_+\|^2 - \|x_-\|^2 = 0$. But $\|x^2\| = 1$; therefore,

$$\|x_+\|^2 = \|x_-\|^2 = \tfrac{1}{2}. \tag{4.2}$$

Conversely, let x satisfy condition (ii) of the theorem. Letting $x = x_j + x_k$, we obtain after an easy calculation,

$$K^2 x = \sigma^2 x = \tfrac{1}{4}(\gamma_j - \gamma_k)^2 x, \tag{4.3}$$

where γ_j, γ_k are the eigenvalues associated with the eigenspaces of x_j and x_k.

Remark 4.2. Condition (4.2) represents the product of two spheres in S^{n-1}, each of radius $\frac{1}{2}$. In particular, if $m_+ = 1$, then $\|x_+\|^2 = \frac{1}{2}$ represents the 0-sphere, that is, just the two points $\pm 1/\sqrt{2}$. In general, every point x of M lies on some product of spheres, $S^{m_j - 1} \times S^{m_k - 1}$, where $x = x_j + x_k$, and m_j and m_k are the multiplicities of the eigenspaces containing x_j and x_k. Taking all such combinations $x_j + x_k$ over all pairs of eigenspaces, we have that M consists of a finite union of products of spheres of the type indicated above.

Example 4.3. For purposes of illustration of the last theorem, we consider the case in which all the eigenvalues of A are distinct. Therefore if $x \in M$, $x = \sum_{i=1}^n \xi_i e_i$, then no more than two of the $\{\xi_i\}_{i=1}^n$ are nonzero. If only one of the $\{\xi_i\}_{i=1}^n$, say ξ_{i_0}, is nonzero, then $\xi_{i_0}^2 = 1$, hence $x = \pm e_{i_0}$ is an eigenvector. If, say, ξ_i and ξ_j are nonzero (for some $i \neq j$), we must have $\lambda_i = \mu + \sigma$ and $\lambda_j = \mu - \sigma$ (or vice versa). Furthermore, $\xi_i^2 = \xi_j^2 = \frac{1}{2}$. Thus x must be of the form

$$\pm \frac{1}{\sqrt{2}} e_i \pm \frac{1}{\sqrt{2}} e_j.$$

This, of course, is equivalent to the product of two 0-spheres. It follows that M consists of all such points for all $i, j = 1, 2, \cdots, n, i \neq j$, along with the eigenvectors of A.

5. The Lyapunov function. We now turn to an examination of a Lyapunov function for (3.5). We choose for this function the quantity $\sigma^2(x)$, and henceforth will usually denote it by $V(x)$. By definition,

$$V(x) = \|Kx\|^2 \geqq 0$$

and with $\alpha = -1$, from (3.3) and (3.4) we obtain

$$\dot{V}(x) = -(\|K^2x\|^2 - \|Kx\|^4) \leqq 0. \tag{5.1}$$

The term $\dot{V}$ represents dV/dt along trajectories $x(t)$. Observe that since

$$\|K^2x - \|Kx\|^2x\|^2 = \|K^2x\|^2 - \|Kx\|^4,$$

the set of $x \in S^{n-1}$ for which $\dot{V}(x)$ vanishes is precisely the set M of stationary points of the differential equation (3.5).

A comment on this equivalence is in order. We know in general that

$$\dot{V}(x) = \langle \mathrm{grad}_x V, \dot{x} \rangle,$$

where $\mathrm{grad}_x V$ is the gradient of V evaluated at x. Suppose, for some $x \in S^{n-1}$ which is not an eigenvector of A, that $\dot{V}(x) = 0$. If $\dot{x} \neq 0$, we conclude $\mathrm{grad}_x V$ is normal to S^{n-1} at x since $\dot{x}$ is tangent to S^{n-1} at x. But it is easily verified for any $x \in S^{n-1}$ that $\mathrm{grad}_x V = 2K^2x$. Thus, if $\dot{V}(x) = 0$ and x is not an eigenvector of A, there exists a nonzero scalar p for which $2K^2x + px = 0$, i.e., K^2x is collinear with x. Taking the inner product with x yields $p = -2\|Kx\|^2$. Thus, $K^2x - \|Kx\|^2x = 0$, or $x \in M$. It follows that we must have $\dot{x} = 0$.

Now consider the set Z of critical points of $V(x)$ on S^{n-1}. Let $g(x) = \|x\|^2 - 1$.

$$Z = \{x \in S^{n-1} : \mathrm{grad}_x(V + \eta g) = 0 \text{ for some } \eta \in R\}.$$

Z contains all those points of S^{n-1} where V has a relative maximum, relative minimum, or saddle point. Moreover, we know in general that Z is characterized by those points at which $\mathrm{grad}_x V$ is normal to S^{n-1}. Therefore, $Z = M$.

Remark 5.1. We observe the following:

$$V(x) = \|Kx\|^2 = \langle Kx, Kx \rangle = \langle K^2x, x \rangle.$$

When x is not an eigenvector of A, we use the Cauchy–Schwarz inequality on $\langle K^2x, x\rangle$ to obtain

$$V(x) \leqq \|K^2x\|.$$

Moreover, we get equality whenever K^2x and x are collinear. That is, $V(x)$ is extremal when $\mathrm{grad}_x V$ is outwardly normal to S^{n-1}. This yields, as usual, $K^2x = \sigma^2x$.

By a straightforward, albeit long calculation, we can derive an alternative expression for $V(x)$, more useful than (2.1). When $x \in S^{n-1}$ we obtain

$$V(x) = \sum_{j=1}^{n} \sum_{i=1}^{j} (\lambda_j - \lambda_i)^2 \xi_i^2 \xi_j^2, \tag{5.2}$$

a homogeneous polynomial of degree 4. Restricting ourselves to M, we can compute the extremal values for $V(x)$. For any x not an eigenvector, we have

$x = x_j + x_k$ uniquely, with γ_j and γ_k the associated (distinct) eigenvalues. Then (4.3) provides

$$V(x) = \tfrac{1}{4}(\gamma_j - \gamma_k)^2.$$

In particular, $V(x)$ is constant on each product sphere of M.

6. Proof that the eigenvectors of A are asymptotically stable. Define for the distinct eigenvalues $\gamma_1, \gamma_2, \cdots, \gamma_d$ of A,

$$G = \bigcap_{\substack{j,k=1 \\ j \neq k}}^{d} \{x \in S^{n-1} : V(x) < \tfrac{1}{4}(\gamma_j - \gamma_k)^2\}.$$

Then $G = \bigcup_{i=1}^{d} G_i$, where G_i is the open (in S^{n-1}) component of G which contains $E_i \cap S^{n-1}$. (Except in the case of S^0, this set is connected.)

THEOREM 6.1. *The set of unit eigenvectors of A is asymptotically stable. Equivalently, suppose $x(t; x_0)$ is a solution of* (3.5) *for any initial value $x_0 \in G_i$, so that $x(0; x_0) = x_0$. Then $\lim_{t \to \infty} x(t; x_0) \in E_i \cap S^{n-1}$.*

Proof. Letting S_i denote $E_i \cap S^{n-1}$, we have $V(x) = 0$ on S_i, $V(x) > 0$ on $G_i \setminus S_i$, $\dot{V}(x) = 0$ on S_i, and $\dot{V}(x) < 0$ on $G_i \setminus S_i$. Then V is a Lyapunov function for (3.5) and by a result of Lyapunov (see [2, pp. 296–297]) every solution $x(t; x_0)$ of (3.5) with $x(0; x_0) = x_0 \in G_i$ tends to S_i as $t \to \infty$.

7. Addendum. Our choice for the differential equation rested on selecting $\alpha = -1$. This is convenient, but by no means is unique with respect to finding a differential equation which admits asymptotically stable solutions of the form: $x =$ eigenvector of A. In fact, let

$$\alpha = -\frac{1}{2}\frac{\|Kx\|^2}{\|K^2x\|^2 - \|Kx\|^4}. \tag{7.1}$$

This yields a simple differential equation for σ^2 (or V) along trajectories of (3.2). From (3.3) we obtain

$$\frac{d}{dt}\sigma^2 = -\sigma^2.$$

Thus

$$V(t) = \sigma^2(t) = \sigma_0^2 e^{-t}, \tag{7.2}$$

where σ_0^2 is a positive constant. Thus V is strictly decreasing along all (nonstationary) solutions of (3.2). The author has considered this case for a 2×2 matrix A, and obtained, not unexpectedly, more rapid convergence to eigenvectors.

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SIAM J. MATH. ANAL.
Vol. 2, No. 3, August 1971

APPELL FUNCTIONS AND MULTIPLE AVERAGES*

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Abstract. If f is an analytic function of one complex variable, two applications of an averaging process produce from f an analogous analytic function $\mathscr{F}(b, Z, \beta)$ depending on a rectangular matrix Z of complex variables and on two sets of complex parameters, one b-parameter being associated with each row of Z and one β-parameter with each column. Special cases of $\mathscr{F}$ include the Appell functions F_1, F_2, and F_3 and the Lauricella functions F_B and F_D. Transformations of these functions are shown to be equivalent to the symmetry of $\mathscr{F}$ under permutations of the rows or columns of Z. Differential equations, series expansions, and a Cauchy integral formula are given for $\mathscr{F}$. A different type of multiple average, obtained by averaging a function of several complex variables $f(z_1, \cdots, z_n)$ with respect to each variable separately, is denoted by $F(B, Z)$, where B and Z are matrices of parameters and variables, respectively. The transformations of Lauricella's F_A are equivalent to permutation symmetries of $F(B, Z)$.

1. Introduction. By averaging an analytic function $f(z)$ of one complex variable over the convex hull of $\{z_1, \cdots, z_k\}$, we produce an analytic function of several complex variables with properties quite analogous to those of f [4]. Because the weight function used in the averaging involves complex parameters $b_1, \cdots, b_k$ (the parameter b_i being closely associated with the variable z_i), the average is denoted by $F(b, z)$, where b and z are k-tuples. If f is taken to be a power or exponential function, F is a hypergeometric or confluent hypergeometric function. In particular we obtain in this way the special functions known as ${}_2F_1$, ${}_1F_1$, Appell's F_1, Lauricella's F_D, and Legendre's elliptic integrals of all three kinds. In [5] it is shown that many known transformations of these functions are equivalent to the symmetry of $F(b, z)$ with respect to simultaneous permutations of the components of b and z.

In the present paper we extend these considerations to multiple averages, especially a double average which changes $f(z)$ into a function $\mathscr{F}(b, Z, \beta)$, where one b-parameter is associated with each row and one β-parameter with each column of the rectangular matrix Z. Among the special functions reached in this way are Appell's F_2 and F_3 and Lauricella's F_B. The transformations of F_2 into itself are shown to be equivalent to the symmetry of $\mathscr{F}$ under permutations of rows and columns. Moreover, F_1 and F_D also have representations in which Z is a matrix with two rows, and symmetry under interchange of these rows is equivalent to the Euler transformation which was not identified as a permutation symmetry in [5]. Some properties of $\mathscr{F}$ and its special cases are discussed, including differential equations, a number of series expansions, and an analogue of Cauchy's integral formula.

In § 7 we consider more briefly a different type of multiple average, starting from an analytic function $f(z_1, \cdots, z_n)$ of several complex variables and averaging with respect to each variable separately. The result is a function $F(B, Z)$ in which each element of the matrix B is a complex parameter associated with the corresponding element of the matrix Z of complex variables. Lauricella's F_A can be

* Received by the editors September 22, 1970.

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represented as either type of multiple average, but the notation afforded by this second type is much more convenient. The transformations of F_A into itself are shown to be equivalent to permutation symmetries.

A quite different approach to special functions of several complex variables, which is closely connected with homogeneous convex cones but apparently not with Appell functions, can be found in [7] and the earlier papers cited therein.

2. Double averages of functions of one variable. Let Z be a $k \times \varkappa$ matrix with complex elements Z_{ij}. Let $u = (u_1, \cdots, u_k)$ be an ordered k-tuple of real nonnegative weights with $\sum u_i = 1$, and similarly for $v = (v_1, \cdots, v_\varkappa)$. We define

$$u \cdot Z \cdot v = \sum_{i=1}^{k} \sum_{j=1}^{\varkappa} u_i Z_{ij} v_j, \tag{2.1}$$

$$u \cdot Z_j = \sum_{i=1}^{k} u_i Z_{ij}, \qquad j = 1, 2, \cdots, \varkappa, \tag{2.2}$$

$${}_iZ \cdot v = \sum_{j=1}^{\varkappa} Z_{ij} v_j, \qquad i = 1, 2, \cdots, k. \tag{2.3}$$

If Z_{ij} is regarded as a point of the complex plane, all these convex combinations are points in the convex hull of $\{Z_{11}, \cdots, Z_{k\varkappa}\}$, denoted by $H(Z)$.

Let $b = (b_1, \cdots, b_k)$ be an ordered k-tuple of complex numbers with positive real parts (signified by $\operatorname{Re} b > 0$), and similarly for $\beta = (\beta_1, \cdots, \beta_\varkappa)$. We define

$$c = \sum_{i=1}^{k} b_i, \qquad \gamma = \sum_{j=1}^{\varkappa} \beta_j. \tag{2.4}$$

Let E be the standard $(k-1)$-simplex with vertices $(0, 0, \cdots, 0), (1, 0, \cdots, 0), \cdots, (0, 0, \cdots, 1)$. On E we define the measure

$$d\mu_b(u) = \prod_{i=1}^{k} u_i^{b_i - 1}\, du_1 \cdots du_{k-1} \left(\int_E \prod_{i=1}^{k} u_i^{b_i - 1}\, du_1 \cdots du_{k-1} \right)^{-1}, \tag{2.5}$$

and similarly for $d\mu_\beta(v)$. By $\int g(u)\, d\mu_b(u)$ we shall always mean $\int_E g(u)\, d\mu_b(u)$.

DEFINITION 1. Let f be holomorphic on a domain D in the complex plane. If $\operatorname{Re} b > 0$, $\operatorname{Re} \beta > 0$, and $H(Z) \subset D$, we define

$$\mathscr{F}(b, Z, \beta) = \int \int f(u \cdot Z \cdot v)\, d\mu_b(u)\, d\mu_\beta(v) \tag{2.6}$$

for any $k, \varkappa = 2, 3, 4, \cdots$. We define $\mathscr{F}'$, or in general $\mathscr{F}^{(n)}$, $n = 0, 1, 2, \cdots$, by replacing f in (2.6) by $f' = df/dz$ or $f^{(n)} = d^n f/dz^n$, respectively, where $f^{(0)} = f$ and $f^{(1)} = f'$. If k or $\varkappa$ is unity, the corresponding integration is omitted and $\mathscr{F}$ is therefore independent of b_1 or β_1, respectively. If $k = \varkappa = 1$ we define $\mathscr{F} = f$.

As in [4, Theorem 1] it follows immediately that $\mathscr{F}$ is holomorphic in the elements of b, Z, and β on its domain of definition. Also as in [4], $\mathscr{F}$ can be continued analytically in the parameters and variables so long as $c \neq 0, -1, -2, \cdots$, $\gamma \neq 0, -1, -2, \cdots$, and all Z_{ij} remain in D, provided that D is simply connected. Note that

$$\mathscr{F}(b, Z, \beta) = f(\zeta) \quad \text{if } Z_{ij} = \zeta \quad \text{for all } i \text{ and } j. \tag{2.7}$$

If $\varkappa = 1$, then $\mathscr{F}(b, Z, \beta) = F(b_1, \cdots, b_k; Z_{11}, \cdots, Z_{k1})$, where F is the single average discussed in [4], and similarly if $k = 1$. Moreover, by performing one of the integrations in (2.6), we find

$$\begin{aligned} \mathscr{F}(b, Z, \beta) &= \int F(b_1, \cdots, b_k;\ {}_1Z \cdot v, \cdots, {}_kZ \cdot v)\, d\mu_\beta(v) \\ &= \int F(\beta_1, \cdots, \beta_\varkappa; u \cdot Z_1, \cdots, u \cdot Z_\varkappa)\, d\mu_b(u). \end{aligned} \tag{2.8}$$

From the symmetry of F it follows that $\mathscr{F}$ has these properties.

PROPERTIES 2.9.

(i) Row symmetry (symmetry in the indices $1, \cdots, k$ which label the b-parameters and the rows of Z);

(ii) Column symmetry (symmetry in the indices $1, \cdots, \varkappa$ which label the β-parameters and the columns of Z);

(iii) Transposition symmetry: $\mathscr{F}(b, Z, \beta) = \mathscr{F}(\beta, \tilde{Z}, b)$,

where $\tilde{Z}$ is the transposed matrix with elements $\tilde{Z}_{ij} = Z_{ji}$. Other properties of F imply corresponding properties of $\mathscr{F}$. In particular [4, p. 128], a vanishing b-parameter can be omitted along with the corresponding row of Z, and two or more identical rows can be replaced by a single row if the corresponding b-parameters are replaced by their sum. Similar statements hold for β-parameters and columns.

Corresponding to the particular functions z^t and e^z, we define

$$\mathscr{R}_t(b, Z, \beta) = \int\int (u \cdot Z \cdot v)^t\, d\mu_b(u)\, d\mu_\beta(v), \tag{2.10}$$

$$\mathscr{S}(b, Z, \beta) = \int\int e^{u \cdot Z \cdot v}\, d\mu_b(u)\, d\mu_\beta(v). \tag{2.11}$$

In (2.10), t is any complex constant and the domain D in Definition 1 is the complex plane cut along the nonpositive real axis. If λ is a complex constant, let λZ and $Z + \lambda$ denote the matrices with elements λZ_{ij} and $Z_{ij} + \lambda$, respectively; note that the rectangular matrix with all elements unity is to be denoted here by 1. The following homogeneity properties are then obvious:

$$\mathscr{R}_t(b, \lambda Z, \beta) = \lambda^t \mathscr{R}_t(b, Z, \beta), \qquad \mathscr{S}(b, Z + \lambda, \beta) = e^\lambda \mathscr{S}(b, Z, \beta). \tag{2.12}$$

So also is the binomial theorem for $\mathscr{R}$-polynomials,

$$\mathscr{R}_n(b, Z + \lambda, \beta) = \sum_{m=0}^{n} \binom{n}{m} \lambda^{n-m} \mathscr{R}_m(b, Z, \beta), \qquad n = 0, 1, 2, \cdots, \tag{2.13}$$

and the relation of confluence,

$$\mathscr{S}(b, Z, \beta) = \lim_{t \to \infty} \mathscr{R}_t(b, 1 + t^{-1}Z, \beta). \tag{2.14}$$

3. Hidden symmetry of Appell's F_2. For the notations and integral representations used in this and later sections for various hypergeometric functions, we refer to [1] and [6]. Comparison with (2.10) gives the following identifications, wherein $\mathcal{R}_t(b, Z, \beta)$ is written in the more explicit form $\mathcal{R}_t(b_1, \cdots, b_k; Z; \beta_1, \cdots, \beta_x)$:

(3.1)
$$F_2(a, b, \beta; c, \gamma; x, y) = \mathcal{R}_{-a}(b, c-b; Z; \beta, \gamma-\beta),$$
$$Z = \begin{bmatrix} 1-x-y & 1-x \\ 1-y & 1 \end{bmatrix};$$

(3.2)
$$_3F_2(a, b, \beta; c, \gamma; z) = \mathcal{R}_{-a}(b, c-b; Z; \beta, \gamma-\beta),$$
$$Z = \begin{bmatrix} 1-z & 1 \\ 1 & 1 \end{bmatrix}.$$

Although (3.2) fails to display the symmetry of $_3F_2$ in its parameters, (3.1) and (2.12) show that

(3.3)
$$\mathcal{R}_{-a}(b_x, b_y; Z; \beta_z, \beta_w) = (y+w)^{-a}F_2\left(a, b_x, \beta_z; b_x+b_y, \beta_z+\beta_w; \frac{y-x}{y+w}, \frac{w-z}{y+w}\right),$$
$$Z = \begin{bmatrix} x+z & x+w \\ y+z & y+w \end{bmatrix}.$$

The row and column symmetries of $\mathcal{R}$ (see Properties 2.9) imply that the expression containing F_2 must be symmetric in x and y and symmetric in z and w. These two symmetries, together with their product, are readily shown to be equivalent to the three transformations of F_2 into itself [6, § 5.11].

The symmetry of F_2 will be treated again in § 7 from a different point of view which allows an extension to Lauricella's F_A.

4. Properties of restricted $\mathcal{R}$-functions. The function $\mathcal{R}_t(b, Z, \beta)$ has some special properties if $t = -c$ or $t = -\gamma$ (see (2.4)). From the formula [4, (3.21)]

(4.1)
$$R_{-c}(b, z) = \prod_{i=1}^{k} z_i^{-b_i},$$

it follows by (2.8) that

(4.2)
$$\mathcal{R}_{-c}(b, Z, \beta) = \int \prod_{i=1}^{k} ({}_iZ \cdot v)^{-b_i}\, d\mu_\beta(v), \qquad \operatorname{Re}\beta > 0.$$

This function has two obvious properties.

PROPERTY 4.3 (Row homogeneity). $\mathcal{R}_{-c}$ is homogeneous of degree $-b_i$ in the elements of the ith row of Z, for $i = 1, 2, \cdots, k$.

PROPERTY 4.4 (Unit row property). $\mathcal{R}_{-c}(b, Z, \beta) = \mathcal{R}_{-c'}(b', Z', \beta)$, where Z' is distinguished from Z by an extra row with all elements equal to unity and where $b' = (b_1, \cdots, b_k, c'-c)$. The value of c' is arbitrary except that $c' \neq 0, -1, -2, \cdots$.

Column homogeneity and a unit column property for $\mathcal{R}_{-\gamma}$ follow by Properties 2.9.

If $t = -c = -\gamma$, then $\mathcal{R}_t$ has both row and column homogeneity and will be denoted by $\mathcal{R}$ with no subscript. That is, if $c = \gamma$ we define

(4.5)
$$\mathcal{R}(b, Z, \beta) = \mathcal{R}_{-c}(b, Z, \beta), \qquad c = \gamma.$$

This function provides a useful standard form, which we shall call the bare form, in which to express $\mathscr{R}_t$ whenever $t = -c$, for we can apply the unit row property and choose $c' = \gamma$. Similar remarks apply whenever $t = -\gamma$. The bare $\mathscr{R}$-function has the integral representations

$$\begin{aligned}\mathscr{R}(b, Z, \beta) &= \int \prod_{i=1}^{k} ({}_iZ \cdot v)^{-b_i}\, d\mu_\beta(v) \qquad (\operatorname{Re}\beta > 0)\\ &= \int \prod_{j=1}^{\varkappa} (u \cdot Z_j)^{-\beta_j}\, d\mu_b(u) \qquad (\operatorname{Re} b > 0).\end{aligned} \tag{4.6}$$

5. Applications to hypergeometric functions. From the integral representation [6, (2.1(10))], we find

$$ {}_2F_1(a, b; c; z) = \mathscr{R}_{-a}(a; 1 - z, 1; b, c - b), \tag{5.1}$$

where the elements of the row matrix Z are displayed and we have chosen the immaterial row parameter to be a. We can therefore use the bare form:

$$ {}_2F_1(a, b; c; z) = \mathscr{R}(a, c - a; Z; b, c - b), \qquad Z = \begin{bmatrix} 1 - z & 1 \\ 1 & 1 \end{bmatrix}. \tag{5.2}$$

The symmetry of ${}_2F_1$ in a and b, which is conspicuous in the series representation but not in the usual integral representation, is plain from the transposition symmetry, Property 2.9 (iii), of $\mathscr{R}$. Substitution of (2.10) gives a manifestly symmetric integral representation of ${}_2F_1$ which is due to Erdélyi [6, (2.4(1))].

The homogeneity properties of $\mathscr{R}$ imply with (5.2) that

$$\begin{aligned}\mathscr{R}(a_x, a_y; Z; b_1, b_2) &= y_1^{-b_1} y_2^{-b_2} \left(\frac{x_2}{y_2}\right)^{-a_x} {}_2F_1\left(a_x, b_1; c; 1 - \frac{x_1 y_2}{x_2 y_1}\right),\\ Z &= \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}, \qquad a_x + a_y = c = b_1 + b_2.\end{aligned} \tag{5.3}$$

Row and column symmetries of $\mathscr{R}$ show that the members of the first equation are symmetric in x and y and symmetric in the indices 1 and 2. These two symmetries and their product are equivalent to three transformations of ${}_2F_1$ [7, (2.1 (22), (23))].

From the representations of Appell's F_3 and F_1 by double integrals [1, p.28], we find

$$\begin{aligned} F_3(\alpha, \alpha', \beta, \beta'; \gamma; x, y) &= \mathscr{R}_{-\alpha-\alpha'}(\alpha, \alpha'; Z; \beta, \beta', \gamma - \beta - \beta'),\\ Z &= \begin{bmatrix} 1 - x & 1 & 1 \\ 1 & 1 - y & 1 \end{bmatrix};\end{aligned} \tag{5.4}$$

$$F_1(\alpha, \beta, \beta'; \gamma; x, y) = \mathscr{R}_{-\alpha}(\alpha; 1 - x, 1 - y, 1; \beta, \beta', \gamma - \beta - \beta'). \tag{5.5}$$

In the first case we have $t = -c = -\alpha - \alpha'$. In the second case the elements of the row matrix Z are displayed and we have chosen the immaterial row parameter so that $t = -c = -\alpha$. Thus the bare form can be used in both cases:

$$F_3(\alpha, \alpha', \beta, \beta'; \gamma; x, y) = \mathcal{R}(\alpha, \alpha', \gamma - \alpha - \alpha'; Z; \beta, \beta', \gamma - \beta - \beta'),$$
(5.6)
$$Z = \begin{bmatrix} 1 - x & 1 & 1 \\ 1 & 1 - y & 1 \\ 1 & 1 & 1 \end{bmatrix};$$

$$F_1(\alpha, \beta, \beta'; \gamma; x, y) = \mathcal{R}(\alpha, \gamma - \alpha; Z; \beta, \beta', \gamma - \beta - \beta'),$$
(5.7)
$$Z = \begin{bmatrix} 1 - x & 1 - y & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Equations (5.6) and Properties 2.9 exhibit the symmetry of F_3 under interchange of α, α' with β, β', a symmetry which is conspicuous in the series representation but not in the usual integral representation nor in (5.4). Equations (5.7) and (4.6) show that F_1 has a representation by a single integral as well as one by a double integral.

If F_3 is restricted by the condition $\gamma = \alpha + \alpha'$, the third row of Z in (5.6) can be omitted because the corresponding row parameter vanishes. We then use column homogeneity to obtain

$$F_3(\alpha, \alpha', \beta, \beta'; \alpha + \alpha'; x, y) = (1 - y)^{-\beta'}\mathcal{R}(\alpha, \alpha'; Z; \beta, \beta', \alpha + \alpha' - \beta - \beta'),$$
(5.8)
$$Z = \begin{bmatrix} 1 - x & (1 - y)^{-1} & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Comparison with (5.7) puts the restricted F_3 in terms of F_1 and thus gives a very simple proof of [6, (5.11(11))].

The homogeneity properties of $\mathcal{R}$ imply also that

$$\mathcal{R}(a_x, a_y; Z; b_1, b_2, b_3) = \prod_{i=1}^{3} y_i^{-b_i}\left(\frac{x_3}{y_3}\right)^{-a_x} F_1\left(a_x, b_1, b_2; c; 1 - \frac{x_1 y_3}{y_1 x_3}, 1 - \frac{x_2 y_3}{y_2 x_3}\right),$$
(5.9)
$$Z = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}, \qquad a_x + a_y = c = b_1 + b_2 + b_3.$$

Row and column symmetries of $\mathcal{R}$ show that the expression containing F_1 must be symmetric in x and y and symmetric in the indices 1, 2, 3. These symmetries are readily shown to imply the five transformations of F_1 into itself [6, § 5.11].

If F_2 is restricted by the condition $\alpha = \gamma'$ (or $\alpha = \gamma$), (3.1) can be replaced by the bare form:

$$F_2(\alpha, \beta, \beta'; \gamma, \alpha; x, y) = \mathcal{R}(\beta, \gamma - \beta; Z; \beta', \alpha - \beta', \gamma - \alpha),$$
(5.10)
$$Z = \begin{bmatrix} 1 - x - y & 1 - x & 1 \\ 1 - y & 1 & 1 \end{bmatrix}.$$

Using homogeneity in the elements of the first column of Z and comparing with (5.7), we can put the restricted F_2 in terms of F_1, as in [1, (10), p. 35].

The generalizations of F_3 and F_1 provided by the Lauricella functions F_B and F_D can be put in the bare form also:

$$(5.11)\quad \begin{aligned} &F_B(\alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_n; \gamma; x_1, \cdots, x_n) \\ &\quad = \mathscr{R}(\alpha_1, \cdots, \alpha_n, \gamma - \alpha_1 - \cdots - \alpha_n; Z; \beta_1, \cdots, \beta_n, \gamma - \beta_1 - \cdots - \beta_n), \\ &\qquad Z = \begin{bmatrix} 1 - x_1 & 1 & \cdots & & 1 \\ 1 & 1 - x_2 & & & \vdots \\ \vdots & & \ddots & & \\ & & & 1 - x_n & 1 \\ 1 & & \cdots & 1 & 1 \end{bmatrix}; \end{aligned}$$

$$(5.12)\quad \begin{aligned} &F_D(\alpha, \beta_1, \cdots, \beta_n; \gamma; x_1, \cdots, x_n) = \mathscr{R}(\alpha, \gamma - \alpha; Z; \beta_1, \cdots, \beta_n, \gamma - \beta_1 - \cdots - \beta_n), \\ &\qquad Z = \begin{bmatrix} 1 - x_1 & \cdots & 1 - x_n & 1 \\ 1 & \cdots & 1 & 1 \end{bmatrix}. \end{aligned}$$

The expression for F_B shows the symmetry in α and β which is conspicuous in the series representation but not in the usual integral representation [1, p. 115]. The transformations of F_D [1, p. 116] can be identified as row and column symmetries in the same way as in (5.9).

In place of F_D or F_1 it seems preferable to use the equivalent R-function in which symmetry has already replaced all but one of the transformations. Since the R-function is an $\mathscr{R}$-function in which Z is a matrix with one row, it can be put in the bare form as follows:

$$(5.13)\quad \begin{aligned} R_{-a}(b_1, \cdots, b_k; z_1, \cdots, z_k) &= \mathscr{R}_{-a}(a; z_1, \cdots, z_k; b_1, \cdots, b_k) \\ &= \mathscr{R}(a, c - a; Z; b_1, \cdots, b_k), \\ Z &= \begin{bmatrix} z_1 & \cdots & z_k \\ 1 & & 1 \end{bmatrix}. \end{aligned}$$

Replacement of $\mathscr{R}$ by the second integral representation in (4.6) gives the representation of R by a single integral [4, (4.22)]. It follows from the column homogeneity of $\mathscr{R}$ that

$$(5.14)\quad \begin{aligned} \mathscr{R}(a, c - a; Z; b_1, \cdots, b_k) &= \prod_{i=1}^{k} w_i^{-b_i} R_{-a}\left(b_1, \cdots, b_k; \frac{z_1}{w_1}, \cdots, \frac{z_k}{w_k}\right), \\ Z &= \begin{bmatrix} z_1 & \cdots & z_k \\ w_1 & \cdots & w_k \end{bmatrix}. \end{aligned}$$

While the column symmetry of $\mathscr{R}$ is the same as the permutation symmetry of R, the row symmetry of $\mathscr{R}$ is equivalent to Euler's transformation [4, (4.23)]. Replacement of $\mathscr{R}$ in (5.14) by the first integral representation in (4.6) gives at once a formula [3, (6.12)] which has practical applications in evaluating elliptic integrals connected with ellipsoids.

6. Properties of double averages. If f has the Taylor series

$$f(z) = \sum_{n=0}^{\infty} \alpha_n (z - \lambda)^n, \qquad |z - \lambda| < \rho, \tag{6.1}$$

we find from (2.8) and [4, Theorem 2] that

$$\mathscr{F}(b, Z, \beta) = \sum_{n=0}^{\infty} \alpha_n \mathscr{R}_n(b, Z - \lambda, \beta), \qquad |Z - \lambda| < \rho, \tag{6.2}$$

where $|Z| = \max \{|Z_{11}|, \cdots, |Z_{k\kappa}|\}$ and $c, \gamma \neq 0, -1, -2 \cdots$. In particular we have

$$\mathscr{R}_{-a}(b, Z, \beta) = \sum_{n=0}^{\infty} \frac{(a, n)}{n!} \mathscr{R}_n(b, 1 - Z, \beta), \qquad |1 - Z| < 1, \tag{6.3}$$

$$\mathscr{S}(b, Z, \beta) = \sum_{n=0}^{\infty} \frac{1}{n!} \mathscr{R}_n(b, Z, \beta), \qquad |Z| < \infty. \tag{6.4}$$

It is obvious from (2.10) and (2.11) that

$$\mathscr{R}_t(b, Z, \beta) = R_t(b, z) R_t(\beta, \zeta), \qquad Z_{ij} = z_i \zeta_j, \tag{6.5}$$

$$\mathscr{S}(b, Z, \beta) = S(b, z) S(\beta, \zeta), \qquad Z_{ij} = z_i + \zeta_j, \tag{6.6}$$

where the stated conditions apply to all matrix elements Z_{ij}. By (6.6) and [4, (4.18)], a product of two Bessel functions $J_\mu(x) J_\nu(y)$ can be expressed as an $\mathscr{S}$-function. From (6.3), (6.4) and (6.5) we obtain the bilateral generating relations

$$\mathscr{R}_{-a}(b, 1 - Z, \beta) = \sum_{n=0}^{\infty} \frac{(a, n)}{n!} R_n(b, z) R_n(\beta, \zeta), \qquad Z_{ij} = z_i \zeta_j, \tag{6.7}$$

where $|Z| < 1$, and

$$\mathscr{S}(b, Z, \beta) = \sum_{n=0}^{\infty} \frac{1}{n!} R_n(b, z) R_n(\beta, \zeta), \qquad Z_{ij} = z_i \zeta_j. \tag{6.8}$$

In both cases it is assumed that $c, \gamma \neq 0, -1, -2, \cdots$.

In the first equation of (2.8) we put $Z_{ij} = z_i + \zeta_j$ and use [4, Theorem 6] to obtain

$$\mathscr{F}(b, Z, \beta) = \sum_{n=0}^{\infty} \frac{1}{n!} F^{(n)}(b, z) R_n(\beta, \zeta), \qquad Z_{ij} = z_i + \zeta_j. \tag{6.9}$$

It is assumed here that f is holomorphic on a simply connected domain D containing the ρ-neighborhoods of $z_1, \cdots, z_k$, where ρ is a positive number such that $|\zeta|_{\max} < \rho$. A special case is the generalized binomial series,

$$\mathscr{R}_t(b, Z, \beta) = \sum_{n=0}^{\infty} \binom{t}{n} R_{t-n}(b, z) R_n(\beta, \zeta), \qquad Z_{ij} = z_i + \zeta_j, \tag{6.10}$$

where D is now the complex plane cut along the nonpositive real axis.

From (2.8) and [4, Theorem 5] we conclude, with the help of analytic continuation, that $\mathscr{F}$ also has a generalized Cauchy formula,

$$\mathscr{F}^{(n)}(b, Z, \beta) = \frac{n!}{2\pi i}\int_\gamma f(t)\mathscr{R}_{-n-1}(b, t - Z, \beta)\, dt, \qquad n = 0, 1, 2, \cdots. \tag{6.11}$$

All the matrix elements of Z are required to lie in the inner region of the positively oriented rectifiable Jordan curve γ, and f is assumed to be holomorphic on γ and its inner region.

If we denote briefly by $\mathscr{F}(b_i + 1)$ the result of replacing one parameter b_i by $b_i + 1$ in $\mathscr{F}(b, Z, \beta)$, we find from (2.8) and [4, (2.9)] that

$$\begin{aligned} \mathscr{F} &= \sum_{i=1}^{k} \frac{b_i}{c}\mathscr{F}(b_i + 1) = \sum_{j=1}^{\varkappa} \frac{\beta_j}{\gamma}\mathscr{F}(\beta_j + 1) \\ &= \sum_{i=1}^{k}\sum_{j=1}^{\varkappa} \frac{b_i\beta_j}{c\gamma}\mathscr{F}(b_i + 1, \beta_j + 1). \end{aligned} \tag{6.12}$$

If $D_{ij} = \partial/\partial Z_{ij}$, the analogue of [4, (2.8)] is

$$D_{ij}\mathscr{F} = \frac{b_i\beta_j}{c\gamma}\mathscr{F}'(b_i + 1, \beta_j + 1). \tag{6.13}$$

Equations (6.12) and (6.13) imply

$$\begin{gathered} \sum_{i=1}^{k} D_{ij}\mathscr{F} = \frac{\beta_j}{\gamma}\mathscr{F}'(\beta_j + 1), \qquad \sum_{j=1}^{\varkappa} D_{ij}\mathscr{F} = \frac{b_i}{c}\mathscr{F}'(b_i + 1), \\ \sum_{i=1}^{k}\sum_{j=1}^{\varkappa} D_{ij}\mathscr{F} = \mathscr{F}'. \end{gathered} \tag{6.14}$$

From the last equation we deduce as in [4, Theorem 6] that

$$\mathscr{F}(b, Z + \lambda, \beta) = \sum_{n=0}^{\infty} \mathscr{F}^{(n)}(b, Z, \beta)\frac{\lambda^n}{n!}, \qquad |\lambda| < \rho, \tag{6.15}$$

provided f is holomorphic on a simply connected domain containing the ρ-neighborhoods of $Z_{11}, \cdots, Z_{k\varkappa}$.

We mention finally the system of second order differential equations satisfied by $\mathscr{F}$. It follows from (6.13) that

$$D_{im}D_{jn}\mathscr{F} = D_{in}D_{jm}\mathscr{F}, \qquad i, j = 1, \cdots, k, \quad m, n = 1, \cdots, \varkappa. \tag{6.16}$$

Also, a calculation which is too long to reproduce here shows that, for $i, j = 1, \cdots, k$,

$$\sum_{m=1}^{\varkappa}\sum_{n=1}^{\varkappa} (Z_{im} - Z_{jn})D_{im}D_{jn}\mathscr{F} + b_i \sum_{n=1}^{\varkappa} D_{jn}\mathscr{F} - b_j \sum_{m=1}^{\varkappa} D_{im}\mathscr{F} = 0. \tag{6.17}$$

Transposition symmetry (Property 2.9(iii)) then implies, for $m, n = 1, \cdots, \varkappa$,

$$\sum_{i=1}^{k} \sum_{j=1}^{k} (Z_{im} - Z_{jn}) D_{im} D_{jn} \mathscr{F} + \beta_m \sum_{j=1}^{k} D_{jn} \mathscr{F} - \beta_n \sum_{i=1}^{k} D_{im} \mathscr{F} = 0. \tag{6.18}$$

7. Other types of multiple averages. The double average in (2.6) is readily generalized to a triple average,

$$\int \int \int \left(\sum_i \sum_j \sum_k u_i v_j w_k Z_{ijk} \right) d\mu_b(u)\, d\mu_{b'}(v)\, d\mu_{b''}(w), \tag{7.1}$$

or even higher averages. Many properties of the double average have straightforward analogues for these cases, but the notations are rather cumbersome. Appell's F_4 can be written as a triple average of a power of z by using a representation by a double integral [6, (5.8(4))] together with (4.1), but the result seems to have no very simple features to recommend it.

A second kind of multiple average is reached by starting with a function of several variables, say $f(z_1, \cdots, z_n)$, and averaging each variable separately:

$$F(B, Z) = \int \cdots \int f(u^{(1)} \cdot z^{(1)}, \cdots, u^{(n)} \cdot z^{(n)})\, d\mu(b^{(1)}, u^{(1)}) \cdots d\mu(b^{(n)}, u^{(n)}), \tag{7.2}$$

where $u^{(j)} \cdot z^{(j)} = \sum_i u_i^{(j)} z_i^{(j)}$ and $\sum_i u_i^{(j)} = 1$ for $j = 1, \cdots, n$. We write $d\mu(b, u)$ in place of $d\mu_b(u)$ for typographical convenience. In general, B and Z are arrays of parameters and variables with columns $b^{(1)}, \cdots, b^{(n)}$ and $z^{(1)}, \cdots, z^{(n)}$, respectively. However, in the cases of principal interest every column has the same number of elements, say k, and B and Z are then rectangular matrices with elements $B_{ij} = b_i^{(j)}$ and $Z_{ij} = z_i^{(j)}$ for $i = 1, \cdots, k$ and $j = 1, \cdots, n$.

The second type of multiple average can be expressed in terms of the first if $f(z_1, \cdots, z_n)$ depends only on the sum $z_1 + \cdots + z_n$, for we then have

$$\begin{gathered} u^{(1)} \cdot z^{(1)} + \cdots + u^{(n)} \cdot z^{(n)} = \sum_i \cdots \sum_r u_i^{(1)} \cdots u_r^{(n)} W_{i \cdots r}, \\ W_{i \cdots r} = z_i^{(1)} + \cdots + z_r^{(n)}. \end{gathered} \tag{7.3}$$

However, the notation used in (7.2) seems preferable if $n > 2$.

Choosing $f(z_1, \cdots, z_n) = (z_1 + \cdots + z_n)^t$, we define

$$R_t(B, Z) = \int \cdots \int [u^{(1)} \cdot z^{(1)} + \cdots + u^{(n)} \cdot z^{(n)}]^t \; d\mu(b^{(1)}, u^{(1)}) \cdots d\mu(b^{(n)}, u^{(n)}), \tag{7.4}$$

where the elements of B have positive real parts and the convex hull of the set of elements of Z is contained in the plane cut along the nonpositive real axis. In addition to being homogeneous of degree t in the elements of Z, the R-function plainly has column symmetry if the columns of B and Z are permuted together. It has also symmetry within columns: the elements of the ith columns of B and Z may be permuted together without changing the value of the function. This implies row symmetry but not symmetry within rows. Finally, no change is produced by adding λ to all the elements of one column of Z and subtracting λ from all the elements of another column.

Because the structure of the matrix Z in (3.1) is like that of W in (7.3) with $n = 2$, Appell's F_2 can be expressed in the form (7.4):

$$F_2(\alpha, \beta, \beta'; \gamma, \gamma'; x, y) = R_{-\alpha}(B, Z),$$
$$B = \begin{bmatrix} \beta & \beta' \\ \gamma - \beta & \gamma' - \beta' \end{bmatrix}, \qquad Z = \begin{bmatrix} 1 - x & -y \\ 1 & 0 \end{bmatrix}. \tag{7.5}$$

The generalization of F_2 to n variables known as Lauricella's F_A has an integral representation [1, p. 115] which shows that

$$F_A(\alpha, \beta_1, \cdots, \beta_n; \gamma_1, \cdots, \gamma_n; x_1, \cdots, x_n) = R_{-\alpha}(B, Z),$$
$$B = \begin{bmatrix} \beta_1 & \cdots & \beta_n \\ \gamma_1 - \beta_1 & \cdots & \gamma_n - \beta_n \end{bmatrix}, \qquad Z = \begin{bmatrix} 1 - x_1 & -x_2 & \cdots & -x_n \\ 1 & 0 & \cdots & 0 \end{bmatrix}. \tag{7.6}$$

Conversely, we have

$$R_{-a}(B, Z) = \left(\textstyle\sum y_i\right)^{-a}$$
$$\cdot F_A\left(a, b_1, \cdots, b_n; b_1 + b_1', \cdots, b_n + b_n'; (y_1 - x_1)/\textstyle\sum y_i, \cdots, (y_n - x_n)/\textstyle\sum y_i\right),$$
$$B = \begin{bmatrix} b_1 & \cdots & b_n \\ b_1' & \cdots & b_n' \end{bmatrix}, \qquad Z = \begin{bmatrix} x_1 & \cdots & x_n \\ y_1 & \cdots & y_n \end{bmatrix}, \tag{7.7}$$

where $\sum y_i = y_1 + \cdots + y_n$. To show this, we subtract y_i from both elements of the ith column of Z and add it to both elements of the first column for $i = 2, 3, \cdots, n$. We then use homogeneity and compare with (7.6). The symmetry within columns of the R-function is equivalent to the transformations of F_A into itself [1, p. 116].

Because $F(B, Z)$ and $R_t(B, Z)$ are obtained by averaging one variable at a time, their general properties closely resemble those of the case $n = 1$, which was discussed in detail in [4], and the methods used there can be used again here. We mention only the system of differential equations satisfied by $F(B, Z)$, in which $D_{im} = \partial/\partial Z_{im}$:

$$(Z_{im} - Z_{jm})D_{im}D_{jm}F + B_{im}D_{jm}F - B_{jm}D_{im}F = 0, \tag{7.8}$$

for $i, j = 1, \cdots, k$ and $m = 1, \cdots, n$.

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SIAM J. MATH. ANAL.
Vol. 2, No. 3, August 1971

DISTRIBUTION DERIVATIVES OF FUNCTIONS HARMONIC IN THE UNIT DISC*

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Abstract. Let $u(r, \theta)$ be harmonic for $r < 1$ and continuous for $r \leqq 1$. Without additional smoothness assumptions on $f(\theta) = u(1, \theta)$ we cannot infer anything about the normal derivative of u on $r = 1$; $\lim_{r\to 1-} (f(\theta) - u(r, \theta))/(1 - r)$. The purpose of this paper is to show that this limit exists, for arbitrary continuous 2π-periodic f, as an element of a Sobolev space of index -1. We also show that $(\partial/\partial r)u(r, \theta)$ tends to the same limit in the same sense.

1. Introduction. Let $u(r, \theta)$ be harmonic for $r < 1$ and continuous for $r \leqq 1$. Without additional smoothness assumptions on $f(\theta) = u(1, \theta)$ we cannot infer anything about the normal derivative of u on $r = 1$; $\lim_{r\to 1-} (f(\theta) - u(r, \theta))/(1 - r)$. The purpose of this paper is to show that this limit exists, for arbitrary continuous 2π-periodic f, as an element of a Sobolev space of index -1. We will also show that we obtain the same result if we consider $\lim_{r\to 1-} (\partial/\partial r)u(r, \theta)$.

2. The spaces W_2^1 and W_2^{-1}.

DEFINITION. $W_2^1 = \{f | f$ is absolutely continuous on $[-\pi, \pi]$, $f(-\pi) = f(\pi)$, $f' \in L_2[-\pi, \pi]\}$.

For $f \in W_2^1$ we define $\|f\|_{W_2^1}^2 = \|f\|_{L_2}^2 + \|f'\|_{L_2}^2$. Standard arguments show that W_2^1 is complete under this norm and hence is a Hilbert space. In what follows we will be dealing with Fourier series. The symbol $\sum$ will denote that the sum is taken over all integers while the symbol $\sum'$ will denote that the sum is taken over all nonzero integers.

THEOREM 1. *Let $g \in L_2$, $g = \sum c_n e^{inx}$. Then $g \in W_2^1$ if and only if*

$$\sum n^2|c_n|^2 < \infty. \tag{2.1}$$

Proof. Suppose (2.1) holds. Consider $h = \sum' inc_n e^{inx}$. By (2.1), $h \in L_2$ hence $h \in L_1$. A computation shows $g(x) = g(-\pi) + \int_{-\pi}^x h(t)\,dt$. Thus $g' = h$ and $g(\pi) = g(-\pi)$. Therefore $g \in W_2^1$.

Conversely, suppose $g \in W_2^1$. Then $g' \in L_2$, $g' = \sum d_n e^{inx}$ with $\sum |d_n|^2 < \infty$. But $d_n = inc_n$. Thus $\sum n^2|c_n|^2 = \sum |d_n|^2 < \infty$.

COROLLARY 1. *If $g \in W_2^1$,*

$$\|g\|_{W_2^1}^2 = 2\pi \sum |c_n|^2(1 + n^2). \tag{2.2}$$

COROLLARY 2. *If $g \in W_2^1$, $g = \sum c_n e^{inx}$, the series converges to g in the topology of W_2^1.*

DEFINITION. We define W_2^{-1} to be the *topological dual* of W_2^1. For $f \in W_2^1$, $\phi \in W_2^{-1}$ we write $\phi(f)$ as $\langle f, \phi\rangle$.

If $f \in W_2^1$, $g \in L_2$,

$$\left|\int_{-\pi}^{\pi} f(t)g(t)\,dt\right| \leqq \|f\|_{L_2}\|g\|_{L_2} \leqq \|f\|_{W_2^1}\|g\|_{L_2}.$$

* Received by the editors June 30, 1970, and in revised form February 5, 1971.

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Thus the mapping $f \to \int_{-\pi}^{\pi} f(t)g(t)\,dt$ defines an element of W_2^{-1} which of course we denote by g; $\langle f, g\rangle = \int_{-\pi}^{\pi} f(t)g(t)\,dt$. In particular, if $g = (1/(2\pi))e^{-inx}$, then

$$\langle f, g\rangle = c_n. \tag{2.3}$$

If $f \in W_2^1$, $g \in L_2$, then $f' \in L_2$ and

$$\left| - \int f'(t)g(t)\,dt \right| \leqq \|f'\|_{L_2}\|g\|_{L_2} \leqq \|f\|_{W_2^1}\|g\|_{L_2}.$$

Thus the mapping $f \to -\int_{-\pi}^{\pi} f'(t)g(t)\,dt$ defines an element of W_2^{-1} which we denote by g';

$$\langle f, g'\rangle = -\int_{-\pi}^{\pi} f'(t)g(t)\,dt.$$

THEOREM 2. *Let $\phi \in W_2^{-1}$. Then ϕ is the weak limit of*

$$\phi_n = \frac{1}{2\pi}\sum_{|k|\leqq n} \langle e^{ikx}, \phi\rangle e^{-ikx}.$$

Proof. Let $f \in W_2^1$, $f = \sum c_n e^{inx}$. Then using (2.3), we have

$$\begin{aligned}\langle f, \phi\rangle &= \lim_{n\to\infty}\left\langle \sum_{|k|\leqq n} c_k e^{ikx}, \phi\right\rangle \\ &= \lim_{n\to\infty}\left\langle f, \frac{1}{2\pi}\sum_{|k|\leqq n}\langle e^{ikx}, \phi\rangle e^{-ikx}\right\rangle \\ &= \lim_{n\to\infty}\langle f, \phi_n\rangle.\end{aligned}$$

THEOREM 3. *Let $\{b_k\}_{k\in Z}$ be a set of constants. For $n = 0, 1, 2, \cdots$ set*

$$\phi_n = \frac{1}{2\pi}\sum_{|k|\leqq n} b_k e^{-ikx}.$$

Then:

(a) *If $\{\phi_n\}$ converges in the weak topology of W_2^{-1}, then*

$$\sum{}' |b_n|^2/n^2 < \infty. \tag{2.4}$$

(b) *If* (2.4) *holds, then $\{\phi_n\}$ converges in the strong topology of W_2^{-1}.*

COROLLARY. *Let $\phi \in W_2^{-1}$. Then*

$$\phi = \frac{1}{2\pi}\sum b_k e^{-ikx}, \tag{2.5}$$

$$b_k = \langle e^{ikx}, \phi\rangle. \tag{2.6}$$

The series converges in the strong topology of W_2^{-1}. The constants b_k satisfy (2.4). Conversely, if we are given a set of constants satisfying (2.4), then (2.5) defines an element of W_2^{-1}.

Proof of Theorem 3. (a) Let $f = \sum c_n e^{inx}$. Then

$$\langle f, \phi_n \rangle = \frac{1}{2\pi} \sum_{|k| \leqq n} b_k \langle f, e^{-ikx} \rangle = \sum_{|k| \leqq n} b_k c_k.$$

Thus $\sum b_k c_k$ converges for every set $\{c_k\}_{k \in Z}$ satisfying (2.1). This implies (2.4).

(b) Let $f = \sum c_k e^{ikx}$. Then if $n > m$,

$$\begin{aligned} |\langle f, \phi_n - \phi_m \rangle|^2 &= |\sum_{m < |k| \leqq n} b_k c_k|^2 \\ &\leqq \left(\sum_{m < |k| \leqq n} |b_k|^2 k^{-2} \right) \left(\sum_{m < |k| \leqq n} |c_k|^2 k^2 \right) \\ &\leqq \sum_{m < |k| \leqq n} |b_k|^2 k^{-2} \| f \|_{W_2^1}. \end{aligned}$$

Thus $\|\phi_n - \phi_m\|^2_{W_2^{-1}} \leqq \sum_{m < |k| \leqq n} |b_k|^2 k^{-2} \to 0$ as $n, m \to \infty$.

We can now derive an expression for the norm of an element of W_2^{-1}. Let $\phi \in W_2^{-1}$, $\phi = (1/(2\pi)) \sum b_k e^{-ikx}$. Let $f \in W_2^1$, $f = \sum c_k e^{ikx}$. Then

$$|\langle f, \phi \rangle|^2 = |\sum b_k c_k|^2 \leqq \frac{1}{2\pi} \sum \frac{|b_k|^2}{1 + k^2} \cdot 2\pi \sum |c_k|^2 (1 + k^2)$$

with equality holding if $c_k = \lambda \bar{b}_k / (1 + k^2)$ for all $k \in Z$. Hence

$$\|\phi\|^2_{W_2^{-1}} = \frac{1}{2\pi} \sum \frac{|b_k|^2}{1 + k^2}. \tag{2.7}$$

THEOREM 4. *Let $\phi \in W_2^{-1}$ have the representation* (2.5). *Then $\phi = (1/(2\pi)) b_0 + g'$, where $g \in L_2$.*

Proof. Let

$$g = \frac{1}{2\pi} \sum{}' \, i \frac{b_k}{k} e^{-ikx}.$$

By (2.4), $g \in L_2$. If $f \in W_2^1$, $f = \sum c_k e^{ikx}$, then

$$\left\langle f, g' + \frac{1}{2\pi} b_0 \right\rangle = \sum c_k b_k = \langle f, \phi \rangle.$$

3. Generalized normal derivatives of harmonic functions.

THEOREM 5. *Let $u(r, t)$ be harmonic for $r < 1$ and continuous for $r \leqq 1$. Let $f(t) = u(1, t)$ have the Fourier series $(1/(2\pi)) \sum c_n e^{-int}$. Let*

$$\phi = \frac{1}{2\pi} \sum{}' \, |n| c_n e^{-int} \in W_2^{-1}. \tag{3.1}$$

Then

$$\lim_{r \to 1-} \frac{\partial u}{\partial r}(r, \cdot) = \phi \quad \text{in } W_2^{-1}, \tag{3.2}$$

$$\lim_{r \to 1-} \frac{f(\cdot) - u(r, \cdot)}{1 - r} = \phi \quad \text{in } W_2^{-1}. \tag{3.3}$$

Proof. For $r < 1$ we have

$$u(r, t) = \frac{1}{2\pi} \sum c_n e^{-int} r^{|n|}.$$

Thus

$$\frac{\partial u}{\partial r}(r, t) = \frac{1}{2\pi} \sum' |n| c_n e^{-int} r^{|n|-1}.$$

Hence

$$\frac{\partial u}{\partial r}(r, \cdot) - \phi = \frac{1}{2\pi} \sum' |n| c_n e^{-int} \{ r^{|n|-1} - 1 \}.$$

By (2.7),

$$\left\| \frac{\partial u}{\partial r}(r, \cdot) - \phi \right\|^2_{W_2^{-1}} \leqq \frac{1}{2\pi} \sum' |c_n|^2 |r^{|n|-1} - 1|^2.$$

For $0 < r < 1$, $|r^{|n|-1} - 1|^2 < 1$. Given $\varepsilon > 0$ we can find N such that $\sum_{|n| \geqq N} |c_n|^2 \leqq \pi\varepsilon$. Thus

$$\left\| \frac{\partial u}{\partial r}(r, \cdot) - \phi \right\|_{W_2^{-1}} \leqq \frac{1}{2\pi} \sum_{0 < |n| < N} |c_n|^2 |r^{|n|-1} - 1|^2 + \frac{\varepsilon}{2}.$$

The first term on the right can be made less than $\varepsilon/2$ by taking r sufficiently close to 1, proving (3.2).

To prove (3.3) we note that

$$\frac{f(\cdot) - u(r, \cdot)}{1 - r} = \frac{1}{2\pi} \sum' c_n e^{-int} \frac{1 - r^{|n|}}{1 - r}$$

$$= \frac{1}{2\pi} \sum' c_n e^{-int} |n| r_n^{|n|-1}, \qquad r < r_n < 1.$$

The proof of (3.3) then proceeds as above.

Remark 1. The distribution ϕ given by (3.1) is the distribution derivative of the function represented by the Fourier series conjugate to that of f (see [1]); i.e., $\phi = g'$, where $g = (1/(2\pi)) \sum' i \operatorname{sgn}(n) c_n e^{-int} \in L_2$.

Remark 2. It is clear that we get the same results if we assume only that the function $u(r, t)$ is harmonic for $r < 1$ and has boundary values in L_2.

Acknowledgment. The author would like to thank the reviewer for his suggestions on how to simplify the presentation. The proof of Theorem 5 presented here is based on a suggestion made by the reviewer.

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SIAM J. MATH. ANAL.
Vol. 2, No. 3, August 1971

ASYMPTOTICALLY ALMOST PERIODIC SOLUTIONS OF A NONLINEAR VOLTERRA SYSTEM*

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Abstract. This paper studies forced almost periodic oscillations in a nonlinear system of two Volterra integral equations. It improves certain results in an earlier paper on the same topic in two ways. First it is shown that the oscillatory solution is Lyapunov stable under small perturbations in the coefficients of the equation. Secondly, it is shown that whenever the coefficients are quasi-periodic and analytic, the almost periodic oscillation is in the same class.

1. Introduction. In this paper we study forced oscillations in a nonlinear system of Volterra integral equations of the form

$$
\begin{aligned}
x_1(t) &= f_1(t) - \int_0^t a_1(t-s)g_1(s, x_1(s), x_2(s))\,ds - \int_0^t a_2(t-s)g_2(s, x_1(s), x_2(s))\,ds,\\
x_2(t) &= f_2(t) - \int_0^t a_2(t-s)g_1(s, x_1(s), x_2(s))\,ds - \int_0^t a_1(t-s)g_2(s, x_1(s), x_2(s))\,ds,
\end{aligned} \tag{1.1}
$$

where the functions $f_i(t)$ and $g_i(t, x)$ are asymptotically almost periodic in t. In an earlier paper [1] sufficient conditions were given so that the solutions $x_1(t)$ and $x_2(t)$ tend to certain almost periodic limiting functions $P_1(t)$ and $P_2(t)$ as $t \to \infty$. In this paper we shall improve the previous results in two ways. First, in § 2 it will be shown that this oscillatory behavior is stable under small perturbations in the functions f_i and g_i; that is, the solution of the perturbed problem is oscillatory and is near the solution of the unperturbed problem. Second, in § 3 below we give rather weak sufficient conditions in order that the limiting functions $P_1(t)$ and $P_2(t)$ be analytic in t.

System (1.1) arises in a natural way from the initial boundary value problem

$$
\begin{aligned}
&u_t = u_{xx}, \qquad t > 0, \quad 0 < x < \pi,\\
&u(0, x) = F(x), \quad 0 < x < \pi,\\
&\quad u_x(t, 0) = g_1(t, u(t, 0), u(t, \pi)),\\
&\quad u_x(t, \pi) = -g_2(t, u(t, 0), u(t, \pi))
\end{aligned} \tag{1.2}
$$

for all $t > 0$. In particular, we have in mind boundary conditions motivated by C. C. Lin's theory of superfluidity (see [2], [3] or [5]):

$$
\begin{aligned}
g_1(t, x_1) &= B_1(x_1 - c_1 \sin k_1 t)^3,\\
g_2(t, x_2) &= B_2(x_2 - c_2 \sin k_2 t)^3.
\end{aligned} \tag{1.3}
$$

As an application of the results proved here and in [1] we shall prove the following result.

* Received by the editors October 1, 1970, and in revised form February 1, 1971.

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THEOREM 1. *Consider the problem* (1.2)–(1.3) *with* $F_0 \in C^2[0, \pi]$. *Then given any* $B > 0$ *there exists* $\varepsilon > 0$ *such that if* $|B_i - B| < \varepsilon$ *for* $i = 1, 2$ *and if* $F \in C^2[0, \pi]$ *with*

$$\sum_{j=0}^{2} \max_{x} |F_0^{(j)}(x) - F^{(j)}(x)| < \varepsilon,$$

then the boundary functions $u(t, 0)$ *and* $u(t, \pi)$ *tend asymptotically as* $t \to \infty$ *to almost periodic limiting functions* $P_1(t)$ *and* $P_2(t)$. *The functions* $u(t, 0)$, $u(t, \pi)$, $P_1(t)$ *and* $P_2(t)$ *all vary continuously with* F, B_1, B_2, C_1 *and* C_2. *Moreover, each* $P_i(t)$ *has the form* $P_i(t) = P_i(k_1 t, k_2 t)$, *where* $P_i(\theta_1, \theta_2)$ *is real analytic in* (θ_1, θ_2) *and is* 2π-*periodic in each of its two variables.*

An outline of the proof appears in § 4 below.

2. Perturbation results. System (1.1) may be written in the vector form

$$x(t) = f(t) - \int_0^t A(t - s)G(s, x(s))\, ds, \tag{E}$$

where $A(t)$ is the appropriate 2×2 matrix (see (A1) below) and $x, f(t)$ and $G(t, x)$ are the appropriate two-dimensional column vectors. The vector norm used in this paper will always be

$$|x| = \max\{|x_1|, |x_2|\} \quad \text{when } x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

The symbol Q will always denote the special matrix

$$Q = 2^{-1/2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Note that $Q = Q^* = Q^{-1}$ and also that $\|Q\| = \sqrt{2}$, where $\|\cdot\|$ is the matrix norm compatible with the vector norm given above. Furthermore, if $A(t)$ is any matrix satisfying assumption (A1) below, then

$$QA(t)Q = \operatorname{diag}(a_1(t) + a_2(t), a_1(t) - a_2(t))$$

is a diagonal matrix. For any $N > 0$, let $A_N(t) = NQA(t)Q$. Let $R_N(t)$ be the resolvent of $A_N(t)$, that is, $R_N(t)$ satisfies the matrix resolvent equation

$$R(t) = A(t) - \int_0^t A(t - s)R(s)\, ds \tag{RE}$$

when $A(t) \equiv A_N(t)$. The following result was proved in [1, Lemma 1].

LEMMA 1. *Suppose* $A_N(t)$ *and* $R_N(t)$ *are the functions defined above. Then* $R_N(t)$ *exists, is continuous and is positive definite for all* t *in the interval* $0 < t < \infty$. *Moreover,* $R_N(t) = \operatorname{diag}(\lambda_{1N}(t), \lambda_{2N}(t))$ *is a diagonal matrix with components which satisfy the relations*

$$\int_0^\infty \lambda_{1N}(t)\, dt = 1, \qquad \int_0^\infty \lambda_{2N}(t)\, dt < 1.$$

The change of variables $x = Qy + f(t)$ may be used to transform (E) to

$$y(t) = -Q\int_0^t A(t-s)G(s, Qy(s) + f(s))\,ds$$
$$= -\int_0^t NQA(t-s)Q\{QG(s, Qy(s) + f(s))/N\}\,ds$$

or

$$y(t) = -\int_0^t A_N(t-s)G_N(s, y(s))\,ds,$$

where $A_N(t)$ is the matrix defined above and

$$G_N(t, y) = QG(t, Qy + f(t))/N.$$

The resolvent $R_N(t)$ and the variation of constants formula for integral equations may now be used to rewrite this equation in the equivalent form

$$(\mathrm{E}_N) \qquad y(t) = \int_0^t R_N(t-s)\{y(s) - G_N(s, y(s))\}\,ds.$$

Assume that the coefficient functions f, G and A in (E) satisfy the following hypotheses:

(A1) $a_1(t) = 1 + 2\sum_{n=1}^{\infty} \exp(-n^2 t), \quad a_2(t) = 1 + 2\sum_{n=1}^{\infty} (-1)^n \exp(-n^2 t)$

and

$$A(t) = \begin{pmatrix} a_1(t) & a_2(t) \\ a_2(t) & a_1(t) \end{pmatrix}.$$

(A2) $f(t)$ is continuous and bounded on $0 \leqq t < \infty$.

(A3) $G(t, x)$ is continuous in (t, x) for all $t \geqq 0$, $|x| < \infty$, and G is locally Lipschitz continuous in x.

(A4) The function $G(t, x_1, x_2)$ has the special form

$$G(t, x_1, x_2) = \begin{pmatrix} g(t, x_1) \\ g(t, x_2) \end{pmatrix},$$

where $g(t, y)$ is an odd nondecreasing function of y which is bounded in $t \in (-\infty, \infty)$ uniformly for y on any compact subset of $(-\infty, \infty)$.

The following hypotheses are related to (A4):

(A5) There exist positive numbers N and K such that if $|y| \leqq K$, then $|y - G_N(t, y)| < K$ uniformly for all $t \in (-\infty, \infty)$. Here $y \in R^2$ is a two-dimensional column vector and G_N is the function defined above (E_N).

(A6) There exist positive numbers N, K_0 and K_1 such that if $|y| \leqq K$, then $|y - G_N(t, y)| \leqq K_0 < K_1$ uniformly for all $t \in (-\infty, \infty)$.

The proof of Lemma 3 in [1] is actually a proof of the following stronger result.

LEMMA 2. *Suppose G satisfies* (A3) *and* (A4) *and $b \geqq \sup\{|f(t)| : t \geqq 0\}$. Then for any $M > \sqrt{2}\, b$ and for any ε in the interval $0 < \varepsilon < b$ there exists a number $N > 0$ such that* (A5) *is true with $K = M + \varepsilon$. Moreover, if L is a constant such that*

$$L \geqq \sup\{|G(t, y)| : -\infty < t < \infty, |x| \leqq 5M\},$$

then N depends only on the numbers M, ε and L.

Using Lemma 2 we now prove the following lemma.

LEMMA 3. *Suppose G satisfies* (A3) *and* (A4). *Then G satisfies* (A6), *where N and $K = K_1$ are the numbers obtained in Lemma 2 above.*

Proof. Let $K = K_1$ and N be given by Lemma 2. We must show that there exists a number $K_0 < K_1$ such that (A6) is true. For a contradiction we suppose there is no such K_0. Then for each positive integer n there exist numbers y_n and t_n such that

$$|y_n - G_N(t_n, y_n)| \geqq K - 1/n.$$

By possibly taking a subsequence we may assume that $y_n \to y_0$, $G_N(t_n, y_n) \to g_0$ and $f(t_n) \to f_0$ as $n \to \infty$. Note that $|y_0| \leqq K$. Define $x_n = Qy_n + f(t_n)$ so that $x_n \to x_0 = Qy_0 + f_0$.

Write x_0 and $\gamma = NQg_0$ in terms of their components,

$$x_0 = \operatorname{col}(x_{01}, x_{02}), \qquad \gamma = NQg_0 = \operatorname{col}(\gamma_1, \gamma_2).$$

Define

$$g^*(z) = \begin{cases} 0 & \text{if } z = 0, \\ |\gamma_1| & \text{if } z = |x_{01}|, \\ |\gamma_2| & \text{if } z = |x_{02}|. \end{cases}$$

Extend $g^*(z)$ linearly between the points $0, |x_{01}|$ and $|x_{02}|$, extend g^* as a constant on the remaining part of the half-line $z \geqq 0$ and let $g^*(-z) = -g^*(z)$ when $z < 0$. Define $G^*(x_1, x_2) = \operatorname{col}(g^*(x_1), g^*(x_2))$ for all x_1 and x_2. Note that $G^*(x_0) = NQg_0$.

The function G^* defined in this way satisfies (A3) and (A4) and has the same upper bound L as the function G in Lemmas 2 and 3. Since $|y_0| \leqq K$, then Lemma 2 implies that

$$|y_0 - QG^*(Qy + f_0)N^{-1}| < K.$$

On the other hand, it follows by the construction of G^* and the choice of y_0 and g_0 that

$$\begin{aligned} |y_0 - QG^*(Qy_0 + f_0)N^{-1}| &= |y_0 - QG^*(x_0)N^{-1}| \\ &= |y_0 - g_0| \geqq K. \end{aligned}$$

This contradiction completes the proof.

The following boundedness result was proved in [1, Theorem 4]. Its proof depends on Lemma 1, the equivalence of (E) and (E_N), and a fixed-point theorem.

THEOREM 2. *Suppose* (A1)–(A3) *and* (A5) *are true. Then the unique solution $x(t)$ of* (E) *exists and satisfies $|x(t)| \leqq K$ for all $t \geqq 0$.*

This result may be improved as follows.

THEOREM 3. *Suppose* (A1)–(A4) *are true and* $b = \sup\{|F(t)| : t \geqq 0\}$. *Consider the perturbation problem*

$$\text{(PE)} \qquad X(t) = F(t) - \int_0^t A(t-s)\{G(s, X(s)) + P(s, X(s))\}\, ds,$$

where P is continuous in (t, x). *Then for any* $\varepsilon > 0$ *there exists a number* $\delta > 0$ *such that if*

$$\sup\{|P(t, x)| : t \geqq 0, |Q(x - F(t))| < \sqrt{2}b + \varepsilon\} < \delta,$$

then the solution $X(t)$ *of* (PE) *exists for all* $t \geqq 0$ *and satisfies the inequality*

$$|Q(X(t) - F(t))| \leqq \sqrt{2}b + \varepsilon.$$

Proof. Define $H(t, x) = G(t, x) + P(t, x)$ and let $H_N(t, y) = QH(t, Qy + F(t))N^{-1}$. Given $K_1 = \sqrt{2}b + \varepsilon$, choose N and K_0 using Lemma 3. Choose $\delta < N(K_1 - K_0)|Q|^{-1}$ so that

$$|QP(t, Qy + F(t))N^{-1}| \leqq |Q|\delta N^{-1} < K_1 - K_0.$$

If $|y| \leqq K_1$, then by the choice of δ one has

$$\begin{aligned} |H_N(t, y)| &\leqq |G_N(t, y)| + |QP(t, Qy + F(t))N^{-1}| \\ &< K^0 + (K_1 - K_0) = K_1. \end{aligned}$$

Thus $H(t, x)$ satisfies (A3) and (A5). Now apply Theorem 2 above.

THEOREM 4. *Suppose the coefficients f, A and G of* (E) *satisfy* (A1)–(A4). *Define* $\|f\| = \sup\{|f(t)| : t \geqq 0\}$. *Suppose that given any* $A > 0$ *there exists a positive, continuous, increasing function* $\alpha(u)$ *such that*

$$\{g(t, u + x) - g(t, x)\}/u \geqq \alpha(|u|), \qquad |u| \geqq A,$$

uniformly for all $t \geqq 0$ *and all x such that* $|Q(x - f(t))| \leqq \sqrt{2}\|f\| + 4$. *Then given any* $\varepsilon > 0$ *there exists a positive number* δ *such that whenever*:

(i) $F(t)$ *is any continuous function satisfying* $\|f - F\| = \sup\{|f(t) - F(t) : t \geqq 0\} < \delta$,

(ii) $P(t, x)$ *is any continuous function satisfying* $\sup\{|P(t, x)| : t \geqq 0, |Q(x - f(t))| \leqq \sqrt{2}\|f\| + 4\} < \delta$,

(iii) $X(t)$ *is the unique solution of* (PE);

then $|x(t) - X(t)| \leqq \varepsilon$ *for all* $t \geqq 0$.

Proof. Define $y(t) = x(t) - X(t)$, $\varphi(t) = f(t) - F(t)$ and $H(t, y) = G(t, y + X(t)) - G(t, X(t))$. Then one has

$$y(t) = \varphi(t) - \int_0^t A(t-s)\{H(s, y(s)) - P(s, X(s))\}\, ds,$$

or, symbolically,

$$y = \varphi - A * \{H(y) - P(X)\}.$$

Let $Y = Qy$, $A_N = NQAQ$ and $H_N(t, Y) = QH(t, QY)N^{-1}$ so that

$$\text{(2.1)} \qquad Y = (Q\varphi + A_N * \{Y - H_N(Y) + QP(X)\}N^{-1}) - A_N * Y.$$

If R_N is the resolvent of A_N, that is

$$R_N = A_N - A_N * R_N, \tag{2.2}$$

then any equation of the form $Y = S - A_N * Y$ may be written in the equivalent form $Y = S - R_N * S$. Applying this to (2.1) and using the relation (2.2) one can calculate

$$Y = Q\varphi + A_N * \{Y - H_N(Y) + QP(X)N^{-1}\} - R_N * Q\varphi$$
$$- R_N * A_N * \{Y - H_N(Y) + QP(X)N^{-1}\},$$

$$Y = Q\varphi - R_N * Q\varphi + R_N * \{Y - H_N(Y) + QP(X)N^{-1}\} \tag{2.3}$$

or

$$Y(t) = Q\varphi(t) - Q\int_0^t R_N(t-s)\varphi(s)\,ds \tag{2.3'}$$
$$+ \int_0^t R_N(t-s)\{Y(s) - H_N(s, Y(s)) + QP(s, X(s))N^{-1}\}\,ds.$$

Define $S_0 = \{(t, x): t \geqq 0, |Q(x - F(t))| \leqq \sqrt{2}\|F\| + 1\}$ and let $S_1 = \{(t, x): t \geqq 0, |Q(x - f(t))| \leqq \sqrt{2}\|f\| + 4\}$. If $\|\varphi\| = \|f - F\| < 1$, and if $(t, x) \in S_0$, then

$$|Q(x - f(t))| \leqq |Q(x - F(t))| + |Q|\,\|f - F\|$$
$$\leqq \sqrt{2}\|F\| + 1 + \sqrt{2}\cdot 1 \leqq \sqrt{2}(\|f\| + 1) + 1 + \sqrt{2}$$
$$\leqq \sqrt{2}\|f\| + 4.$$

Therefore, $S_0 \subset S$, if $\|\varphi\| < 1$. By Theorem 3, there exists a number $\delta_0 > 0$ such that if $|P(t, x)| < \delta_0$ on S_0, then $X(t)$ exists for all $t \geqq 0$ and $(t, X(t)) \in S_0$.

Write $H(t, x)$ in the form $H(t, x) = \mathrm{col}\,(M_1x_1, M_2x_2)$, where

$$M_j(t, x) = \{g(t, x_j + X_j(t)) - g(t, X_j(t))\}/x_j, \qquad j = 1, 2.$$

Then $Y - H_N(t, Y)$ can be written in the form

$$Y - H_N(t, Y) = A(t, Y)Y, \qquad A = \begin{pmatrix} 1 - \dfrac{M_1 + M_2}{2N} & \dfrac{M_1 - M_2}{2N} \\[2ex] \dfrac{M_1 - M_2}{2N} & 1 - \dfrac{M_1 + M_2}{2N} \end{pmatrix}.$$

If $M(t, x) = \mathrm{col}\,(M_1(t, x_1), M_2(t, x_2))$ and if $|M(t, QY)| < N$, then the norm of the matrix A is $|A| = 1 - |M(t, QY)|/N = \max\{1 - M_1/N,\ 1 - M_2/N\}$. For any number $K > 0$ if $|Y| \leqq K/2$, then since $|A| < 1$ one has $|AY| \leqq K/2$. If $K/2 \leqq |Y| \leqq K$, then either $|Y_1 + Y_2|$ or $|Y_1 - Y_2| \geqq K/2$. Therefore, the hypotheses of the theorem imply that $|M(t, QY)| \geqq \alpha(K/2) > 0$ for some function $\alpha(u)$. This means that

$$|Y - H_N(t, Y)| \leqq 1 - \alpha(K/2)/N, \qquad K/2 \leqq |Y| \leqq K.$$

Consequently, for any given $\varepsilon > 0$, if $K = \varepsilon/\sqrt{2}$, then there exist positive numbers N and K_0 such that if $|Y| \leqq \varepsilon/\sqrt{2}$, then $|Y - H_N(t, Y)| \leqq K_0 < \varepsilon/\sqrt{2}$. The number δ in the conclusion of the present theorem will be chosen so that $\delta \leqq \min\{\delta_0, 1\}$ and such that $4\|f - F\| + 2|P(t,x)|N^{-1} \leqq 6\delta \leqq \varepsilon - \sqrt{2}K_0$ for all $(t, x) \in S_1$. For this choice of δ we shall show that $|y(t)| = |x(t) - X(t)| \leqq \varepsilon$ for all $t \geqq 0$. Equivalently, since $Q = Q^{-1}$ and $|Q| = \sqrt{2}$, then we may show that $|Y(t)| = |Q(x(t) - X(t))| \leqq \varepsilon/\sqrt{2}$.

Let $W = \{z \in C[0, \infty): |z(t)| \leqq \varepsilon/\sqrt{2}$ for all $t \geqq 0\}$ and let TZ be the map defined by the right-hand side of (2.3), that is,

$$TZ(t) = Q\varphi(t) - Q\int_0^t R_N(t-s)\varphi(s)\,ds$$
$$+ \int_0^t R_N(t-s)\{Z(s) - H_N(s, Z(s)) - QP(s, X(s))N^{-1}\}\,ds.$$

By Lemma 1 above, $R_N \in L^1(0, \infty)$ and $\int_0^t |R_N(t-s)|\,ds \leqq 1$ for all $t \geqq 0$. Therefore, if $z \in W$,

$$\begin{aligned}|TZ(t)| &\leqq \sqrt{2}\|\varphi\| + \sqrt{2}\|\varphi\| \int_0^t |R_N(t-s)|\,ds \\ &\quad + N^{-1}\sqrt{2}\max_{S_1}|P(t,x)| \int_0^t |R_N(t-s)|\,ds + \int_0^t |R_N(t-s)|K_0\,ds \\ &\leqq 2\sqrt{2}\|\varphi\| + \sqrt{2}\max_{S_1}|P(t,x)|N^{-1} + K_0 \\ &\leqq 6\delta + K_0 \leqq \varepsilon/\sqrt{2}.\end{aligned}$$

This shows that $Tz \in W$ if $z \in W$. If the space $C[0, \infty)$ is given the topology of uniform convergence on bounded subsets of $[0, \infty)$, then it becomes a locally convex linear topological space with the additional property that $T: C[0, \infty) \to C[0, \infty)$ is a completely continuous map. Since W is a closed bounded convex subset of $C[0, \infty)$ and T maps W into itself, then the Schauder fixed-point theorem applies. This means that (2.3′) has at least one solution $Y(t)$ such that $|Y(t)| \leqq \varepsilon/\sqrt{2}$ for all $t \geqq 0$. But the function $H(t, Y)$ is locally Lipschitz continuous in Y so that the solution of (2.3′) is unique, that is, $Y(t) = Q(x(t) - X(t))$.

3. Quasi-periodic functions. Let $k_1, k_2, \cdots, k_m$ be positive constants which are linearly independent over the integers. Let k denote the vector $k = (k_1, k_2, \cdots, k_m)$ with $m \geqq 1$.

DEFINITION. A function $\varphi(t)$ will be called *quasi-periodic with fundamental frequencies* k if and only if there exists a function $\Phi(\theta) = \Phi(\theta_1, \theta_2, \cdots, \theta_m)$ continuous in θ and periodic in each variable θ_j of period 2π such that

$$\varphi(t) = \Phi(kt) = \Phi(k_1t, k_2t, \cdots, k_mt), \qquad -\infty < t < \infty.$$

Each quasi-periodic function is easily seen to be almost periodic. If $m = 1$ so that $k = k_1$, then the quasi-periodic function is actually periodic.

According to the results in [1] if $x = Qy + f(t)$, then for any $N > 0$ the function $y(t)$ solves (E_N). Conditions are given in [1] which guarantee that $y(t)$ tends asymptotically to an almost periodic function $p(t)$, where

$$(3.1) \qquad p(t) = \int_{-\infty}^{t} R_N(t-s)\{p(s) - G_N(s, p(s))\}\, ds, \qquad -\infty < t < \infty.$$

The function p is the unique solution of (3.1) if N is sufficiently large.

The aim in this section is to give sufficient conditions in order that $p(t) = P(kt)$ is quasi-periodic and $P(\theta)$ is analytic in θ. Assume:

(A7) $G(t, x) = \gamma(kt, x)$ and $f(t) = \varphi(kt)$ are quasi-periodic in t with fundamental frequencies k. Moreover, $\gamma(\theta, x)$ and $\varphi(\theta)$ are real analytic functions of (θ, x) and θ respectively in regions

$$(3.2) \qquad U(\delta_0) = \{(\theta, x) : |\mathrm{Im}\, \theta_j|, |\mathrm{Im}\, x_i| < \delta_0, -\infty < \mathrm{Re}\, x_i, \mathrm{Re}\, \theta_j < \infty \text{ for } 1 \leqq j \leqq m \text{ and } i = 1, 2\}$$

and

$$(3.3) \qquad D(\delta_0) = \{\theta : |\mathrm{Im}\, \theta_j| < \delta_0, -\infty < \mathrm{Re}\, \theta_j < \infty \text{ for } 1 \leqq j \leqq m\}.$$

Under this assumption it follows that the function

$$\gamma_N(kt, y) = Q\gamma(kt, Qy + \varphi(kt))N^{-1}$$

is also quasi-periodic and analytic in $U(\delta_0)$. If the solution of (3.1) was quasi-periodic, say $p(t) = P(kt)$, then (3.1) could be rewritten as

$$\begin{aligned} P(kt) &= \int_{-\infty}^{t} R_N(t-s)\{P(ks) - \gamma_N(ks, P(ks))\}\, ds \\ &= \int_0^{\infty} R_N(s)\{P(kt - ks) - \gamma_N(kt - ks, P(kt - ks))\}\, ds. \end{aligned}$$

Since $k = (k_1, k_2, \cdots, k_n)$ is a vector of linearly independent frequencies and $P(\theta)$ is continuous in θ, then this is equivalent to the equation

$$(3.4) \qquad P(\theta) = \int_0^{\infty} R_N(s)\{P(\theta - ks) - \gamma_N(\theta - ks, P(\theta - ks))\}\, ds.$$

Conversely, if $P(\theta)$ is any continuous solution of (3.4) such that $P(\theta)$ is 2π-periodic in each variable θ_j, then $p(t) = P(kt)$ will solve (3.1). Therefore, our problem is reduced to finding an analytic and periodic solution of (3.4).

For any $\delta > 0$ the symbols $D(\delta)$ or $U(\delta)$ will denote regions defined in the manner of (3.2) and (3.3). Using this notation we now prove the following theorem.

THEOREM 5. *Suppose* (A1)–(A3) *and* (A6)–(A7) *are true. Then there exists a* $\delta > 0$ *such that* (3.4) *has a solution* $P(\theta)$ *which is real analytic in* $\theta \in D(\delta)$ *and* 2π-*periodic in each variable* θ_j.

Proof. Let N, K_0 and K_1 be the numbers given by (A6). For any δ in the interval $0 < \delta < \delta_0$ let $\mathscr{F}(\delta)$ denote the set of functions $Z(\theta)$ real analytic in $\theta \in D(\delta)$ and 2π-periodic in each variable θ_j. If $\mathscr{F}(\delta)$ is given the topology of uniform convergence on compact subsets of $D(\delta)$, then this family becomes a locally convex linear topological space over the real numbers. Define

$$S = \{Z \in \mathscr{F}(\delta) : |Z(\theta)| \leqq K_1 \text{ for all } \theta \in D(\delta)\},$$

where K_1 is the constant in (A6). Then S is a closed, convex, nonempty and compact subset of $\mathscr{F}(\delta)$. Since (A6) is true for $G(t, x) = \gamma(kt, x)$ and since $k = (k_1, k_2, \cdots, k_m)$ is a vector with linearly independent components, then

$$|y - \gamma(\theta, y)| \leqq K_0 < K_1 \quad \text{if } |y| \leqq K_1, \qquad (0, y) \in U(\delta),$$

and (θ, y) is real. By continuity there exists a number δ with $0 < \delta < \delta_0$ such that

$$|y - \gamma(\theta, y)| \leqq K_1 \quad \text{if } |y| \leqq K_1 \quad \text{and} \quad (\theta, y) \in U(\delta),$$

where (θ, y) is now allowed to be complex. This is the appropriate δ.

For any $Z \in S$ define

$$TZ(\theta) = \int_0^\infty R_N(s)\{Z(\theta - ks) - \gamma_N(\theta - ks, Z(\theta - ks))\}\, ds, \qquad \theta \in D(\delta).$$

By Lemma 1 above, the matrix $R_N(t) \in L^1(0, \infty)$ with $\int_0^\infty |R_N(t)|\, dt \leqq 1$. This means that $TZ(\theta)$ is well-defined, $TZ \in \mathscr{F}(\delta)$ and

$$|TZ(\theta)| \leqq \int_0^\infty |R_N(s)| K_1\, ds \leqq K_1.$$

In particular, T maps S into S continuously. By the Schauder fixed-point theorem T has a fixed point.

4. Outline of the proof of Theorem 1. The results in § 2 of [1] show that (1.2) is equivalent to (E) with $x_1(t) = u(t, 0)$ and $x_2(t) = u(t, \pi)$, with

$$f_1(t) = \frac{F_0}{2} + \sum_{n=1}^\infty F_n \exp(-n^2 t), \qquad f_2(t) = \frac{F_0}{2} + \sum_{k=1}^\infty F_n(-1)^n \exp(-n^2 t)$$

and with

$$F_n = \frac{2}{\pi} \int_0^\pi F(x) \cos nx\, dx.$$

It is easy to prove that $f_1(t)$ and $f_2(t)$ vary continuously in the uniform norm over $0 \leqq t < \infty$ as F varies in the norm of $C^2[0, \pi]$. The results in § 2 above show that $x_1(t)$ and $x_2(t)$ vary continuously (again in the uniform norm over $0 \leqq t < \infty$) as f and g vary.

The results in § 6 of [1] are sufficient to see that $x_1(t)$ and $x_2(t)$ are asymptotic to almost periodic functions $p_1(t)$ and $p_2(t)$ such that $p(t) = \operatorname{col}(p_1(t), p_2(t))$ solves (3.1). If k_1 and k_2 are linearly independent, then Theorem 5 above implies that $p(t)$ is analytic and quasi-periodic with fundamental frequencies k_1 and k_2. Finally, note that since $|p(t) - x(t)| \to 0$ as $t \to \infty$, where $p(t)$ is almost periodic and $x(t)$ varies continuously with f and g, then $p(t)$ varies continuously (in the uniform norm over $-\infty < t < \infty$) with f and g. If k_1 and k_2 are linearly dependent over the integers, then the same conclusion follows but with $p(t)$ a periodic function.

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SIAM J. MATH. ANAL.
Vol. 2, No. 3, August 1971

FIELDS DUE TO ELECTRONS ON AN ANALYTIC CURVE*

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Abstract. Let D be the interior of a simple closed analytic curve C and let $z_{n1}, \cdots, z_{nn}$ be points on C. Assuming a logarithmic potential, the electrostatic field due to electrons (charges $-\varepsilon$) at the points z_{nk} may be represented as (the complex conjugate of)

$$\varepsilon E_n(z) = \varepsilon \sum_{k=1}^{n} \frac{1}{z_{nk} - z}.$$

The authors give necessary and sufficient conditions under which $E_n(z) \to 0$ as $n \to \infty$ uniformly on the compact subsets of D. For equilibrium distributions of electrons (thinking of C as a conductor) the fields $\varepsilon E_n(z)$ tend to a limit $\varepsilon E(z)$ holomorphic in the closure of D. The limiting field is identically zero if and only if C is a circle. For general analytic C, the limiting field is of the same order of smallness as the field due to a single electron outside D.

1. Introduction and results. Let D be a bounded simply connected region in the z-plane, C its outer boundary (that is, C is the boundary of the unbounded component of the complement of the closure of D). Suppose we place electrons (charges $-\varepsilon$) at points z_{nk}, $k = 1, \cdots, n$, of C. Assuming attractive forces inversely proportional to the distance and using appropriate normalization, the force on a charge 1 at the point z is given by the complex conjugate of the expression

$$\varepsilon E_n(z) = \varepsilon \sum_{k=1}^{n} \frac{1}{z_{nk} - z}. \tag{1.1}$$

Ignoring the complex conjugation and the factor ε, we shall usually refer to $E_n(z)$ itself as the field due to electrons at the points z_{nk}.

It follows from work of G. R. MacLane [5], M. D. Thompson [9] and the first author [3], [4] that electrons placed at conformal images of nth roots of unity, or shifted nth roots of unity, produce a small field in D in the sense that

$$E_n(z) \to 0 \quad \text{as } n \to \infty \tag{1.2}$$

uniformly on every compact subset of D. If C is not a rectifiable Jordan curve, n may have to run through a subsequence $\{n_j\}$ of the positive integers [3]. The conformal mapping used goes from the exterior of the unit circle Γ in the w-plane to the exterior of C and takes infinity to infinity.

A family of sequences of points

$$z_{n1}, \cdots, z_{nn} \quad \text{on } C, \quad \text{with } n = n_j \to \infty \tag{1.3}$$

and such that (1.2) holds is called *asymptotically neutral* relative to D [3], [4], [10]. In this paper we characterize the asymptotically neutral families for those sets D which are the interior of an *analytic Jordan curve* C. It turns out that the asymptotically neutral families on C are precisely the conformal images of the asymptotically neutral families on the unit circle Γ.

* Received by the editors November 2, 1970, and in revised form January 18, 1971.

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Thinking of the analytic curve C as a conductor, one will be particularly interested in the fields $E_n(z)$ produced by equilibrium distributions of n electrons on C. In this case the points (1.3) are nth Fekete points, that is, points $z_1, \cdots, z_n$ on C for which $\prod_{j<k}|z_j - z_k|$ is a maximum (the potential energy is a minimum for such a configuration). Using recent work of C. Pommerenke [7], [8], we show that with the fields $E_n(z)$ corresponding to nth Fekete points there is associated a function $E(z)$, holomorphic in the closure of D, such that

$$E_n(z) \to E(z) \quad \text{as } n \to \infty \tag{1.4}$$

uniformly on every compact subset of D. We show also that $E(z)$ is the zero function if and only if C is a circle. Thus if C is not a circle, the limiting field $E(z)$ corresponding to equilibrium distributions is not as small as the limiting field corresponding to conformal images of roots of unity!

However, the limiting field in the case of equilibrium distributions is quite small from a physical point of view. Reintroducing the factor ε as in (1.1) when discussing fields due to electrons, the limiting field will be given by $\varepsilon E(z)$; hence it is of the same order of smallness as the field $\varepsilon/(z_0 - z)$ due to a single electron at a point z_0 outside D. One should be perfectly safe inside a two-dimensional, analytic Faraday cage!

It would be interesting to consider the extension of the above results to nonanalytic curves. The corresponding three-dimensional problems appear to be much harder, although some initial results exist [6].

2. An auxiliary polynomial representation theorem. Let C be an analytic Jordan curve in the z-plane, D its interior. Let

$$\Phi(w) = aw + a_0 + \frac{a_1}{w} + \cdots \tag{2.1}$$

be one of the analytic functions which map the exterior of the unit circle Γ in the w-plane one-to-one and conformally onto the exterior of C in such a way that infinity goes into infinity. There will be numbers $r < 1$ such that $\Phi(w)$ is analytic and univalent for $|w| > r$ (cf. [2, p. 346]); the smallest such r will be called ρ. For $r > \rho$, the positively oriented circle $|w| = r$ will be denoted by Γ_r, its image under the mapping $z = \Phi(w)$ will be called C_r, and the interior of C_r will be called D_r.

For z in D_r and $|w| \geqq r$ one has the Laurent expansion

$$\frac{1}{\Phi(w) - z} = \sum_{p=0}^{\infty} K_p(z) w^{-p-1}, \tag{2.2}$$

where the coefficients

$$K_p(z) = \frac{1}{2\pi i} \int_{\Gamma_r} \frac{1}{\Phi(w) - z} w^p \, dw \tag{2.3}$$

are independent of r. It is easy to see that $K_p(z)$ is a polynomial of degree p. For $r > \rho$ and z in D_r,

$$|K_p(z)| \leqq \frac{1}{\delta_r(z)} r^{p+1}, \tag{2.4}$$

where $\delta_r(z)$ is the distance between z and C_r.

We shall prove the following expansion theorem of which certain aspects could be obtained from the general results in the book by R. P. Boas and R. C. Buck [1, Faber polynomials and related material, pp. 57–60].

THEOREM 1. *Every holomorphic function $F(z)$ in D can be represented in the form*

$$F(z) = \sum_{p=0}^{\infty} b_p K_p(z) \tag{2.5}$$

with coefficients b_p such that

$$b_p = O(e^{\delta p}) \quad as \quad p \to \infty \tag{2.6}$$

for every number $\delta > 0$. A given $F(z)$ has only one such representation, and for $\rho < r < 1$,

$$b_p = \frac{1}{2\pi i}\int_{\Gamma_r} F\{\Phi(w)\}\Phi'(w)w^{-p-1}\,dw. \tag{2.7}$$

Proof. (i) Let $F(z)$ be holomorphic in D. Then for $\rho < r < 1$ and z in D_r,

$$\begin{aligned} F(z) &= \frac{1}{2\pi i}\int_{C_r} \frac{F(\zeta)}{\zeta - z}d\zeta \\ &= \frac{1}{2\pi i}\int_{\Gamma_r} \frac{F\{\Phi(w)\}}{\Phi(w) - z}\Phi'(w)\,dw. \end{aligned}$$

Substituting the uniformly convergent expansion (2.2) one obtains (2.5) with coefficients b_p given by (2.7). The coefficients b_p are clearly independent of r, hence (2.5) holds for all z in D. Since r may be taken arbitrarily close to 1, the b_p of (2.7) satisfy the estimate (2.6) for every $\delta > 0$.

(ii) It remains to prove the uniqueness of the representation (2.5) subject to condition (2.6). Suppose that

$$\sum_{0}^{\infty} b_p K_p(z) = 0 \quad \text{in } D,$$

where the b_p satisfy (2.6) for every $\delta > 0$. Then by (2.3), taking $\rho < r < 1$ and introducing the function $G(w) = \sum_0^\infty b_p w^p$, we have

$$\int_{\Gamma_r} \frac{1}{\Phi(w) - z}G(w)\,dw = 0, \qquad z \text{ in } D_r. \tag{2.8}$$

It is no restriction to assume that the origin $z = 0$ lies in $\bigcap D_r$. Expanding $1/\{\Phi(w) - z\}$ in powers of z, equation (2.8) then implies that $G(w)$ is "orthogonal" to all negative integral powers of $\Phi(w)$ on Γ_r. We shall show that as a consequence, $G(w)$ is orthogonal to all negative integral powers of w on Γ_r.

Let $\rho < s < r$ and let V denote the exterior of C_s. By Runge's theorem, every holomorphic function in V that vanishes at infinity can be approximated, uniformly on C_r, by linear combinations of negative powers of z. Thus by conformal mapping, the negative powers of w can be uniformly approximated on Γ_r by linear combinations of negative powers of $\Phi(w)$. It follows that $G(w)$ is orthogonal to w^{-p-1} on Γ_r; hence

$$b_p = \frac{1}{2\pi i}\int_{\Gamma_r} w^{-p-1}G(w)\,dw = 0, \qquad p = 0, 1, 2, \cdots.$$

3. Corollary: a convergence theorem. Theorem 1 implies a convenient necessary and sufficient condition for uniform convergence of holomorphic functions $F_n(z)$ to $F(z)$ on the compact subsets of D. We introduce the representation for $F(z)$ given by Theorem 1 and the corresponding representations

$$F_n(z) = \sum_{p=0}^{\infty} b_{np} K_p(z). \tag{3.1}$$

THEOREM 2. *For analytic C, holomorphic functions $F_n(z)$ converge to $F(z)$ uniformly on every compact subset of D if and only if*

$$b_{np} \to b_p \quad \text{as } n \to \infty \quad \text{for } p = 0, 1, 2, \cdots \tag{3.2}$$

and for every number $\delta > 0$ there is a constant B_δ such that

$$|b_{np}| \leqq B_\delta e^{\delta p} \text{ for all } n \text{ and } p. \tag{3.3}$$

Proof. (i) Suppose $F_n(z) \to F(z)$ uniformly on the compact subsets of D. Then $F_n\{\Phi(w)\} \to F\{\Phi(w)\}$ uniformly on Γ_r, where $\rho < r < 1$. Hence by (2.7), the coefficients b_{np} satisfy conditions (3.2) and (3.3) for every $\delta > 0$.

(ii) Suppose the coefficients b_{np} satisfy conditions (3.2) and (3.3) for all $\delta > 0$. We choose r arbitrarily between ρ and 1, and select $\delta > 0$ such that $e^\delta < 1/r$. It then follows from (2.4) that the functions $F_n(z)$ converge to $F(z)$ uniformly on every compact subset of D_r.

4. Fields due to electrons on C. Asymptotically neutral families. For arbitrary fixed n, we shall place electrons (charges $-\varepsilon$) at the n points of the analytic Jordan curve C given by

$$z_{nk} = \Phi\{\exp(i\theta_{nk})\}, \qquad k = 1, \cdots, n, \tag{4.1}$$

where Φ is as in § 2. Ignoring complex conjugates and the factor ε, we have that the resulting field will be represented by the sum

$$E_n(z) = \sum_{k=1}^{n} \frac{1}{z_{nk} - z} = \sum_{1}^{n} \frac{1}{\Phi\{\exp(i\theta_{nk})\} - z}.$$

Hence by (2.2),

$$\begin{aligned} E_n(z) &= \sum_{k=1}^{n} \sum_{p=0}^{\infty} K_p(z) \exp\{-(p+1)i\theta_{nk}\} \\ &= \sum_{p=0}^{\infty} \bar{s}_{n,p+1} K_p(z), \end{aligned} \tag{4.2}$$

where

$$s_{nv} = \sum_{k=1}^{n} e^{iv\theta_{nk}}. \tag{4.3}$$

Applying Theorem 2 we thus obtain the following characterization of asymptotically neutral families (see § 1).

THEOREM 3. *The family of sequences of points* (4.1), $n = 1, 2, \cdots$, *on the analytic Jordan curve C is asymptotically neutral relative to the interior D; in other words,* $E_n(z) \to 0$ *uniformly on every compact subset of D, if and only if*

$$s_{n\nu} \to 0 \quad \text{as } n \to \infty \quad \text{for } \nu = 1, 2, \cdots \tag{4.4}$$

and for every number $\delta > 0$ *there is a constant M such that*

$$|s_{n\nu}| \leqq M e^{\delta \nu} \quad \text{for all } n \quad \text{and} \quad \nu \geqq 1. \tag{4.5}$$

Note that the above conditions are independent of the curve C. Thus asymptotically neutral families on analytic curves are preserved under conformal mappings of the exterior onto the exterior such that infinity goes into infinity.

COROLLARY 3.1. *The asymptotically neutral families on an analytic Jordan curve C are precisely the conformal images of the asymptotically neutral families on the unit circle* Γ.

A standard type argument shows that conditions (4.4) and (4.5) for all $\delta > 0$ are equivalent to the following. For every sequence d_ν, $\nu = 1, 2, \cdots$, such that $d_\nu = O(e^{-\gamma\nu})$ for *some* number $\gamma > 0$, the sums $\sum_{\nu=1}^{\infty} s_{n\nu} d_\nu$ exist and

$$\sum_{\nu=1}^{\infty} s_{n\nu} d_\nu \to 0 \quad \text{as } n \to \infty. \tag{4.6}$$

We now consider the class of all analytic functions $\psi(\theta)$ of the real variable θ which have period 2π. In terms of their Fourier series,

$$\psi(\theta) = \sum_{-\infty}^{\infty} d_\nu e^{i\nu\theta},$$

such functions are characterized by the fact that there is a constant $\gamma > 0$ such that $d_\nu = O(e^{-\gamma|\nu|})$. One has

$$\sum_{k=1}^{n} \psi(\theta_{nk}) - n d_0 = \sum_{\nu=1}^{\infty} (s_{n\nu} d_\nu + \bar{s}_{n\nu} d_{-\nu}).$$

Thus by the preceding remarks we have the following corollary.

COROLLARY. 3.2. *The family of sequences of points* (4.1), $n = 1, 2, \cdots$, *on the analytic curve C is asymptotically neutral if and only if*

$$\sum_{k=1}^{n} \psi(\theta_{nk}) - \frac{n}{2\pi} \int_0^{2\pi} \psi(\theta)\, d\theta \to 0 \quad \text{as } n \to \infty \tag{4.7}$$

for every analytic function $\psi(\theta)$ *of the real variable* θ *of period* 2π.

Condition (4.7) first arose in a conversation with M. D. Thompson who considered the case where C is a circle. The condition is analogous to (but much more demanding than) a classical condition for uniformly distributed families of points (cf. [11, p. 164]).

We end by considering the special case where the z_{nk} are conformal images of nth roots of unity,

$$\theta_{nk} = 2\pi k/n, \qquad k = 1, \cdots, n.$$

In this case,

$$s_{n\nu} = \begin{cases} 0 & \text{for } \nu \not\equiv 0 \ (\mathrm{mod}\, n), \\ n & \text{for } \nu \equiv 0 \ (\mathrm{mod}\, n). \end{cases} \tag{4.8}$$

It is clear that conditions (4.4) and (4.5) are satisfied for all $\delta > 0$. Formulas (4.8), (4.2) and (2.4) show furthermore that for $\rho < r < 1$ and z in D_r,

$$\begin{aligned} |E_n(z)| &= \left| \sum_1^\infty \bar{s}_{n\nu} K_{\nu-1}(z) \right| \\ &\leqq n\{|K_{n-1}(z)| + |K_{2n-1}(z)| + \cdots\} \\ &\leqq \frac{1}{\delta_r(z)} \frac{nr^n}{1 - r^n}. \end{aligned} \tag{4.9}$$

COROLLARY 3.3. *On an analytic curve C, conformal images of the n-th roots of unity, $n = 1, 2, \cdots$, form an asymptotically neutral family with the property that the corresponding fields $E_n(z)$ tend exponentially to zero on every fixed compact subset of D.*

5. Equilibrium families of electrons on C. In this section we assume that for each n, the n electrons on the analytic curve C are in equilibrium configuration. In other words, the points z_{nk} in (4.1) will be nth Fekete points. For such points it has recently been shown by C. Pommerenke [8] that

$$\theta_{nk} = \frac{2\pi k}{n} + \alpha_n + \frac{1}{n}\varphi\left(\frac{2\pi k}{n} + \alpha_n\right) + \tau_{nk}, \qquad k = 1, \cdots, n, \tag{5.1}$$

where $\varphi(\theta)$ is a real analytic function of period 2π as described below and

$$\sum_{k=1}^{n} |\tau_{nk}| = O\left(\frac{1}{n}\right).$$

One considers the following expansions for functions related to the mapping function $\Phi(w)$ (§ 2):

$$\begin{aligned} \log \frac{\Phi(u) - \Phi(v)}{u - v} &= \log a - \sum_{p,q=1}^{\infty} a_{pq} u^{-p} v^{-q}, \\ \log \Phi'(w) &= \log a - \sum_{\nu=1}^{\infty} c_\nu w^{-\nu}, \end{aligned} \tag{5.2}$$

where $|u|, |v|, |w| > 1$. With ρ defined as in § 2,

$$a_{pq} = O(\rho^{p+q}), \qquad c_\nu = O(\rho^\nu)$$

[7]. The function $\varphi(\theta)$ is given by the Fourier series

$$\varphi(\theta) = i \sum_{\nu=1}^{\infty} \left(\frac{\bar{s}_\nu}{\nu} e^{i\nu\theta} - \frac{s_\nu}{\nu} e^{-i\nu\theta} \right) \tag{5.3}$$

[7], where $\{s_\nu\}$ is the unique solution of the system of equations

$$s_\nu + \nu \sum_{\mu=1}^{\infty} a_{\nu\mu}\bar{s}_\mu = \tfrac{1}{2}\nu c_\nu, \qquad \nu = 1, 2, \cdots, \tag{5.4}$$

such that

$$s_\nu = O(\nu\rho^\nu). \tag{5.5}$$

It can be derived from (5.1) by some calculation (cf. [7]) that

$$s_{n\nu} = \sum_{k=1}^{n} e^{i\nu\theta_{nk}} = s_\nu + \nu^2 O\left(\frac{1}{n}\right). \tag{5.6}$$

If the nth Fekete points are not unique, we select one set. The field due to electrons placed at the nth Fekete points will be denoted by $E_n(z)$; we again have the representation (4.2). In view of (5.6) it is natural also to introduce the function

$$E(z) = \sum_{p=0}^{\infty} \bar{s}_{p+1} K_p(z). \tag{5.7}$$

By (5.5) and (2.4) with $r < 1/\rho$, formula (5.7) can be used to define $E(z)$ as a holomorphic function throughout the region D_σ, where $\sigma = 1/\rho$.

It is clear that $s_{n\nu} \to s_\nu$ as $n \to \infty$ for each $\nu \geqq 1$ and that for every $\delta > 0$ there is a constant M_δ such that $|s_{n\nu}| \leqq M_\delta e^{\delta\nu}$ for all n and ν. We may thus apply Theorem 2 of § 3.

THEOREM 4. *For an analytic Jordan curve C, the fields $E_n(z)$ due to electrons placed at n-th Fekete points tend to a limit function $E(z)$ as $n \to \infty$ uniformly on every compact subset of D; the limit function is holomorphic in a neighborhood of the closure of D.*

Can it happen that $E(z)$ is the zero function? By Theorem 1 of §2 one has $E = 0$ if and only if $\bar{s}_{p+1} = 0$ for all $p \geqq 0$. By (5.4) the last condition is equivalent to the requirement that c_ν be zero for all $\nu \geqq 1$, and by (5.2) the latter condition is satisfied if and only if $\Phi'(w) = a$, or

$$\Phi(w) = aw + a_0. \tag{5.8}$$

The mapping function $\Phi(w)$ has the form (5.8) if and only if C is a circle.

COROLLARY 4.1. *The limiting field $E(z)$ is identically zero if and only if C is a circle.*

Theorem 4 and Corollary 4.1 answer a question raised at the Symposium on Analytic Function Theory held at the University of Kentucky, May 28–June 1, 1965 (cf. [12]; the comment printed with the problem is incorrect).

In the case of a circle the consecutive nth Fekete points are equidistant and $E_n(z)$ tends exponentially to zero on every compact subset of the interior. In the general case it follows from (5.6) and (2.4) that on every compact subset of D,

$$E_n(z) = E(z) + O(1/n).$$

It is possible to express $E(z)$ in terms of the mapping function. Assuming that 0 is in D (this is no restriction) and integrating from 0 to z along a path in D, we find that

$$\int_0^z E_n(\zeta)\,d\zeta = -\sum_1^n \log\left(1 - \frac{z}{z_{nk}}\right) \to \int_0^z E(\zeta)\,d\zeta$$

as n $\to \infty$. We now set $e^{i\theta_{nk}} = w_{nk}$, so that $z_{nk} = \Phi(w_{nk})$ and take $z = \Phi(w)$, where $\rho < |w| < 1$. Then for suitable values of the logarithms,

$$-\sum_1^n \log\left(1 - \frac{z}{z_{nk}}\right) = -\sum_1^n \log\left(1 - \frac{w}{w_{nk}}\right) + \left\{n \log a - \sum_1^n \log \frac{\Phi(w) - \Phi(w_{nk})}{w - w_{nk}}\right\}$$

$$+ \sum_1^n \log \frac{z_{nk}}{a w_{nk}} = T_1 + T_2 + T_3,$$

say. By (4.3) and (5.6),

$$T_1 = \sum_1^\infty \frac{1}{\nu} \bar{s}_{n\nu} w^\nu \to \sum_1^\infty \frac{1}{\nu} \bar{s}_\nu w^\nu,$$

and by (5.2),

$$T_2 = \sum_{p,q} a_{pq} w^{-p} \bar{s}_{nq} \to \sum_p \left(\sum_q a_{pq} \bar{s}_q\right) w^{-p}.$$

It follows that the constants T_3 tend to a limit which we denote by $-\gamma - \frac{1}{2}\log a$.

Using (5.4) we thus find that

$$\text{(5.9)} \quad H(z) = \gamma + \int_0^z E(\zeta)\,d\zeta = \sum_1^\infty \frac{1}{\nu}(\bar{s}_\nu w^\nu - s_\nu w^{-\nu}) - \frac{1}{2}\log a + \frac{1}{2}\sum_1^\infty c_\nu w^{-\nu}.$$

Letting $w \to e^{i\theta}$ and using (5.2), we conclude that for $z = \Phi(w)$ on C,

$$\operatorname{Re} H(z) = -\tfrac{1}{2}\log|\Phi'(w)| = \tfrac{1}{2}\log|\Psi'(z)|,$$

where Ψ is the inverse of Φ.

COROLLARY 4.2. *One has*

$$E(z) = H'(z),$$

where $H(z)$ *is holomorphic in the closure of* D *and such that*

$$\operatorname{Re} H(z) = \tfrac{1}{2}\log|\Psi'(z)| \quad \text{on } C.$$

Remark. Formula (5.9) shows that Pommerenke's function $\varphi(\theta)$ is equal to

$$-\operatorname{Im}\left[H\{\Phi(e^{i\theta})\} + \tfrac{1}{2}\log \Phi'(e^{i\theta})\right].$$

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SIAM J. MATH. ANAL.
Vol. 2, No. 3, August 1971

TIME REVERSAL IN ABSTRACT CAUCHY PROBLEMS*

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Abstract. Using the fact that under homogeneous endpoint conditions on a finite interval the eigenvalues of the operator d^2/dx^2 tend to $-\infty$, a standard construction shows that initial boundary value problems for the backward heat equation $u_t = -u_{xx}$ are not well-posed in the sense that the solutions do not depend continuously on the data. Corresponding problems for the forward heat equation are well-posed. Abstract considerations show that the pathology of heat conduction problems is essentially due to the unboundedness of d^2/dx^2 rather than to the distribution of its eigenvalues. Indeed, if A is a closed, unbounded operator in a Banach space X, and if the initial value problem $u'(t) = Au(t), t > 0, u(0) = f$, is well-posed in X, then the backward initial value problem $u'(t) = -Au(t)$, $t > 0$, $u(0) = f$, is not well-posed, even with the data f restricted to the domain $D(A)$ of A.

1. Introduction. Let A be a closed, densely defined operator in a Banach space X. Consider the initial value problem

$$u'(t) = Au(t), \qquad 0 < t < T,$$
$$u(0) = f,$$

and the terminal value problem

$$v'(t) = Av(t), \qquad 0 < t < T,$$
$$v(T) = g.$$

If A is bounded or X is finite-dimensional, the initial and terminal value problems are essentially Cauchy problems for first order linear ordinary differential equations, and they are well-posed; that is, they have the properties of existence, uniqueness, and continuous dependence on the data.

If $u'(t) = Au(t)$ represents a partial differential equation, the situation is entirely different. In this case A is an unbounded operator in an infinite-dimensional Banach space X, and we shall show that if the initial value problem for A is well-posed in X (in a sense to be made precise), then the terminal value problem for A is not well-posed, even if the terminal data g is restricted to the domain $D(A)$ of A.

2. Preliminaries. Let X be a Banach space. A function $u:(a, b) \to X$, defined on a real interval (a, b), is *differentiable on* (a, b) if there exists a function $u':(a, b) \to X$ such that for each $t \in (a, b)$,

$$\|u(t + h) - u(t) - hu'(t)\| = o(|h|) \quad \text{as } h \to 0.$$

Let A be a linear operator in X, $c \in [a, b]$, $f \in X$. A *solution* of the Cauchy problem

$$u'(t) = Au(t), \qquad a < t < b, \tag{1}$$

$$u(c) = f, \tag{2}$$

* Received by the editors November 19, 1970, and in revised form January 25, 1971.

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is a continuous function $u:[a,b] \to X$, differentiable on (a,b), such that $u(t) \in D(A)$ for $t \in (a,b)$, and (1)–(2) are satisfied.

If V is a linear manifold in X, the Cauchy problem is said to be *well-posed in V* if the following three conditions are satisfied.

(A) *Existence.* For each $f \in V$ there exists a solution u.

(B) *Uniqueness.* For each $f \in V$ there is at most one solution u.

(C) *Continuous dependence on the data.* For each $t \in [a,b]$ there exists a constant M_t such that for each $f \in V$ the solution u of the Cauchy problem satisfies $\|u(t)\| \leqq M_t \|f\|$.

Note that if the Cauchy problem is well-posed and V is closed, then the constant M_t in (C) can be taken to be independent of t. Indeed, if (A)–(C) hold, the solution u of the Cauchy problem with data f can be written in the form $u(t) = U(t)f$, where $U(t): V \to X$ is a bounded operator for each $t \in [a,b]$. Since each solution u is continuous on $[a,b]$, there exists a constant M_f such that $\|U(t)f\| \leqq M_f$, $a \leqq t \leqq b$, for each $f \in V$. Hence by the uniform boundedness principle [3, p. 46] there exists a constant M such that $\|U(t)\| \leqq M$ for $a \leqq t \leqq b$.

The graph $G(A)$ of A will always be equipped with the graph norm

$$\|[u, Au]\| = \|u\| + \|Au\|;$$

so if A is closed, $G(A)$ is a Banach space.

If Y and Z are Banach spaces, a linear mapping $T: Y \to Z$ will be called *compact* if the image under T of any bounded subset of Y is totally bounded in Z.

3. The effect of time reversal. Assume now that X is an infinite-dimensional Banach space, and A is an unbounded, closed, densely defined linear operator in X.

THEOREM 1. *If the initial value problem*

$$u'(t) = Au(t), \qquad 0 < t < T, \tag{3}$$

$$u(0) = f, \tag{4}$$

is well-posed in X, then the terminal value problem

$$v'(t) = Av(t), \qquad 0 < t < T, \tag{5}$$

$$v(T) = g, \tag{6}$$

is not well-posed in $D(A)$.

Proof. Suppose (3)–(4) is well-posed in X. Let $U(t): X \to X$ be the solution operator $u(t) = U(t)f$. By (C), $U(t)$ is bounded for $0 \leqq t \leqq T$. Since the range of $U(t)$ is contained in $D(A)$ for $0 < t < T$, we can define a mapping $J(t): X \to G(A)$ by

$$J(t)f = [U(t)f, AU(t)f], \qquad 0 < t < T.$$

We claim that for each fixed $t \in (0, T)$, $J(t)$ is bounded. Indeed, suppose $f_n \to f$ in X, and $J(t)f_n \to [v, Av]$ in $G(A)$. Then $U(t)f_n \to v$ in X, and $AU(t)f_n \to Av$ in X. But by continuity, $U(t)f_n \to U(t)f$. Hence $v = U(t)f$; so $[v, Av] = J(t)f$. Thus $J(t)$ is closed and therefore bounded [3, p. 45].

Assume now that existence and uniqueness hold for the terminal value problem (5)–(6) in $D(A)$. Let v be the solution of (5)–(6) with terminal data $g \in D(A)$.

Fix $t \in (0, T)$, and let $u(s) = v(s + t)$. Then u satisfies $u'(s) = Au(s)$, $u(0) = v(t)$. Since (3)–(4) is well-posed, u must be given by $u(s) = U(s)v(t)$. In particular $g = u(T - t) = U(T - t)v(t)$. Hence for $0 < t < T$, $U(T - t)$ has an inverse defined on $D(A)$, and $v(t) = U(T - t)^{-1}g$. We claim that $U(T - t)^{-1}$ is unbounded on $D(A)$ for $0 < t < T$ (and hence problem (5)–(6) is not well-posed on $D(A)$). Define the operator $P: G(A) \to D(A)$ by $P[u, Au] = u$. Since A is unbounded, there exists a sequence (g_n) in $D(A)$ such that $g_n \to 0$, but $\|P^{-1}g_n\| \to \infty$. Obviously $U(T - t) = PJ(T - t)$. Since P and $U(T - t)$ are invertible, so is $J(T - t)$, and we have $U(T - t)^{-1} = J(T - t)^{-1}P^{-1}$. Since $J(T - t)$ is bounded, if $U(T - t)^{-1}g_n$ converged to 0, we would have $P^{-1}g_n \to 0$, which is not the case. Hence for $0 < t < T$, $U(T - t)^{-1}$ is an unbounded operator on $D(A)$; so the terminal value problem (5)–(6) is not well-posed.

4. A priori bounds. If A is a symmetric, negative definite operator in a Hilbert space H, and A^{-1} is compact, then the initial value problem (3)–(4) is well-posed in H. If $\phi_1, \phi_2, \cdots$ is a complete orthonormal sequence of eigenvectors of A, then the solution u of (3)–(4) with initial data $f \in H$ is given by

$$u(t) = \sum_{k=1}^{\infty} a_k \exp(\lambda_k t)\phi_k,$$

where $a_k = (f, \phi_k)$, and λ_k is the eigenvalue of ϕ_k. Differentiation of $(u(t), u(t))$ shows that $\|u(t)\| \leqq \|f\|$. Thus Theorem 1 shows that the terminal value problem (5)–(6) for A cannot be well-posed. Failure of continuous dependence on the data is illustrated by the following example. The unique solution u_n of (5)–(6) with terminal data $g_n = \phi_n/\lambda_n$ is given by

$$u_n(t) = \frac{e^{\lambda_n(t - T)}}{\lambda_n}\phi_n.$$

Since $\lambda_n \to -\infty$, $\|g_n\| \to 0$. But for any $t < T$, $\|u_n(t)\| \to \infty$. This unbounded behavior of solutions is analogous to the pathology encountered in backward heat conduction problems [6, p. 228].

If u satisfies $u'(t) = Au(t)$, and A is symmetric, then $\log\|u(t)\|$ is a convex function. Hence if $r < s < t$,

$$\log\|u(s)\| \leqq \frac{s - r}{t - r}\log\|u(t)\| + \frac{t - s}{t - r}\log\|u(r)\|. \tag{7}$$

This shows that if v_n satisfies (5)–(6) with terminal data $v_n(T) = g_n$, and $\|g_n\| \to 0$, then $\|v_n(t)\| \to 0$ for $0 < t \leqq T$, provided there exists a constant M such that $\|v_n(0)\| \leqq M$, $n = 1, 2, \cdots$. In this case we say that (5)–(6) has *continuous dependence on the data under a prescribed bound.*

Continuity with a bound holds for more general problems, but the sharp estimate (7) is no longer valid. Let A be a closed linear operator in an infinite-dimensional Banach space X, and assume that the mapping $[u, Au] \to u$ of the graph of A into X is compact. Assume also that X is reflexive.

THEOREM 2. *If the initial value problem* (3)–(4) *is well-posed in* X, *and if the terminal value problem* (5)–(6) *has uniqueness in* X, *then the terminal value problem has continuous dependence on the data under a prescribed bound.*

Results analogous to Theorem 2 are needed for the numerical treatment of a wide variety of improperly posed problems in partial differential equations [4], [5]. The proof of Theorem 2 depends on the following lemma, of which a special case is well known.

LEMMA 1. *Let X, Y, Z be Banach spaces, X reflexive. Let $T: X \to Y$ be a compact linear operator, $S: Y \to Z$ a bounded linear operator. Assume that $ST: X \to Z$ is injective. Then for each $\varepsilon > 0$ there exists a constant $C = C(\varepsilon)$ such that*

$$\|Tx\| \leqq \varepsilon\|x\| + C\|STx\|$$

for all $x \in X$.

Proof. Suppose the lemma is false. Then there is an $\varepsilon > 0$ and a sequence (x_n) in X such that

$$\|Tx_n\| > \varepsilon\|x_n\| + n\|STx_n\|.$$

By normalization in X we can assume that $\|x_n\| = 1$; so

$$\|Tx_n\| > \varepsilon + n\|STx_n\| \geqq \varepsilon > 0. \tag{8}$$

Since $\|x_n\| = 1$, and X is reflexive, (x_n) has a weakly convergent subsequence [3, p. 30] which we still denote by (x_n). Let $x \in X$ be the weak limit of this sequence. Since T is compact, $Tx_n \to Tx$ in Y. Hence $STx_n \to STx$ in Z. But (8) shows that $STx = 0$. Hence $x = 0$; so $Tx = 0$. But this contradicts (8).

Proof of Theorem 2. Assume that (3)–(4) is well-posed, and let $U(t)$ be the solution operator. Let $0 < t < T$. Since $U(t) = PJ(t)$ and P is compact (by hypothesis), $U(t)$ is compact. Note that $U(T) = U(T - t)U(t)$. If (5)–(6) has uniqueness in X, then $U(T)$ is injective. Hence by Lemma 1, for each $\varepsilon > 0$ there is a constant C such that

$$\|U(t)f\| \leqq \varepsilon\|f\| + C\|U(T)f\|.$$

That is,

$$\|u(t)\| \leqq \varepsilon\|u(0)\| + C\|u(T)\| \tag{9}$$

for any solution u of $u'(t) = Au(t)$, $0 < t < T$, which is continuous on $[0, T]$. Now suppose (u_n) is a sequence of solutions of (5)–(6) with terminal data $u_n(T) = g_n$. Then by (9),

$$\|u_n(t)\| \leqq \varepsilon\|u_n(0)\| + C\|g_n\|;$$

so if $\|g_n\| \to 0$ and $\|u_n(0)\| \leqq M$, then $\limsup \|u_n(t)\| \leqq \varepsilon M$. Since ε is arbitrary, $\lim \|u_n(t)\| = 0$. Since $t \in (0, T)$ is arbitrary, $\lim \|u_n(t)\| = 0$ for $0 < t \leqq T$.

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SIAM J. MATH. ANAL.
Vol. 2, No. 3, August 1971

A CLASS OF COMPLETE ORTHOGONAL SEQUENCES OF PERIODIC FUNCTIONS*

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Abstract. A class of orthogonal sequences of functions which are linear combinations of certain complete systems of the type $\{h(nx)\}$ is defined. Each member of the class is shown to be complete in $L_2(0, 1)$ and two general class-wide pointwise convergence theorems are obtained for the Fourier expansions relative to these sets. This class includes the orthogonal set of step functions defined by Harrington and Cell and also the saw-tooth functions of Duffin.

1. Introduction. Harrington and Cell [7] and Duffin [4] have constructed complete orthogonal sequences of functions which are linear combinations of the functions of certain complete systems of the type $\{h(nx)\}$, $n = 1, 2, \cdots$. Szäsz [9] has shown that a complete system $\{h(nx)\}$ is generated by any function $h \in L_2(a, b)$ whose Fourier coefficients are mutliplicative.

THEOREM 1 (Szäsz). *Let $g(x)$ be bounded, $-\infty < x < \infty$, and such that $\{g(nx)\}$, $n = 1, 2, \cdots$, is an orthogonal sequence on $[a, b]$ which is complete in $L_2(a, b)$ (or a subspace $S \subset L_2(a, b)$). If $h \in L_2(a, b)$ has Fourier coefficients a_n (with respect to $\{g(nx)\}$) which satisfy $a_1 = 1$, $a_m a_n = a_{mn}$ for all integers m and n, then the sequence $\{h(nx)\}$, $n = 1, 2, \cdots$, is complete in $L_2(a, b)$ (or S).*

In this paper we define a class of complete orthogonal sets on $[0, 1]$ composed of functions which are linear combinations of the $h(nx)$ functions generated as in Theorem 1. This class includes the set of square wave functions defined by Harrington and Cell, the saw-tooth functions of Duffin, and (in a trivial sense) the familiar $\{\sin n\pi x\}$. Two theorems are obtained giving sufficient conditions for pointwise convergence of Fourier expansions relative to orthogonal sets in the class.

It should be noted at the outset that the function $g(x)$ of Theorem 1 can, for practical purposes, be taken to be $\sqrt{2} \sin \pi x$ (or $\sqrt{2} \cos \pi x$ in the case of orthonormal sets of the type $\{1, g(nx)\}$), since Bourgin and Mendel [2] have shown that $\{g(nx)\}$ is a complete orthonormal set in $L_2(0, 1)$ if and only if $g(x) = \pm\sqrt{2} \sin \pi x$ a.e. (and analogously $g(x) = \pm\sqrt{2} \cos \pi x$ a.e. for complete orthonormal sets $\{1, g(nx)\}$).

2. Definitions and fundamental properties. Let g denote a bounded function of period 2 on $(-\infty, \infty)$ such that $\{g(nx)\}$, $n = 1, 2, \cdots$, is a complete orthonormal sequence in $L_2(0, 1)$ or a subspace (such as that spanned by $\{\cos n\pi x\}$). Consider g to be either an odd function or an even function.

Let $h \in L_2(0, 1)$ have the Fourier expansion with respect to $\{g(nx)\}$,

$$h(x) \sim \sum_{n=1}^{\infty} a_n g(nx), \tag{1}$$

such that $a_1 = 1$ and $a_{mn} = a_m a_n$ for all positive integers m and n. Considering (1) as a half-range expansion of h, it will be understood that the definition of h is

* Received by the editors August 7, 1969, and in final revised form January 22, 1971.

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extended first to the domain $[-1, 1]$ as the odd or even extension according as g is odd or even and then to the domain $(-\infty, \infty)$ by requiring that $h(x + 2) = h(x)$.

It follows from Theorem 1 that the sequence of functions $\{H_n\}$ $\{h(nx)\}$ is complete in $L_2(0, 1)$ or S. In general, the H_n will not be orthogonal on $[0, 1]$. In order to obtain an orthogonal sequence, a related sequence $\{\phi_n\}$ is defined in terms of the H_n as follows:

$$\phi_n(x) = \sum_{d|n} \mu(d) a_d h\left(\frac{nx}{d}\right), \tag{2}$$

where the sum is taken over all positive integers d which divide n, a_d is the dth Fourier coefficient of h, and $\mu(d)$ is the Möbius function defined by $\mu(1) = 1$, $\mu(d) = 0$ if d contains a square factor, and $\mu(d) = (-1)^k$ if d is the product of k distinct primes. We note that since $a_1 = 1$, $\phi_1(x) = h(x)$.

Several lemmas are needed to show that the sequence $\{\phi_n\}$ is orthogonal on $[0, 1]$ and complete in $L_2(0, 1)$. The proofs, in most cases, are omitted either because they are quite straightforward or because they are analogous to the corresponding proofs in [7] with the specialized sequence of coefficients used in [7] replaced by a general multiplicative sequence $\{a_n\}$. The essential number-theoretic properties of the Möbius function can be found in [6].

LEMMA 1. *If m and n are relatively prime integers,*

$$\int_0^1 h(nx)h(mx)\,dx = a_m a_n \|h\|^2 = a_m a_n \sum_{k=1}^{\infty} a_k^2.$$

LEMMA 2. *If n is an integer, f and g are periodic of period 2, and f and g are both even (or both odd), then*

$$\int_0^1 f(nx)g(nx)\,dx = \int_0^1 f(x)g(x)\,dx.$$

LEMMA 3. *If m and n are relatively prime and $n > 1$, then*

$$\int_0^1 h(mx)\phi_n(x)\,dx = 0.$$

LEMMA 4. *If m and n are relatively prime and $m \neq n$, then*

$$\int_0^1 \phi_m(x)\phi_n(x)\,dx = 0.$$

LEMMA 5. (a) *If $n = p^\alpha N$, where p is a prime and $\alpha > 1$, then for each β, $1 \leqq \beta < \alpha$,*

$$\phi_n(x) = \phi_M(p^{\alpha-\beta}x),$$

where $M = p^\beta N$.

(b) *If $n = p^\alpha N$, where p^α ($\alpha \geqq 1$) is the highest power of prime p dividing n, then*

$$\phi_n(x) = \phi_N(p^\alpha x) - a_p \phi_N(p^{\alpha-1}x).$$

LEMMA 6. *If $m = p^\alpha M$ and $n = p^\alpha N$, where p is a prime and $\alpha > 1$, then*

$$\int_0^1 \phi_m(x)\phi_n(x)\,dx = \int_0^1 \phi_{pM}(x)\phi_{pN}(x)\,dx.$$

By the repeated application of Lemma 6 one may reduce any integral of the type $\int_0^1 \phi_m(x)\phi_n(x)\,dx$ to one of the same type in which the greatest common divisor of the subscripts has only first powers of primes as factors.

LEMMA 7. *Suppose that p is a prime such that p, but not p^2, is a factor of the greatest common divisor of the integers m and n. Then the integral $\int_0^1 \phi_m(x)\phi_n(x)\,dx$ can be expressed as a linear combination of integrals of the type $\int_0^1 \phi_j(x)\phi_k(x)\,dx$, where $j|m$ and $k|n$ but p is not a common divisor of j and k.*

It is now possible to prove that the sequence $\{\phi_n\}$ is orthogonal on $[0, 1]$.

THEOREM 2. *If m and n are distinct integers,*

$$\int_0^1 \phi_m(x)\phi_n(x)\,dx = 0.$$

Proof. If m and n are relatively prime, the result is that of Lemma 4. Otherwise if

$$(m, n) = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k},$$

where the p_i are distinct primes, the repeated application of Lemma 6 will reduce the integral to the form

$$\int_0^1 \phi_{HM}(x)\phi_{HN}(x)\,dx,$$

where $H = p_1p_2\cdots p_k$ and $(M, N) = 1$. Since the greatest common divisor of HM and HN involves only first powers of primes, the successive application of Lemma 7 (corresponding to each p_i) reduces this integral to a sum of the type $\int_0^1 \phi_s(x)\phi_t(x)\,dx$ where no one of the p_i is a common divisor of s and t, hence $(s, t) = 1$. These integrals all vanish by Lemma 4.

THEOREM 3.

$$\|\phi_n\|^2 = \int_0^1 \phi_n^2(x)\,dx = \|\phi_1\|^2 \prod_{p|n} (1 - a_p^2), \tag{3}$$

where the product is taken over all primes which divide n and where

$$\|\phi_1\|^2 = \|h\|^2 = \sum_{n=1}^\infty a_n^2.$$

The proof is like that of Theorem 10 in [7] where $\|S_1\| = 1$.

Since the sequence $\{H_n\} \equiv \{h(nx)\}$ is complete in $L_2(0, 1)$ or in S, the completeness of $\{\phi_n\}$ can be established by showing that each H_n can be expressed as a linear combination of the ϕ_n. This is shown by the following theorem.

THEOREM 4. *If n has no prime divisor p for which $a_p = 0$, then*

$$h(nx) = a_n \sum_{m|n} \frac{1}{a_m}\phi_m(x);$$

otherwise if $n = PN$ with $a_p = 0$ for each prime p that divides P and $a_d \neq 0$ for all $d|N$ (there is always one such d since $a_1 = 1$), then

$$h(nx) = a_N \sum_{m|N} \frac{1}{a_m}\phi_{Pm}(x).$$

The proof follows from the multiplicative property of the $\{a_n\}$ and from the Möbius inversion formula [6].

The results obtained to this point are collected in the next theorem.

THEOREM 5. *The sequence of functions* $\{\phi_n\}$, $n = 1, 2, \cdots$, *is complete in* $L_2(0, 1)$ *and orthogonal on* $[0, 1]$ *with norms as given by Theorem* 3.

From Theorem 5 we see that any f in $L_2(0, 1)$ or in S has the Fourier ϕ_n expansion (convergent to f in the norm of $L_2(0, 1)$),

$$f(x) \sim \sum_{n=1}^{\infty} c_n\phi_n(x), \tag{4}$$

where

$$c_n = \frac{1}{\|\phi_n\|^2}\int_0^1 f(x)\phi_n(x)\,dx. \tag{5}$$

Since $h(nx)$ is a simpler function than $\phi_n(x)$, it is of interest to note that the Fourier ϕ_n coefficients of f can be expressed as follows:

$$c_n = \frac{1}{\|\phi_n\|^2}\sum_{d|n}\mu(d)a_d\int_0^1 f(x)h\left(\frac{nx}{d}\right)dx. \tag{6}$$

One may also derive a formula for the c_n in terms of the Fourier $g(nx)$ coefficients of f and h.

3. The Fourier expansion of h. In the definition of the sequence $\{\phi_n\}$ it was assumed that the function h had an expansion in terms of the orthonormal set $\{g(nx)\}$ with multiplicative coefficients. The Bourgin–Mendel theorems mentioned in § 1, of course, imply that this expansion can be taken to be either the half-range sine or cosine expansion without loss of generality. Since the function h is also assumed to be in $L_2(0, 1)$, the sequence of coefficients $\{a_n\}$ of (1) must satisfy $\sum a_n^2 < \infty$, and conversely if $\{a_n\}$ is any completely multiplicative sequence such that $\sum a_n^2 < \infty$, the expansion (1) represents a function which satisfies the requirements of our definition.

The general sequence of coefficients which satisfies $a_1 = 1$, $a_m a_n = a_{mn}$ can be constructed as follows [9]:

Take $a_1 = 1$; for each prime p let a_p equal an arbitrary real number and if n is any integer with the prime factorization $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$, let

$$a_n = a_{p_1}^{\alpha_1}\ a_{p_2}^{\alpha_2}\cdots a_{p_k}^{\alpha_k}.$$

As a simple example, suppose P is a particular prime and let $a_P = r, 0 < |r| < 1$, and take $a_p = 0$ for all primes $p \neq P$. Then $a_1 = 1, a_n = 0$ if n has a prime factor other than P, and $a_n = r^k$ if $n = P^k$, $k \geqq 1$. Since $\sum a_n^2$ converges, $\sum a_n \sin n\pi x$ represents a function $h \in L_2(0, 1)$ which can be used to determine a complete orthogonal sequence $\{\phi_n\}$.

Also, it is obvious that $\{a_n\} = \{n^\alpha\}$ (α a constant) is a completely multiplicative sequence; thus if $\alpha < -\frac{1}{2}$, the sequence $\{n^\alpha\}$ determines a function $h \in L_2(0, 1)$. Similarly, given any completely multiplicative $\{a_n\}$ with $\sum a_n^2 < \infty$, it is clear that $\{a_n/n^\beta\}$, $\beta > 0$, is also completely multiplicative and thus determines a function h that satisfies the requirements of the definition of $\{\phi_n\}$.

4. Construction of complete orthogonal sequences on $[-1, 1]$. Let $\phi_1 \in L_2(0, 1)$ have the Fourier expansion $\sum a_n \sin n\pi x$, where $\{a_n\}$ is a completely multiplicative sequence. Then the related orthogonal sequence $\{\phi_n\}$, as constructed in § 2, is complete and orthogonal in the space of odd functions of $L_2(-1, 1)$. Now construct a second function in $L_2(0, 1)$,

$$\psi_1(x) \sim \sum_{n=1}^{\infty} b_n \cos n\pi x,$$

with $b_1 = 1$ and $b_n = \pm a_n, n \geqq 1$, subject to the condition that $\{b_n\}$ be completely multiplicative.[1] Then the sequence $\{1, \psi_1(nx)\}$ is complete in the space of even functions of $L_2(-1, 1)$ and the functions $\{1, \psi_n\}$ will be orthogonal on $[-1, 1]$. Thus the combined set $\{1, \psi_n, \phi_n\}$ is complete and orthogonal in $L_2(-1, 1)$.

In the above construction, $\|\psi_1\| = \|\phi_1\|$, and by Theorem 3, $\|\psi_n\| = \|\phi_n\|$. Of course, one can use a ψ_1 with any completely multiplicative $\{b_n\}$ and construct a complete orthogonal set $\{1, \psi_n, \phi_n\}$ on $[-1, 1]$. However, there would be the inconvenience of involving two different norms, $\|\phi_1\|$ and $\|\psi_1\|$, and it would seem more natural to have the even functions $\psi_1(nx)$ closely related to the odd functions $\phi_1(nx)$.

5. Examples. (A) The S_n and C_n functions defined by Harrington and Cell [7] appear to be the simplest examples of ϕ_n-type sets that can be constructed. These functions are linear combinations of the square wave functions $S_1(nx)$, where the basic square wave function

$$S_1(x) = \operatorname{sgn}(\sin \pi x) = \frac{4}{\pi} \sum_{n \text{ odd}} \frac{1}{n} \sin n\pi x.$$

In the notation of this paper, one would use $h(x) = \phi_1(x) = (\pi/4)S_1(x)$ with $\{a_n\} = \{1/n\}$. It is of interest to note that the well-known orthogonal (but incomplete) set of Rademacher functions is the subset $\{S_1(2^n x) = S_{2^n}(x)\}$ of the Harrington–Cell system.

(B) The members of the set $\{r_n(x)\}$ constructed by Duffin [4] are linear combinations of the saw-tooth functions $r(nx)$, where

$$\pi r(x) = \frac{\pi}{2}(1 - 2x) = \sum_{n=1}^{\infty} \frac{1}{n} \sin 2n\pi x.$$

Thus the set $r(nx)$ is complete in the space spanned by $\{\sin 2\pi nx\}$, that is, $L_2(0, \frac{1}{2})$, and the sequence $\{\pi r_n\}$ constructed as in § 2 will be orthogonal and complete on $[0, \frac{1}{2}]$.

(C) Choosing $a_1 = 1$, $a_2 = \frac{1}{2}$, and $a_p = 0$ for all primes $p > 2$, one can construct

$$f_1(x) = \sum_{k=0}^{\infty} 2^{-k} \sin 2^k \pi x.$$

[1] In the case of the square wave functions in [7], $a_n = 0$ for n even, and the natural choice (for geometric analogy with sine and cosine) was to have $b_n = (-1)^{(n-1)/2} a_n$.

As shown in § 3, $\{f_1(nx)\}$ is complete in $L_2(0, 1)$, and one finds the related orthogonal functions

$$f_n(x) = \begin{cases} f_1(nx), & n \text{ odd}, \\ f_1(nx) - \tfrac{1}{2}f_1(nx/2), & n \text{ even}. \end{cases}$$

(D) As a final example, consider the well-known Bernoulli polynomials [5] which have the Fourier expansions

$$B_{2m}(x) = \frac{2(-1)^{m-1}(2m)!}{(2\pi)^{2m}} \sum_{n=1}^{\infty} \frac{\cos 2n\pi x}{n^{2m}},$$

$$B_{2m-1}(x) = \frac{2(-1)^{m}(2m-1)!}{(2\pi)^{2m-1}} \sum_{n=1}^{\infty} \frac{\sin 2n\pi x}{n^{2m-1}}.$$

For each fixed m, the sequences $\{B_{2m-1}(nx)\}$ and $\{1, B_{2m}(nx)\}$ are both complete in $L_2(0, \frac{1}{2})$ and determine complete orthogonal sequences of the type constructed in § 2. The Bernoullian transcendents [5] which occur in the Fourier theory of fractional differentiation are defined as certain multiples of the series given for $B_k(x)$ with k not restricted to integer values. If $k > \frac{1}{2}$, each of these generalized Bernoulli functions also determines a complete orthogonal set. Analogous conclusions can be drawn concerning the Euler polynomials.

6. Pointwise convergence of Fourier ϕ_n expansions. Two theorems giving sufficient conditions for the pointwise convergence of the Fourier ϕ_n expansion (4) will be proved in this section. Since it has not been possible to find a closed expression for the Dirichlet kernels associated with the ϕ_n expansions, the proofs of these theorems must be constructed in a rather indirect manner. As a result one might reasonably suspect that stronger convergence theorems are obtainable.

We begin by proving a lemma that gives an order property for the Fourier ϕ_n coefficients of a function in terms of its coefficients relative to the orthogonal set $\{g(nx)\}$, that is, the sine or cosine set.

LEMMA 8. *Let h and f be functions in $L_2(0, 1)$ whose Fourier coefficients relative to the complete orthonormal set $\{g(nx)\}$ are $O(1/n^\alpha)$, $\alpha > \frac{1}{2}$. Then the Fourier coefficients of f relative to $\{\phi_n\}$ are $O(1/n^{\alpha-\varepsilon})$ for every $\varepsilon > 0$.*

Proof. Suppose that the Fourier $g(nx)$ expansions of h and f are

$$h(x) \sim \sum a_n g(nx) \tag{7}$$

and

$$f(x) \sim \sum b_n g(nx), \tag{8}$$

where $a_1 = 1, a_m a_n = a_{mn}$, and both a_n and b_n are $O(1/n^\alpha)$, $\alpha > \frac{1}{2}$; that is, $|a_n| < M/n^\alpha$ and $|b_n| < K/n^\alpha$ for some constants M and K. To obtain an order property for the Fourier ϕ_n coefficients of f, we first find a bound for the integral

$$I_m = \int_0^1 f(x)h(mx)\,dx.$$

Since $h(mx) \sim \sum a_n g(mnx)$, we have from (7) and (8),

$$I_m = \sum_n a_n b_{mn}. \tag{9}$$

The order property assumed above for a_n and b_n with (9) give the inequality

$$|I_m| \leqq \sum_n |a_n b_{mn}| < \frac{C}{m^\alpha}, \tag{10}$$

where $C = MK \sum_n (1/n^{2\alpha})$.

Since the Fourier ϕ_n coefficients of f are given by

$$c_n = \frac{1}{\|\phi_n\|^2} \int_0^1 f(x)\phi_n(x)\,dx = \frac{1}{\|\phi_n\|^2} \sum_{k|n} \mu(k) a_k I_{n/k},$$

it follows from (10) that

$$|c_n| \leqq \frac{1}{\|\phi_n\|^2} \sum_{k|n} |a_k|\,|I_{n/k}| < \frac{MC\,d(n)}{n^\alpha \|\phi_n\|^2}, \tag{11}$$

where $d(n)$ is the total number of divisors of n. It is known [5] that $d(n) = O(n^\varepsilon)$ for every $\varepsilon > 0$. Also, a lower bound for $\|\phi_n\|^2$ can be obtained by use of the following fundamental theorem on the multiplication of Dirichlet series [6].

If $f(n)$ is a completely multiplicative function on the set of positive integers such that $\sum |f(n)| n^{-s}$ is convergent, then

$$\prod_p \frac{1}{1 - f(p)p^{-s}} = \sum_n f(n) n^{-s}, \tag{12}$$

where the product is over the set of all primes and the sum over the positive integers.

Since $\{a_n^2\}$ is completely multiplicative, (12) with $s = 0$ yields

$$\prod_p (1 - a_p^2) = \frac{1}{\|\phi_1\|^2}. \tag{13}$$

Now $|a_p| < 1$ for each prime p since if this were not true, the subsequence $\{a_{p^k}\}$ would not approach zero as required by the convergence of $\sum a_n^2$. Thus

$$\prod_{p|n} (1 - a_p^2) \geqq \prod_p (1 - a_p^2), \tag{14}$$

and by (3), (13) and (14),

$$\|\phi_n\|^2 \geqq \|\phi_1\|^2 \prod_p (1 - a_p^2) = 1. \tag{15}$$

By combining this result with (11) and the order property for $d(n)$ one obtains the inequality

$$|c_n| < \frac{R}{n^{\alpha - \varepsilon}} \quad (R \text{ a constant}) \tag{16}$$

which holds for every $\varepsilon > 0$.

It does not appear that Lemma 8 can be strengthened appreciably since (6) gives, when $n = p$ (a prime),

$$c_p = \frac{1}{\|\phi_p\|^2}(a_1 I_p - a_p I_1).$$

Thus c_p will normally be of the order of magnitude of a_p or I_p, that is, $O(1/p^\alpha)$.

There are a number of well-known theorems that relate smoothness conditions on a function to order properties of its Fourier sine (cosine) coefficients. For example, if f and h are both of bounded variation on $[0, 1]$ or if both satisfy Lipschitz conditions of order $\alpha > \frac{1}{2}$ on $[0, 1]$, the Fourier coefficients of these functions satisfy the hypotheses of Lemma 8 (see [10]). Furthermore, the weaker assumption that f and h both belong to the class Λ_α^p, $\alpha > \frac{1}{2}$, defined by Zygmund also ensures that the Fourier coefficients of these functions are $O(1/n^\alpha)$.

THEOREM 6. *Suppose that Fourier $g(nx)$ coefficients of both f and h are $O(1/n^\alpha)$, $\alpha > \frac{1}{2}$, or in particular, that f and h are in the class Λ_α^p. Then the Fourier expansion of f relative to $\{\phi_n\}$ converges to f almost everywhere in* $[0, 1]$.

Proof. The Rademacher–Menchenoff theorem [1] asserts that any orthonormal series $\sum b_n\phi_n$ for which $\sum b_n^2 \log^2 n < \infty$ converges almost everywhere on the interval of orthogonality of the ϕ_n. If h and f satisfy the hypotheses of Theorem 6, the coefficients c_n of the Fourier ϕ_n expansion of f are $O(1/n^\beta)$ for some $\beta > \frac{1}{2}$. Since $\|\phi_n\| \leqq \|\phi_1\|$, $|c_n|\,\|\phi_n\| = O(1/n^\beta)$ and one deduces that

$$\sum |c_n|^2 \|\phi_n\|^2 \log^2 n < \infty .$$

So it follows that the Fourier ϕ_n expansion of f converges almost everywhere on $[0, 1]$. On the other hand, since $f \in L_2(0, 1)$, $\sum c_n\phi_n$ converges to f in the L_2-norm and it is well known [8] that there is a subsequence of the sequence of partial sums of this series which converges to $f(x)$ almost everywhere in $[0, 1]$. Thus since $\sum c_n\phi_n$ converges almost everywhere in $[0, 1]$, it follows that this series must converge to $f(x)$ almost everywhere on this interval.

The theorem of Carleson[2] [3], which says that if $f \in L_2$, then its Fourier series converges almost everywhere, suggests that the conditions of Theorem 6 may be unnecessarily stringent. It seems likely that results similar to Carleson's are true for some orthogonal systems (in addition to the trigonometric) contained in the class of systems considered in this paper. The authors do not choose to conjecture at this point whether comparable results can be extended to the entire class.

Now suppose that the basic g function is continuous (that is, $\{g(nx)\}$ is either the sine or cosine set) and that f and h have Fourier $g(nx)$ coefficients which are $O(1/n^\alpha)$, $\alpha > 1$. Under these conditions, the expansion $h(x) = \sum a_n g(nx)$ is uniformly convergent on $[0, 1]$ so h is continuous on this interval. It is also clear that the $h(nx)$, $n = 1, 2, \cdots$, are uniformly bounded on $[0, 1]$. Thus ϕ_n is continuous on $[0, 1]$ and $\{\phi_n(x)\}$ is also uniformly bounded on $[0, 1]$. By Lemma 8 it follows that $\sum c_n\phi_n$ converges uniformly on $[0, 1]$. Thus the sum of this series is a continuous function on $[0, 1]$. Now Theorem 6 guarantees that this sum is almost everywhere equal to $f(x)$, so if f itself is continuous on $[0, 1]$, it follows that $f(x) = \sum c_n\phi_n(x)$ for *all* x in $[0, 1]$.

THEOREM 7. *If f and g are continuous and both f and h have Fourier coefficients relative to $\{g(nx)\}$, $n = 1, 2, \cdots$, which are $O(1/n^\alpha)$, $\alpha > 1$, then the Fourier expansion of f relative to $\{\phi_n\}$, $n = 1, 2, \cdots$, converges to f everywhere in* $[0, 1]$.

[2] The authors express their gratitude to the referee for bringing this result to their attention.

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SIAM J. MATH. ANAL.
Vol. 2, No. 3, August 1971

EXTENSIONS OF DIRICHLET'S MULTIPLE INTEGRAL*

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Abstract. A generalization of Dirichlet's multiple integral is derived in a simpler manner than that recently published. Additionally, other extensions are given.

In a recent note, Sivazlian [1] obtained the following extension of Dirichlet's multiple integral:[1]

If $F(x)$ is continuous and $a_i = 0$, $i = 1, 2, \cdots, n$, then

$$\begin{aligned} I(n, r) &= \int\int_R \cdots \int F(t_1 + t_2 + \cdots + t_r) \prod_{i=1}^{n} t_i^{a_i - 1}\, dt_i \\ &= A \int_0^t u^{a-1}(t-u)^b F(u)\, du, \end{aligned} \tag{1}$$

where

$$a = a_1 + a_2 + \cdots + a_r, \qquad b = a_{r+1} + a_{r+2} + \cdots + a_n,$$

$$A = \frac{\Gamma(a_1)\Gamma(a_2)\cdots\Gamma(a_n)}{\Gamma(a)\Gamma(b+1)}$$

and the region R is given by

$$t_1 + t_2 + \cdots + t_n \leqq t, \qquad t_i \geqq 0.$$

Here, we give a simpler derivation plus some still further extensions.

Integrating (1) partially over the variables $t_1, t_2, \cdots, t_r$, which corresponds to evaluating a Dirichlet integral over the region

$$t_1 + t_2 + \cdots + t_r \leqq t - t_{r+1} - t_{r+2} - \cdots - t_n = \bar{t},$$

we obtain

$$B \int_0^{\bar{t}} F(v) v^{a-1}\, dv,$$

where

$$B = \frac{\Gamma(a_1)\Gamma(a_2)\cdots\Gamma(a_r)}{\Gamma(a)}.$$

Integration of the latter integral over the remaining variables, yields

$$I(n, r) = B \int\int_{\bar{t} \geqq 0} \cdots \int t_{r+1}^{a_{r+1}-1} \cdots t_n^{a_n - 1}\, dt_{r+1} \cdots dt_n \int_0^{\bar{t}} F(v) v^{a-1}\, dv.$$

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* Received by the editors September 1, 1970.

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[1] Actually, Sivazlian's result corresponds to the case $t = 1$ but the extension is trivial.

Since the right-hand side is again a Dirichlet integral,

$$I(n, r) = bA \int_0^t u^{b-1}\, du \int_0^{t-u} F(v)v^{a-1}\, dv.$$

Finally, by letting $w = t - u$, interchanging the order of integration and integrating with respect to w, we obtain the desired result (1).

A further extension of (1) is given by

$$\begin{aligned} I_{nr}(F, G) &= \int\int \underset{R}{\cdots} \int F(t_1 + t_2 + \cdots + t_r)G(t - \bar{t}) \prod_i t_i^{a_i - 1}\, dt_i \\ &= bA \int_0^t G(w)(t - w)^{b-1}\, dw \int_0^w F(v)v^{a-1}\, dv. \end{aligned} \tag{2}$$

To obtain this result, we integrate I_{nr} over the variables $t_1, t_2, \cdots, t_r$ as before. Whence,

$$I_{nr}(F, G) = B \int\int \underset{t \geqq 0}{\cdots} \int G(t - \bar{t}) \prod_{i=r+1}^{n} t_i^{a_i - 1}\, dt_i \int_0^{\bar{t}} F(v)v^{a-1}\, dv.$$

On integrating over the remaining variables $t_{r+1}, \cdots, t_n$, we get

$$bA \int_0^t G(t - u)u^{b-1}\, du \int_0^{t-u} F(v)v^{a-1}\, dv$$

which reduces to (2) by letting $w = t - u$.

If the argument $t - \bar{t}$ of G in (2) is replaced by a partial sum of the remaining variables, that is, $t_{r+1} + t_{r+2} + \cdots + t_s\, (r < s < n)$, then (2) can be reduced to the following triple integral:

$$\frac{\Gamma(a_1)\Gamma(a_2) \cdots \Gamma(a_n)}{\Gamma(a)\Gamma(b)\Gamma(c)} \int_0^t u^{c-1}\, du \int_0^{t-u} G(v)v^{b-1}\, dv \int_0^{t-u-v} F(w)w^{a-1}\, dw, \tag{2'}$$

where

$$b = a_{r+1} + a_{r+2} + \cdots + a_s, \qquad c = a_{s+1} + a_{s+1} + \cdots + a_n.$$

The derivation of (2′) is analogous to that of (2) but instead of using the Dirichlet integral, we use the extension in (1). Interchanging the order of integration, we finally obtain

$$\frac{\Gamma(a_1)\Gamma(a_2) \cdots \Gamma(a_n)}{\Gamma(a)\Gamma(b)\Gamma(c + 1)} \int_0^t G(v)v^{b-1}\, dv \int_0^{t-v} F(w)w^{a-1}(t - v - w)^c\, dw. \tag{2''}$$

We can extend (2″) by allowing an overlap in the variables in the arguments of the functions F and G. Proceeding in a similar fashion as before, we get

$$\begin{aligned} I_{rs}(F, G) &= \int\int \underset{R}{\cdots} \int F(x + y)G(y + z) \prod_{i=1}^{n} t_i^{a_i - 1}\, dt_i \\ &= \frac{\Gamma(a_1)\Gamma(a_2) \cdots \Gamma(a_n)}{\Gamma(a)\Gamma(b)\Gamma(c)} \int_0^t u^{c-1}\, du \int_0^{t-u} v^{b-1}\, dv \\ &\quad \cdot \int_0^{t-u-v} w^{a-1} F(u + v)G(v + w)\, dw, \end{aligned} \tag{3}$$

where

$$x = t_1 + t_2 + \cdots + t_r,$$
$$y = t_{r+1} + t_{r+2} + \cdots + t_s,$$
$$z = t_{s+1} + t_{s+2} + \cdots + t_n, \qquad r < s < n.$$

Extensions of (1) in another direction can be obtained by changing the region R of integration to

$$R': \begin{cases} t_1 + t_2 + \cdots + t_r \leqq t, \\ t_{r+1} + t_{r+2} \cdots + t_n \leqq t' \quad \text{and} \quad t_i \geqq 0. \end{cases}$$

For $r < s$, we then obtain

$$I_{rs}(F) = \int\int \underset{R'}{\cdots} \int F(t_1 + t_2 + \cdots + t_s) \prod_{i=1}^{n} t_i^{a_i - 1}\, dt_i$$

$$= \frac{\Gamma(a_1)\Gamma(a_2)\cdots\Gamma(a_n)}{\Gamma(a)\Gamma(b)\Gamma(c+1)} \int_0^{t'} u^{b-1}(t' - u)^c\, du \int_0^t v^{a-1} F(u + v)\, dv.$$

The case $r > s$ leads to nothing new.

Still further extensions can be obtained by considering the more general integral (which includes all previous ones as special cases)

$$\int\int \underset{R_m}{\cdots} \int F_1(x_1)F_2(x_2)\cdots F_n(x_n) \prod_{i=1}^{n} t_i^{a_i - 1}\, dt_i,$$

where the x_i's are various partial sums of the variables $t_1, t_2, \cdots, t_n$ with or without overlap and the region R_m is given by $t_i \geqq 0$ plus m inequalities of the form

$$\sum_j t_{i_j} \leqq M_i$$

again with or without overlap of the t_i's.

Note added in proof. In a more recent note, Sivazlian [this Journal, 2 (1971), pp. 72–75] gives a further extension of his previous result (loc. cit.) by replacing $\prod t_i^{a_i - 1}$ in (1) by $\prod \phi_i(t_i)$ and reduces the multiple integral to a single integral of a repeated convolution. Again his derivation can be simplified and extended in the manner shown here. For example, (2) would now become

$$I_{nr}(F, G) = \int_0^t F(u)[\varphi_1(u)*\varphi_2(u)* \cdots *\varphi_r(u)]*G(u)[\varphi_{r+1}(u)*\varphi_{r+2}(u)* \cdots *\varphi_n(u)]\, du,$$

where $\phi(u)*\psi(u)$ denotes the convolution $\int_0^u \phi(x)\psi(u - x)\, dx$.

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SIAM J. MATH. ANAL.
Vol. 2, No. 3, August 1971

EXTREME EIGENVALUES OF TOEPLITZ MATRICES ASSOCIATED WITH CERTAIN ORTHOGONAL POLYNOMIALS*

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Abstract. Previous results of the author and I. I. Hirschman on the asymptotic behavior of the extreme eigenvalues of truncated Toeplitz matrices associated with Laguerre polynomials are extended using methods, initiated by S. V. Parter, involving the study of related finite difference operators. The same technique also leads to the corresponding results, already obtained by Hirschman, in the case of the Jacobi polynomials.

1. Introduction. In 1953, Kac, Murdoch, and Szegö [11] initiated investigations on the asymptotic behavior of the extreme eigenvalues of truncated Toeplitz matrices whose elements were Fourier coefficients of a real continuous function f on $[-\pi, \pi]$. In the last decade, several authors have continued work of this nature by not only extending the results of [11] (e.g., [13], [14], [15]) but also attacking the analogous problem for Toeplitz matrices associated with the classical orthogonal polynomials (e.g., [3], [7], [8], [12]).

The present effort is an extension of the approach taken in [3] and rests heavily on [2]. We concern ourselves with what is known as the endpoint case for Toeplitz matrices associated with both Laguerre and Jacobi polynomials. Section 4 is devoted to the Jacobi case and the main result (Theorem 4.11) is essentially that obtained by Hirschman [8] by completely different methods. In § 5, the Laguerre case is studied with the main results there (Theorem 5.5–Corollary 5.9) being somewhat less satisfying than in the Jacobi case.

The work here discussed in the Jacobi case was essentially contained in [4], although a mistake there led to the consideration of the operators studied in [1] rather than to the "right" operators G_ω of [2].

2. Some preliminary results. Let $-\infty \leqq a < b \leqq \infty$ be a finite or infinite interval. Let $w(x) > 0$ for $a < x < b$. We assume that $\int_a^b x^n w(x)\,dx < \infty$ for $n = 0, 1, \cdots$. Then the Gram–Schmidt orthogonalization procedure leads to a set of polynomials $p_n(x)$ of degree n, orthonormal in the sense that

$$\int_a^b p_n(x)p_m(x)w(x)\,dx = \delta_{n,m}.$$

We make $p_n(x)$ unique by requiring that the leading coefficient be positive. Let f be any measurable function on (a, b) which is square integrable with respect to $w(x)$:

$$\int_a^b |f(x)|^2 w(x)\,dx < \infty; \tag{2.1}$$

then we say $f \in L^2(w)$. For $f \in L^2(w)$, we define

$$m_{jk}(f) = \int_a^b f(x)p_j(x)p_k(x)w(x)\,dx. \tag{2.2}$$

* Received by the editors August 13, 1970, and in revised form December 15, 1970.

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Then the matrix $T[f] = (m_{jk}(f)), j, k = 0, 1, \cdots$, is the semi-infinite (generalized) Toeplitz matrix associated with f via the orthogonal polynomials $p_n(x)$. If f is real, then the $(n+1) \times (n+1)$ finite section $T_n[f] = (m_{jk}(f)), j, k = 0, 1, \cdots, n$, is real and symmetric, so the eigenvalues of $T_n[f]$ are real. Let

$$\lambda_{1,n}[f] \leqq \lambda_{2,n}[f] \leqq \cdots \leqq \lambda_{n+1,n}[f]$$

be the eigenvalues of $T_n[f]$ arranged in nondecreasing order with repetitions for multiple eigenvalues. We are interested in the behavior of $\lambda_{v,n}[f]$ as $n \to \infty$ for fixed v.

LEMMA 2.1. *Put $p_n(x) = a_n x^n + b_n x^{n-1}$ + (lower order terms). Then the $p_n(x)$ satisfy the three term recurrence formula*

$$xp_n(x) = \frac{a_n}{a_{n+1}} p_{n+1}(x) + \left[\frac{b_n}{a_n} - \frac{b_{n+1}}{a_{n+1}}\right] p_n(x) + \frac{a_{n-1}}{a_n} p_{n-1}(x)$$

for $n = 0, 1, \cdots$, where we agree to put $p_{-1}(x) \equiv 0$, $a_{-1} = 1$, and $b_0 = 0$.

Proof. See [10, p. 157].

LEMMA 2.2. *Let $f(x) = x$. Then $T[f] = (m_{jk}(f))$, $j, k = 0, 1, \cdots$, is given by*

$$m_{jk}(f) = 0 \quad \text{if} \quad |j - k| \geqq 2,$$

$$m_{kk}(f) = \frac{b_k}{a_k} - \frac{b_{k+1}}{a_{k+1}}, \quad m_{k,k+1}(f) = \frac{a_k}{a_{k+1}}, \quad m_{k,k-1}(f) = \frac{a_{k-1}}{a_k}.$$

Proof. Use Lemma 2.1 (cf. [12, Lemma 3.8]).

Suppose henceforth that (a, b) has a finite endpoint; call it x_0. Recalling that all zeros of $p_n(x)$ are in (a, b) (see [10, pp. 159–160]), we define

$$q_1(n) = [p_n(x_0)]^{-1}, \tag{2.3a}$$

$$q_2(n) = p_{n+1}(x_0)p_n(x_0)a_0 a_n a_{n+1}^{-1}, \tag{2.3b}$$

$$q_3(n) = [a_0 p_n(x_0)]^{-1}, \tag{2.3c}$$

for $n = 0, 1, \cdots$. Let Δ^+ be the semi-infinite matrix (a_{ij}), $i, j = 0, 1, \cdots$, where $a_{ii} = -1$, $a_{i,i+1} = 1, i = 0, 1, \cdots$, and all other entries are 0. Let Δ^- be the negative transpose of Δ^+. Let Q_i, $i = 1, 2, 3$, be the semi-infinite diagonal matrix whose diagonal consists of the numbers $q_i(k)$, $k = 0, 1, \cdots$.

The following factorization of $T[x - x_0]$ appeared in [9].

LEMMA 2.3. $T[x - x_0] = Q_1 \Delta^- Q_2 \Delta^+ Q_3$.

Proof. One must verify (for appropriate values of k) that

(i) $q_1(k)q_2(k)q_3(k+1) = \dfrac{a_k}{a_{k+1}}$,

(ii) $q_1(k)q_2(k)q_3(k) + q_1(k)q_2(k-1)q_3(k) = -\dfrac{b_k}{a_k} + \dfrac{b_{k+1}}{a_{k+1}} + x_0$,

(iii) $q_1(k)q_2(k-1)q_3(k-1) = \dfrac{a_{k-1}}{a_k}$.

Equations (i) and (iii) are trivial and (ii) follows from Lemma 2.1 (cf. [3, Lemma 2.2]).

LEMMA 2.4. *Suppose $f, g \in L^2(w)$ and c is a complex scalar. Then*
(a) $T[cf] = cT[f]$;

(b) $T[f+g] = T[f] + T[g]$;

(c) *if F is the closed linear span in $L^2(w)$ of the orthogonal polynomials $p_n(x)$,*

$$T[fg] = T[f]T[g], \quad \text{all } f, g \in F.$$

Proof. Parts (a) and (b) are trivial; (c) follows from the generalized Parseval equation; see [6, p. 122].

We note that the conclusion of Lemma 2.4(c) holds if f and g are polynomials. Of course, if the orthogonal polynomials $p_n(x)$ are complete in $L^2(w)$, then $T[fg] = T[f]T[g]$ holds for all $f, g \in L^2(w)$. In particular, this is true in the Jacobi, Laguerre, and Hermite cases.

In addition, we shall use freely the results in [12], particularly § 3 of that paper.

3. Friedrichs extensions of some differential operators. Our purpose in this section is to describe some operators studied in [2] and state some results which are fundamental for our work here. The reader is referred to [2] for the details. Let

$$\tau u = -\frac{1}{m(x)}[p(x)u']', \qquad 0 < x < 1. \tag{3.1}$$

We assume that $m(x) > 0$, $p(x) > 0$ and both $m(x)$, $p(x)$ are infinitely differentiable on $(0, 1]$. We further assume that

$$\int_0^1 m(x)\left[\int_x^1 [p(t)]^{-1}\,dt\right]dx \equiv K < \infty. \tag{3.2}$$

Thus, τ may be singular at $x = 0$. We now define certain operators induced by τ in the Hilbert space $L^2(m)$ of all measurable functions on $(0,1)$ for which $\int_0^1 |f|^2 m\,dx < \infty$. Let $C^\infty(0,1)$ denote the class of infinitely differentiable functions on $(0,1)$. Put

$$Tu = \tau u, \qquad u \in D(T), \tag{3.3}$$

where $D(T)$ consists of all $u \in C^\infty(0,1)$ such that $u = 0$ near $x = 1$ and $u' = 0$ near $x = 0$. Since T is symmetric and semibounded below, then T has a Friedrichs extension G. Now define

$$T_* u = Gu, \qquad u \in D(T_*), \tag{3.4a}$$

$$D(T_*) = \{u \in D(G) \cap C^\infty(0,1) : u = 0 \text{ near } x = 1\} \tag{3.4b}$$

and further

$$T_\omega u = T_*^\omega u, \qquad u \in D(T_\omega), \tag{3.5a}$$

$$D(T_\omega) = \{u \in D(T_*^\omega) : \tau^\omega u = 0 \text{ near } x = 0\}. \tag{3.5b}$$

Since T_ω is symmetric and semibounded below, T_ω has a Friedrichs extension, which we call G_ω. It is proved in [2] that G_ω^{-1} exists as a compact self-adjoint operator on $L^2(m)$ and that the spectrum of G_ω consists only of eigenvalues which are strictly positive.

4. The Jacobi case. Here $w(x) = (1 - x)^\alpha(1 + x)^\beta$ ($\alpha > -1$, $\beta > -1$) and $(a, b) = (-1, 1)$. The orthogonal polynomials $p_n(x)$ are now the normalized Jacobi polynomials. Indeed,

$$p_n(x) = h_n^{-1/2} P_n^{(\alpha,\beta)}(x),$$

where

$$(2n + \alpha + \beta + 1)n!\,\Gamma(n + \alpha + \beta + 1)h_n = 2^{\alpha+\beta+1}\Gamma(n + \alpha + 1)\Gamma(n + \beta + 1)$$

and $P_n^{(\alpha,\beta)}(x)$ is the usual Jacobi polynomial.

We restrict our attention to the endpoint $x_0 = 1$. If U is any semi-infinite matrix, let U_n denote the $(n + 1) \times (n + 1)$ finite section of U. Let $D_n = (Q_3)_n$ and let

$$M_n[(1 - x)^N] = ([-Q_3Q_1\Delta^- Q_2\Delta^+]^N)_n. \tag{4.1}$$

LEMMA 4.1.

$$T_n[(1 - x)^N] = (T^N[1 - x])_n = D_n^{-1}M_n[(1 - x)^N]D_n.$$

Proof. Use Lemmas 2.3 and 2.4.

We shall investigate the eigenvalues of $T_n[f]$ first for the case that f is a polynomial. Let $h(x)$ be a real polynomial of the form

$$h(x) = (1 - x)^\omega \sum_{i=0}^{J} a_i(1 - x)^i, \qquad \omega \text{ a positive integer}, \tag{4.2}$$

where

$$a_0 > 0, \qquad \sum_{i=0}^{J} a_i(1 - x)^i > 0 \quad \text{for } -1 \leqq x \leqq 1. \tag{4.3}$$

LEMMA 4.2.

$$T_n[h] = D_n^{-1}\left(\sum_{i=0}^{J} a_iM_n[(1 - x)^{i+\omega}]\right)D_n.$$

Proof. Use Lemmas 4.1 and 2.4.

Thus the eigenvalue problem for $T_n[h]$ is the same as that for $\sum_{i=0}^{J} a_iM_n \cdot [(1 - x)^{i+\omega}]$. Our next step is to interpret this new eigenvalue problem as an eigenvalue problem for a finite difference operator.

For each positive integer n, let $\Delta x = (n + 2\omega + 2J + 3)^{-1}$ and let $x_j = j\Delta x$, $j = 0, 1, \cdots, n + 2\omega + 2J + 3$, be the lattice points on the unit interval $[0, 1]$. We define

$$\alpha_n(x_j) = \frac{\alpha + \beta + 2j - 1}{2[\alpha + \beta + j]}\,\frac{d(j)}{j^{2\alpha+1}}\,x_j^{2\alpha+1}, \qquad j = 1, 2, \cdots, \tag{4.4}$$

$$\beta_n(x_j) = \frac{2[\alpha + j]}{\alpha + \beta + 2j}\,\frac{d(j)}{j^{2\alpha+1}}\,x_j^{2\alpha+1}, \qquad j = 1, 2, \cdots, \tag{4.5}$$

where

$$d(j) = \frac{\Gamma(\alpha + j)\Gamma(\alpha + \beta + j + 1)}{\Gamma(j)\Gamma(\beta + j)}, \qquad j = 1, 2, \cdots. \tag{4.6}$$

Let X_n be the space of piecewise linear functions $u^{(n)}(x)$, $0 \leqq x \leqq 1$, determined by their values at the points x_j and for which $u^{(n)}(0) = u^{(n)}(1) = 0$. For $u^{(n)} \in X_n$, define δ_+ and δ_- by

$$(\delta_+ u^{(n)})(x_j) = \begin{cases} \dfrac{1}{\Delta x}[u^{(n)}(x_{j+1}) - u^{(n)}(x_j)], & 0 < x_j < 1, \\ 0, & x_j = 0, \quad x_j = 1; \end{cases} \tag{4.7}$$

$$(\delta_- u^{(n)})(x_j) = \begin{cases} \dfrac{1}{\Delta x}[u^{(n)}(x_j) - u^{(n)}(x_{j-1})], & 0 < x_j \leqq 1, \\ 0, & x_j = 0. \end{cases} \tag{4.8}$$

For $u^{(n)} \in X_n$, we also define S_n by

$$(S_n u^{(n)})(x_j) = \begin{cases} \left(-\dfrac{1}{\alpha_n}\delta_- \beta_n \delta_+ u^{(n)}\right)(x_j), & 0 < x_j \leqq 1, \\ 0, & x_j = 0. \end{cases} \tag{4.9}$$

Let $\mathscr{P}_n$ be the $(n + 1)$-dimensional subspace of X_n consisting of all $u^{(n)} \in X_n$ such that $u^{(n)}(x) = 0$ for $x \geqq x_{n+2}$. For $k = 1, 2, \cdots, 2\omega + 2J + 1$, we define $\sigma_n^{(k)}: \mathscr{P}_n \to \mathscr{P}_n$ by

$$(\sigma_n^{(k)} u^{(n)})(x_j) = \begin{cases} (S_n^k u^{(n)})(x_j) & \text{for } j = 1, 2, \cdots, n + 1, \\ 0 & \text{otherwise}. \end{cases} \tag{4.10}$$

Thus, S_n and $\sigma_n^{(k)}$ are precisely the operators that were defined in [2], where α_n and β_n are given by (4.4) and (4.5). The obvious similarity between δ_+ and Δ^+, δ_- and Δ^-, q_2 and β_n, $q_1 q_3$ and α_n is of course not accidental but designed to give the following lemma.

LEMMA 4.3. *Let $u^{(n)} \in \mathscr{P}_n$. Then*

$$(\sigma_n^{(k)} u^{(n)})(x_j) = 2^k(n + 2\omega + 2J + 3)^{2k}(M_n[(1 - x)^k]V)_j$$

for $j = 1, 2, \cdots, n + 1$, where V is the $(n + 1)$-vector with components $u^{(n)}(x_j)$, $j = 1, 2, \cdots, n + 1$, and $(\cdot)_j$ denotes the j-th component of the vector.

Proof. Specializing (2.3) to the Jacobi case ($x_0 = 1$) and using standard formulas for the Jacobi polynomials (see [10, Chap. 8]), we have, by direct calculation,

$$q_2(n) = \frac{a_0}{[\Gamma(\alpha + 1)]^2\, 2^{\alpha+\beta+1}} \frac{2(\alpha + n + 1)}{\alpha + \beta + 2n + 2} d(n + 1),$$

$$q_1(n) q_3(n) = \frac{[\Gamma(\alpha + 1)]^2\, 2^{\alpha+\beta+1}}{a_0} \frac{\alpha + \beta + n + 1}{\alpha + \beta + 2n + 1} \frac{1}{d(n + 1)},$$

where d is defined in (4.6). The result of the lemma readily follows.

One should observe that the above lemma would be false if we had not required $(\delta_+ u^{(n)})(0) = 0$ in (4.7).

Now define $l_n : \mathscr{P}_n \to \mathscr{P}_n$ by

$$l_n = \sum_{i=0}^{J} 2^{-i}(n + 2\omega + 2J + 3)^{-2i} a_i \sigma_n^{(i+\omega)}, \tag{4.11}$$

where the a_i's are given in (4.2) and (4.3). An immediate calculation using Lemma 4.3 gives

$$(l_n u^{(n)})(x_j) = 2^{\omega}(n + 2\omega + 2J + 3)^{2\omega}\left(\sum_{i=0}^{J} a_i M_n[(1 - x)^{i+\omega}]V\right)_j \tag{4.12}$$

for $j = 1, 2, \cdots, n + 1$ and $u^{(n)} \in \mathscr{P}_n$.

LEMMA 4.4. *Let $\lambda_{v,n}[h]$ be the eigenvalues of $T_n[h]$, where h is defined in* (4.2), *and let $\mu_{v,n}$ be the eigenvalues of the finite difference operator l_n. Then*

$$\mu_{v,n} = 2^{\omega}(n + 2\omega + 2J + 3)^{2\omega} \lambda_{v,n}[h], \quad v = 1, 2, \cdots, n + 1.$$

Proof. Use (4.12) and Lemma 4.2.

Now let $m(x) \equiv p(x) \equiv x^{2\alpha+1}$, $0 < x \leqq 1$, and let τ be the formal differential operator

$$\tau u = -\frac{1}{m}(pu')', \qquad 0 < x < 1.$$

Let G_ω be the strictly positive self-adjoint operator with compact inverse described in § 3 for this τ. Let

$$0 < \Lambda_1(G_\omega) \leqq \Lambda_2(G_\omega) \leqq \cdots \leqq \Lambda_v(G_\omega) \leqq \cdots$$

be the eigenvalues of G_ω arranged in nondecreasing order with repetitions for multiple eigenvalues.

We now state the main theorem of [2] in the present context. That theorem has four hypotheses which we state now as assumptions A1 through A4.

A1. Let $\delta = \delta_-\delta_+$. For each $i = 0, 1, \cdots$ and $k = 0, 1$, assume that

$$\lim_{n\to\infty}\left[\sup_{\varepsilon \leqq x_j \leqq 1} |(\delta_+^k \delta^i \beta_n)(x_j) - (D^{2i+k}p)(x_j)|\right] = 0$$

for each ε such that $0 < \varepsilon < 1$. Similarly, assume that

$$\lim_{n\to\infty}\left[\sup_{\varepsilon \leqq x_j \leqq 1}\left|\left[\delta_+^k \delta^i\left(\frac{1}{\alpha_n}\right)\right](x_j) - \left[D^{2i+k}\left(\frac{1}{m}\right)\right](x_j)\right|\right] = 0$$

for each ε such that $0 < \varepsilon < 1$. Here D denotes differentiation.

A2. Assume that there exist positive constants E_1 and E_2, independent of n, such that

$$p(x_j) \leqq E_1\beta_n(x_j), \qquad \alpha_n(x_j) \leqq E_2 m(x_j)$$

for $0 < x_j \leqq 1$.

A3. Let

$$K_n = \sum_{0 < x_j \leqq 1} m(x_j)\left[\sum_{x_j \leqq x_i \leqq 1} [p(x_i)]^{-1}\,\Delta x\right]\Delta x.$$

Assume that $\lim_{n\to\infty} K_n = K$.

A4. Assume that there exists a positive constant c_0, independent of n, such that

$$[\sigma_n^{(\omega)} h^{(n)}, h^{(n)}] \leqq c_0[l_n h^{(n)}, h^{(n)}], \qquad n = 1, 2, \cdots,$$

for all $h^{(n)} \in \mathscr{P}_n$, where

$$[u^{(n)}, v^{(n)}] \equiv \sum_{j=1}^{n+1} u^{(n)}(x_j)\overline{v^{(n)}(x_j)}\alpha_n(x_j)\Delta x$$

for all $u^{(n)}, v^{(n)} \in \mathscr{P}_n$.

THEOREM (see [2, Theorem 4.3]). *Under assumptions* A1 *through* A4, *if* $\mu_{\nu, n}$ are *the eigenvalues of the finite difference operator* l_n, *arranged in nondecreasing order and counting multiplicities, then*

$$\lim_{n \to \infty} \mu_{\nu,n} = a_0 \Lambda_\nu(G_\omega).$$

THEOREM 4.5. *For each fixed* ν,

$$\lim_{n \to \infty} 2^\omega n^{2\omega} \lambda_{\nu,n}[h] = a_0 \Lambda_\nu(G_\omega).$$

Proof. Using Lemma 4.4 and the above theorem, it suffices to verify assumptions A1 through A4 in the present context. We do this in the following sequence of lemmas.

The first lemma is an extension of the well-known fact

$$\lim_{x \to \infty} \frac{\Gamma(x + a)}{x^a \Gamma(x)} = 1. \tag{4.13}$$

LEMMA 4.6. *If k is a positive integer, then*

$$\lim_{x \to \infty} x^k \frac{d^k}{dx^k}\left[\frac{\Gamma(x + a)}{x^a \Gamma(x)}\right] = 0.$$

Proof. The result follows quickly from [5, (4), p. 47].

LEMMA 4.7. *Assumption* A1 *holds.*

Proof. We extend β_n to all of $[\varepsilon, 1]$ by the equation

$$\beta_n(x) = B_n(x) x^{2\alpha+1}, \qquad \varepsilon \leqq x \leqq 1,$$

where

$$B_n(x) = \left(\frac{2[\alpha + Nx]}{\alpha + \beta + 2Nx}\right) \frac{\Gamma(\alpha + Nx)\Gamma(\alpha + \beta + Nx + 1)}{\Gamma(Nx)\Gamma(\beta + Nx)(Nx)^{2\alpha+1}}$$

and $N = n + 2\omega + 2J + 3$. The extended β_n obviously agrees with the old β_n at the lattice points. By (4.13), we see that $\beta_n(x)$ converges uniformly on $[\varepsilon, 1]$ to $p(x) = x^{2\alpha+1}$. One easily verifies, using Lemma 4.6, that all derivatives of $B_n(x)$ of order one or greater converge uniformly on $[\varepsilon, 1]$ to 0. Thus, all derivatives of $\beta_n(x)$ converge uniformly on $[\varepsilon, 1]$ to the corresponding derivative of $p(x) = x^{2\alpha+1}$. Approximating $(D^{2i+k}p)(x_j)$ by $(D^{2i+k}\beta_n)(x_j)$ and then using Taylor's theorem, we arrive at the desired result for β_n. The case of $1/\alpha_n$ is treated similarly.

LEMMA 4.8. *Assumption* A2 *holds.*

Proof. The coefficients of $\alpha_n(x_j)$ and $\beta_n(x_j)$ in (4.4) and (4.5) both tend to 1 as $j \to \infty$ and are independent of n. The lemma thus follows immediately.

LEMMA 4.9. *Assumption* A3 *holds.*

Proof. A simple argument using advanced calculus techniques and focusing attention on the singular behavior near $x = 0$ yields this lemma.

LEMMA 4.10. *Suppose* $u^{(n)} \in \mathscr{P}_n$. *Let*

$$m_0 = \min_{-1 \leqq x \leqq 1} \left(\sum_{i=0}^{J} a_i (1 - x)^i \right) > 0.$$

Then

$$[\sigma_n^{(\omega)} u^{(n)}, u^{(n)}] \leqq \frac{1}{m_0} [l_n u^{(n)}, u^{(n)}],$$

and hence assumption A4 *holds.*

Proof. Let V be the $(n + 1)$-vector corresponding to $u^{(n)} \in \mathscr{P}_n$, i.e., $(V)_j = u^{(n)}(x_j)$, $j = 1, 2, \cdots, n + 1$. Let $V' = D_n^{-1} V$. Since $h(x) \geqq m_0(1 - x)^\omega$, it follows that (see [12, Lemma 3.1])

$$m_0(T_n[(1 - x)^\omega] V', V') \leqq (T_n[h] V', V'),$$

where $(\cdot, \cdot)$ denotes the ordinary (unweighted) inner product in $(n + 1)$-space. Thus

$$m_0((D_n^{-1})^2 M_n[(1 - x)^\omega] V, V) \leqq \left((D_n^{-1})^2 \left(\sum_{i=0}^{J} a_i M_n[(1 - x)^{i+\omega}] \right) V, V \right).$$

The diagonal elements of $(D_n^{-1})^2$ are the numbers

$$\frac{1}{q_3^2(k-1)} = \frac{q_1(k-1)}{q_3(k-1)} \left[\frac{1}{q_3(k-1) q_1(k-1)} \right]$$
$$= \frac{2a_0^2(n + 2\omega + 2J + 3)^{2\alpha+1}}{[\Gamma(\alpha + 1)]^2 \, 2^{\alpha+\beta+1}} \alpha_n(x_k)$$

for $k = 1, 2, \cdots, n + 1$. The desired result now follows from Lemma 4.3 and (4.12).

Having completed the proof of Theorem 4.5, we can now prove a form of Hirschman's theorem [8, Theorem 14b].

THEOREM 4.11. *Let f be a real continuous function on* $[-1, 1]$. *Assume that*

(i) $f(x) > f(1) \equiv m$ *for* $-1 \leqq x < 1$;

(ii) $f(x) \sim m + \sigma(1 - x)^\omega$ *as* $x \to 1^-$, *where ω is a positive integer.*

Then for fixed v,

$$\lim_{n \to \infty} 2^\omega n^{2\omega} (\lambda_{v,n}[f] - m) = \sigma \Lambda_v(G_\omega).$$

Proof. We may assume that $f(1) \equiv m = 0$. Define

$$g(y) = \begin{cases} \dfrac{f(1 - y)}{y^\omega}, & 0 < y \leqq 2, \\ \sigma, & y = 0. \end{cases}$$

Then g is continuous and positive for $0 \leqq y \leqq 2$. Let $\mu = \min_{0 \leqq y \leqq 2} g(y) > 0$. Let $0 < \varepsilon < \mu$. Define

$$g_1(y) = g(y) - \varepsilon/2 \qquad g_2(y) = g(y) + \varepsilon/2, \qquad 0 \leqq y \leqq 2.$$

By the Weierstrass approximation theorem, there exist polynomials $h_i(y)$ $(i = 1, 2)$, so that

$$|h_i(y) - g_i(y)| < \varepsilon/2, \qquad 0 \leqq y \leqq 2.$$

Then

$$0 < g(y) - \varepsilon < h_1(y) < g(y) < h_2(y) < g(y) + \varepsilon, \qquad 0 \leqq y \leqq 2.$$

Applying Theorem 4.5 to $(1 - x)^\omega h_i(1 - x)$, $i = 1, 2$, and noting that

$$(1 - x)^\omega h_1(1 - x) \leqq f(x) \leqq (1 - x)^\omega h_2(1 - x), \quad -1 \leqq x \leqq 1,$$

and that $\varepsilon > 0$ can be taken arbitrarily small, we obtain the theorem.

5. The Laguerre case. Here $w(x) = x^\alpha e^{-x}$, $\alpha > -1$, and $(a, b) = (0, \infty)$. The resulting orthogonal polynomials $p_n(x)$ are essentially the normalized Laguerre polynomials. In fact,

$$p_n(x) = (-1)^n h_n^{-1/2} L_n^{(\alpha)}(x),$$

where now

$$n! h_n = \Gamma(n + \alpha + 1)$$

and $L_n^{(\alpha)}$ is the usual Laguerre polynomial. The endpoint is now $x_0 = 0$. Again, let $D_n = (Q_3)_n$ and let

$$M_n[x^N] = ([Q_3 Q_1 \Delta^- Q_2 \Delta^+]^N)_n. \tag{5.1}$$

As before, we have the following lemma.

LEMMA 5.1.

$$T_n[x^N] = (T^N[x])_n = D_n^{-1} M_n[x^N] D_n.$$

Let $h(x)$ be a polynomial of the form

$$h(x) = x^\omega \sum_{i=0}^{J} a_i x^i, \qquad \omega \text{ a positive integer}, \tag{5.2}$$

where

$$a_0 > 0, \qquad \sum_{i=0}^{J} a_i x^i > 0 \quad \text{for } 0 \leqq x < \infty. \tag{5.3}$$

LEMMA 5.2.

$$T_n[h] = D_n^{-1} \left(\sum_{i=0}^{J} a_i M_n[x^{i+\omega}] \right) D_n.$$

Proof. Use Lemma 5.1.

Thus the eigenvalue problem for $T_n[h]$ is the same as that for $\sum_{i=0}^{J} a_i M_n[x^{i+\omega}]$.

We now define the relevant finite difference operators. Letting $\Delta x = (n + 2\omega + 2J + 3)^{-1}$ as before, define (as in [3])

$$\alpha_n(x_j) = \frac{\Gamma(j+\alpha)}{j^\alpha \Gamma(j)} x_j^\alpha, \qquad j = 1, 2, \cdots, \tag{5.4}$$

$$\beta_n(x_j) = \frac{\Gamma(j+\alpha+1)}{j^{\alpha+1}\Gamma(j)} x_j^{\alpha+1}, \qquad j = 1, 2, \cdots. \tag{5.5}$$

Let S_n and $\sigma_n^{(k)}$ be the finite difference operators of [2] for the α_n and β_n of (5.4) and (5.5). That is, S_n and $\sigma_n^{(k)}$ are defined as in (4.9) and (4.10) except that α_n and β_n of (5.4) and (5.5) are used.

LEMMA 5.3. *Let $u^{(n)} \in \mathscr{P}_n$ and let V be the corresponding $(n + 1)$-vector. Then*

$$(\sigma_n^{(k)} u^{(n)})(x_j) = (n + 2\omega + 2J + 3)^k (M_n[x^k]V)_j$$

for $j = 1, 2, \cdots, n + 1$.

Proof. See [3, Lemma 3.1], where essentially the same result is proved.

Now define $l_n : \mathscr{P}_n \to \mathscr{P}_n$ by

$$l_n = \sum_{i=0}^{J} a_i (n + 2\omega + 2J + 3)^{-i} \sigma_n^{(i+\omega)}, \tag{5.6}$$

where the a_i's are given in (5.2) and (5.3). An immediate calculation using Lemma 5.3 and (5.6) gives

$$(l_n u^{(n)})(x_j) = (n + 2\omega + 2J + 3)^\omega \left(\sum_{i=0}^{J} a_i M_n[x^{i+\omega}]V \right)_j \tag{5.7}$$

for $j = 1, 2, \cdots, n + 1$ and $u^{(n)} \in \mathscr{P}_n$.

LEMMA 5.4. *Let $\lambda_{\nu,n}[h]$ be the eigenvalues of $T_n[h]$ and let $\mu_{\nu,n}$ be the eigenvalues of the finite difference operator l_n. Then*

$$\mu_{\nu,n} = (n + 2\omega + 2J + 3)^\omega \lambda_{\nu,n}[h], \qquad \nu = 1, 2, \cdots, n + 1.$$

Proof. Use (5.7) and Lemma 5.2.

Now let $m(x) = x^\alpha$, $p(x) = x^{\alpha+1}$, $0 < x \leqq 1$, and let G_ω be the strictly positive self-adjoint operator with compact inverse described in § 3 for the formal differential operator

$$\tau u = -\frac{1}{m}(pu')', \qquad 0 < x < 1.$$

Let

$$0 < \Lambda_1(G_\omega) \leqq \Lambda_2(G_\omega) \leqq \cdots \leqq \Lambda_\nu(G_\omega) \leqq \cdots$$

be the eigenvalues of G_ω arranged in nondecreasing order with repetitions for multiple eigenvalues.

THEOREM 5.5. *For each fixed ν,*

$$\lim_{n\to\infty} n^\omega \lambda_{\nu,n}[h] = a_0 \Lambda_\nu(G_\omega).$$

Proof. Once again, we only need to verify the assumptions A1 through A4 of [2, Theorem 4.3]. Since these verifications are analogous to the Jacobi case and since the work was essentially done in [3, Lemmas 3.3 and 3.5], we omit the verifications.

We shall be interested in functions which satisfy the following condition.

CONDITION $A(\omega)$. Let $f(x)$ be a real continuous function on $[0, \infty)$, for which $f(x) > f(0) \equiv m$ for $x > 0$. Let ω be a positive integer. Let $f(x) \sim m + \sigma x^{\omega}$ as $x \to 0^+$ and let $f(x) = o(x^N)$ as $x \to \infty$ for some positive integer N. Finally, let $\liminf_{x\to\infty} f(x) > m$.

THEOREM 5.6. *Suppose f satisfies condition $A(\omega)$. Then*

$$\limsup_{n\to\infty} n^{\omega}[\lambda_{\nu,n}[f] - m] \leqq \sigma\Lambda_\nu(G_\omega).$$

Proof. We may assume $m = 0$. Let $a_0 > \sigma$. Let β be the larger of N and $\omega + 1$. There exists $K > 0$ such that

$$h(x) \equiv a_0 x^{\omega} + K x^{\beta} > f(x) \quad \text{for } 0 \leqq x < \infty.$$

Thus [12, p. 458]

$$n^{\omega}\lambda_{\nu,n}[f] \leqq n^{\omega}\lambda_{\nu,n}[h],$$

and by Theorem 5.5,

$$\limsup_{n\to\infty} n^{\omega}\lambda_{\nu,n}[f] \leqq a_0\Lambda_\nu(G_\omega).$$

Since a_0 can be taken arbitrarily close to σ, the theorem follows.

We would like to be able to get a polynomial lower bound for f also, as we did in the Jacobi case. Here the added difficulty of an infinite interval of orthogonality is felt, and in general a polynomial lower bound for arbitrary f satisfying Condition $A(\omega)$ is impossible. In the case $\omega = 1$, Hirschman [7] has proved the result for bounded functions in $A(\omega)$, and in [3], we used his result in place of a polynomial lower bound.

We conjecture that for arbitrary f satisfying Condition $A(\omega)$,

$$\lim_{n\to\infty} n^{\omega}[\lambda_{\nu,n}[f] - m] = \sigma\Lambda_\nu(G_\omega), \tag{5.8}$$

but have been unable to prove (5.8) in this generality. However, we easily prove the following theorem.

THEOREM 5.7. *Suppose f satisfies Condition $A(\omega)$ and, in addition,*

$$f(x) \geqq m + \sigma x^{\omega} \quad \text{for } 0 \leqq x < \infty.$$

Then (5.8) *is valid.*

Proof. Since $f(x) \geqq m + \sigma x^{\omega}$, we have

$$n^{\omega}[\lambda_{\nu,n}[f] - m] \geqq n^{\omega}[\lambda_{\nu,n}[m + \sigma x^{\omega}] - m],$$

and hence using Theorem 5.5, we have

$$\liminf_{n\to\infty} n^{\omega}[\lambda_{\nu,n}[f] - m] \geqq \sigma\Lambda_\nu(G_\omega),$$

which combined with Theorem 5.6 yields the desired result.

What more can we say about functions satisfying Condition $A(\omega)$ which are, say, bounded? Although we are not able to prove (5.8), we now get a positive lower bound for $\liminf_{n\to\infty} n^{\omega}[\lambda_{\nu,n}[f] - m]$.

THEOREM 5.8. *Suppose f satisfies Condition $A(\omega)$. Then*

$$\liminf_{n\to\infty} n^{\omega}[\lambda_{\nu,n}[f] - m] \geqq \sigma[\Lambda_{\nu}(G_1)]^{\omega} > 0.$$

Proof. As before, we may assume $m = 0$. Then $f(x) > 0$ for $0 < x < \infty$. Define $g(x) > 0$ for $0 < x < \infty$ by $g^{\omega} = f$. Then g satisfies Condition $A(1)$. It follows from [3, Theorem 2, Lemma 4.3] that

$$\lim_{n\to\infty} n[\lambda_{\nu,n}[g]] = \sigma^{1/\omega}\Lambda_{\nu}(G_1). \tag{5.9}$$

Using [12, Lemma 3.1], we have

$$(V, T_n[g]V) = \int_0^{\infty} g(t)|\hat{V}(t)|^2 w(t)\,dt, \tag{5.10}$$

where $V = \{v_k\}$, $k = 0, 1, \cdots, n$, is an $(n + 1)$-vector, $\hat{V}(t) = \sum_{k=0}^{n} v_k p_k(t)$, $p_k(t)$ is the kth normalized Laguerre polynomial, and $(\cdot, \cdot)$ is the ordinary inner product in $(n + 1)$-space.

Using Hölder's inequality on (5.10), we get

$$(V, T_n[g]V) \leqq \left(\int_0^{\infty} f(t)|\hat{V}(t)|^2 w(t)\,dt\right)^{1/\omega}\left(\int_0^{\infty} |\hat{V}|^2 w(t)\,dt\right)^{1-1/\omega},$$

and hence

$$\left[\frac{(V, T_n[g]V)}{(V, V)}\right]^{\omega} \leqq \frac{(V, T_n[f]V)}{(V, V)} \quad \text{for } V \neq 0. \tag{5.11}$$

Thus, by the Courant–Fischer minimax theorem [12, Lemma 3.4], we have

$$\lambda_{\nu,n}^{\omega}[g] \leqq \lambda_{\nu,n}[f],$$

and hence using (5.9), we have

$$\liminf_{n\to\infty} n^{\omega}\lambda_{\nu,n}[f] \geqq \lim_{n\to\infty} (n\lambda_{\nu,n}[g])^{\omega} = \sigma[\Lambda_{\nu}(G_1)]^{\omega}.$$

COROLLARY 5.9. *Suppose f satisfies Condition $A(\omega)$. Then*

$$\lambda_{\nu,n}[f] = m + O\left(\frac{1}{n^{\omega}}\right)$$

and O cannot be replaced by o.

Note that Theorems 5.6 and 5.8 together imply $[\Lambda_{\nu}(G_1)]^{\omega} \leqq \Lambda_{\nu}(G_{\omega})$. This fact seems plausible if one notes that $[\Lambda_{\nu}(G_1)]^{\omega} = \Lambda_{\nu}(G_1^{\omega})$ and that G_{ω} is the Friedrichs extension of a restriction of G_1^{ω} (see [2]).

Acknowledgment. The author would like to express his appreciation to the referee for his helpful comments and criticism.

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SIAM J. MATH. ANAL.
Vol. 2, No. 4, November 1971

ON WEAK SOLUTIONS OF A MILDLY NONLINEAR DIRICHLET PROBLEM*

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Abstract. This paper investigates minimal conditions on functions of the type $F(x, v(x))$ and on the boundary of a domain R to insure the existence and uniqueness of a weak solution to the mildly nonlinear Dirichlet problem

$$\Delta u(x) = F(x, u(x)), \qquad x \in R,$$

$$u(x) = 0, \qquad x \in \partial R.$$

The principal technique is the construction of an operator by means of a linearization of the nonlinear equation. The operator is shown to satisfy the Schauder–Tikhonov theorem and the resulting fixed point is shown to be a solution of the mildly nonlinear problem.

1. Introduction and summary. In this paper we are concerned with obtaining the existence and uniqueness of a weak solution for a certain nonlinear Dirichlet problem; namely, for $F(x, v)$ a given function, we seek a unique solution of

$$-\int_R (\nabla u \cdot \nabla \phi)(x)\, dx = \int_R \phi(x) F(x, u(x))\, dx$$

for all ϕ in an appropriate space of test functions.

Specifically, in § 2, we introduce notation and state two estimates on derivatives of a Green's function.

In § 3, we present a characterization of the dual of the Sobolev space $\mathring{W}_p^1$. This characterization has been proved for smooth domains by Lions and Magenes [14]. Our method differs in the proof of the so-called "shift theorem" and in the restrictions on the smoothness of the boundary.

The next section is devoted to the existence and uniqueness of a weak solution of a linear problem associated with solving the mildly nonlinear equation. Bramble [3] has considered these questions for Laplace's equation.

In the last section, we present our main result. The basic technique is a linearization process suggested by Lees [12] in one dimension.

2. Notation and preliminary results. Let R be a bounded domain in Euclidean N-space, E^N, with its boundary ∂R of class C^1.

We shall denote by $C_0^\infty(R)$ the class of infinitely differentiable, real-valued functions with support compactly contained in R. As usual, $L_p(R)$ is the space of real-valued functions defined on R such that

$$\left(\int_R |f(x)|^p\, dx\right)^{1/p} = \|f\|_{L_p(R)}$$

* Received by the editors November 16, 1970, and in revised form February 1, 1971.

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is finite. Also, for any function f, we shall use the norm

$$|f|_R = \sup_{x \in R} |f(x)|.$$

Let $V_p(R)$ be the space of $(N+1)$-dimensional vectors, whose components are all elements of $L_p(R)$. If

$$U = (u_0, u_1, \cdots, u_n) \in V_p(R), \quad \text{then} \quad \|U\|_{V_p(R)} = \sum_{i=0}^{N} \|u_i\|_{L_p(R)}.$$

We shall consider also the Sobolev spaces $\mathring{W}_p^1(R)$, which we define as the closure of $C_0^\infty(R)$ under the norm

$$\|u\|_{W_p^1(R)} = \left(\|u\|^p_{L_p(R)} + \sum_{i=0}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|^p_{L_p(R)} \right)^{1/p},$$

where $\partial u / \partial x_i$ represents the distributional (or weak) derivative.

To indicate the duality between the spaces $L_p(R)$ and $L_q(R)$, $1/p + 1/q = 1$, we define for $f \in L_p(R)$, $g \in L_q(R)$,

$$\langle f, g \rangle_{L_p(R), L_q(R)} = \int_R f(x) g(x)\, dx.$$

Similarly, for $U = (u_0, u_1, \cdots, u_n) \in V_p(R)$,

$$W = (\omega_0, \omega_1, \cdots, \omega_n) \in V_q(R),$$

$$\langle U, W \rangle_{V_p(R), V_q(R)} = \sum_{i=0}^{N} \int_R u_i(x) \omega_i(x)\, dx$$

and for $u \in W_p^1(R)$, $v \in W_q^1(R)$,

$$\langle u, v \rangle_{W_p^1(R), W_q^1(R)} = \int_R \left[u(x)v(x) + \sum_{i=1}^{N} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \right] dx,$$

where the derivatives are in the distributional sense. Finally, we shall use C to denote a generic constant which is not necessarily the same in any two places.

In the following sections, we shall need certain L_p estimates on derivatives of a Green's function. Since the proofs are essentially computational, we shall state these results and refer the reader to the author's paper [18, § III] for the details. Let B be the unit ball in E^N and let L be the differential operator defined by

$$Lu \equiv -\Delta u + u.$$

Let $G^L(x, y)$ denote the Green's function for the operator L and $G(x, y)$ denote the Green's function associated with the Laplace operator.

THEOREM 2.1. *Suppose Λ is an operator mapping L_p into L_p, $1 < p < \infty$, defined (in the principal value sense) by*

$$(\Lambda f)(x) = \sum_{i,j=1}^{N} \int_B \frac{\partial G^L(x, y)}{\partial x_j \partial y_i} f(y)\, dy.$$

Then

$$\|\Lambda f\|_{L_p(B)} \leqq c \|f\|_{L_p(B)}.$$

For first order derivatives we may obtain the bound on a more general domain.

THEOREM 2.2. *If $\pi : L_p(R) \to L_p(R)$, where R is a bounded domain with boundary of class C^1, is given by*

$$(\pi f)(x) = \sum_{i=1}^{N} \int_R \frac{\partial G^L(x, y)}{\partial x_i} f(y)\, dy,$$

then

$$\|\pi f\|_{L_p(R)} \leqq c \|f\|_{L_p(R)}.$$

3. The dual of $\mathring{W}_p^1(R)$. The space of continuous linear functionals defined on the Sobolev space $\mathring{W}_p^1(R)$, $(\mathring{W}_p^1(R))'$, is usually identified with the negative norm space W_q^{-1}, where $1/p + 1/q = 1$. We shall now show that under certain conditions on ∂R and for a particular equivalent norm, there is an isomorphism between $(\mathring{W}_p^1(R))'$ and the space $\mathring{W}_q^1(R)$, $1/p + 1/q = 1$. This duality is contained in a result obtained previously for domains with smooth boundaries by Lions and Magenes [14].

Our approach differs in the techniques used in proving Lemma 3.1 and in the preciseness of the boundary conditions. We shall prove the duality for C^1 boundaries. It can be shown that, in general, the duality does not hold for arbitrary domains. For example, one can show that for a sector of a circle the values of p and q for which the duality is valid depend upon the sector angle.

We shall have need of the duality in the next section when solving a linear problem.

We begin with a version of the so-called "shift theorem" over the unit ball of R^N, which we denote by B.

LEMMA 3.1. *Suppose $u_0, u_1, \cdots, u_N$ are in $C_0^\infty(B)$. If u is a solution of the problem*

$$(3.1)\qquad \begin{aligned} Lu(x) &= -\Delta u(x) + u(x) = -\sum_{i=1}^{N} \frac{\partial u_i(x)}{\partial x_i} + u_0(x), && x \in B,\\ u(x) &= 0, && x \in \partial B, \end{aligned}$$

then $u \in \mathring{W}_q^1(B)$, $1 < q < \infty$, and satisfies

$$\|u\|_{W_q^1(B)} \leqq c \sum_{i=0}^{N} \|u_i\|_{L_q(B)}.$$

Proof. Denote by $G^L(x, y)$ the Green's function for the operator L. If u is a solution of (3.1), then we may express u as

$$\begin{aligned} u(x) &= \int_B G^L(x, y)[Lu](y)\, dy \\ &= \int_B G^L(x, y)\left[-\sum_{i=1}^{N} \frac{\partial u}{\partial y_i}(y) + u_0(y)\right] dy. \end{aligned}$$

Integration by parts yields that

$$u(x) = \sum_{i=1}^{N} \int_B \frac{\partial G^L(x, y)}{\partial y_i} u_i(y)\, dy + \int_B G^L(x, y) u_0(y)\, dy \tag{3.2}$$

and from the estimates of the Green's function stated previously, we obtain the inequality

$$\|u\|_{L_q} \leqq c \sum_{i=0}^{N} \|u_i\|_{L_q}.$$

It also follows from (3.2) and the Calderon–Zygmund theorem, that in the principal value sense,

$$\frac{\partial u}{\partial x_j}(x) = \sum_{i=1}^{N} \int_B \frac{\partial^2 G^L(x, y)}{\partial x_j \partial y_i} u_i(y)\, dy + \int_B \frac{\partial G^L(x, y)}{\partial x_j} u_0(y)\, dy, \quad \text{a.e.} \tag{3.3}$$

Again using our knowledge of the Green's function (i.e., the second mixed partial is a singular integral kernel satisfying the Calderon–Zygmund theorem) we have that

$$\left\| \frac{\partial u}{\partial x_j} \right\|_{L_q} \leqq c \sum_{i=0}^{N} \|u_i\|_{L_q}.$$

We now see that since

$$\|u\|_{W_q^1(B)} \leqq \left(\sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{L_q} + \|u\|_{L_q} \right)^{1/q},$$

it follows from (3.2) and (3.3) that

$$\|u\|_{W_q^1(B)} \leqq c \sum_{i=0}^{N} \|u_i\|_{L_q(B)}.$$

Using a reflection argument, we may deduce from Lemma 3.1 that the following lemma holds.

LEMMA 3.2. *Lemma* 3.1 *is valid on the unit hemisphere.*

With these lemmas we are now ready for the representation theorem.

THEOREM 3.1. *Let R be a bounded domain in E^N with boundary of class C^1. Then, $(\mathring{W}_p^1(R))'$ is isomorphic to $\mathring{W}_q^1(R)$, $1/p + 1/q = 1$.*

Proof. Denote by $V_p(R)$, the space of $(N + 1)$-dimensional vector-valued $L_p(R)$ functions; that is, if

$$U = (u_0, u_1, \cdots, u_N) \in V_p(R), \quad \text{then } u_i \in L_p(R), \qquad i = 0, 1, \cdots, N.$$

We may consider $\mathring{W}_p^1(R)$ as the subspace of $V_p(R)$ whose elements have the form $(u, \partial u/\partial x_1, \cdots, \partial u/\partial x_N)$.

If f is a continuous linear functional on $\mathring{W}_p^1(R)$, we may extend f to all of $V_p(R)$. Call $\tilde{f}$ the extended functional, and then it is clear that the class of all such extended functionals is contained in $V_q(R)$, $1/p + 1/q = 1$. Moreover, for $v \in \mathring{W}_p^1(R)$, we obtain the following representation from the Riesz–Fréchet theorem:

$$\tilde{f}(v) = f(v) = \left\langle \left(v, \frac{\partial v}{\partial x_1}, \cdots, \frac{\partial v}{\partial x_N} \right), U \right\rangle_{V_p, V_q} \tag{3.4}$$
$$= \sum_{i=1}^{N} \int_R \frac{\partial v(x)}{\partial x_i} u_i(x)\, dx + \int_R v(x) u_0(x)\, dx$$

for some $U = (u_0, u_1, \cdots, u_N) \in V_q(R)$.

For each $u_i, i = 0, \cdots, N$, there exists a sequence of $C_0^\infty(R)$ functions $\{U_i^n\}$, which converges to u_i in $L_q(R)$. Now consider the following problem:

$$[Lu^n](x) = -\sum_{i=1}^{N} \frac{\partial U_i^n}{\partial x_i}(x) + U_0^n(x), \qquad x \in R, \tag{3.5}$$
$$u^n(x) = 0, \qquad x \in \partial R.$$

For each n, a solution of (3.5) exists in $\mathring{W}_q^1(R)$.

Moreover, the set of solutions is uniformly bounded in $\mathring{W}_q^1(R)$. To show this boundedness, we cover R with a family of open spheres $\{\theta_i\}$, $i = 1, \cdots, \nu$, and we let $\{\eta_i\}$ be a partition of unity subordinate to this open covering. Define $u_j^n(x) = u^n(x)\eta_j(x)$; then

$$[Lu_j^n](x) = \eta_j(x)\left[-\sum_{i=1}^{N} \frac{\partial U_i^n}{\partial x_i}(x) + U_0^n(x) \right] \tag{3.6}$$
$$+ 2\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(u^n \frac{\partial \eta_j}{\partial x_i} \right)(x) - u^n(x)\Delta\eta_j(x), \qquad x \in \theta_j \cap R,$$
$$u_j^n(x) = 0, \qquad x \in \partial(\theta_j \cap R).$$

Note that from the representation of u^n in terms of the Green's function and from Theorem 2.1, we obtain that u^n is bounded in L_q independent of n. Also note that since ∂R is of class C^1, $\theta_i \cap R$, $i = 1, \cdots, \nu$, can be considered the image of either the unit sphere or the unit hemisphere. Therefore we are able to apply Lemmas 3.1 and 3.2 to the problem (3.6) and conclude that for each $j = 1, \cdots, n$, $\|u_j^n\|_{W_q^1} \leqq K$, independent of n. Thus $\|u^n\|_{W_q^1} \leqq K$, independent of n. By reflexivity there is a subsequence of the u^n which converges weakly in $\mathring{W}_q^1(R)$ (let us call the weak limit u). We claim that u is the $\mathring{W}_q^1$ function corresponding to the functional f.

For any $v \in \mathring{W}_p^1(R)$, we see from (3.5) that

$$\int_R vLu^n\, dx = \int_R \left[-\sum_{i=1}^{N} \frac{\partial U_i^n}{\partial x_i} \right] v\, dx + \int_R U_0^n v\, dx.$$

If we integrate by parts on both sides and then take the limit as n tends to infinity, we obtain

$$\sum_{i=1}^{N} \int_R \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\, dx + \int_R uv\, dx = \sum_{i=1}^{N} \int_R u_i \frac{\partial v}{\partial x_i} + \int_R u_0 v\, dx.$$

By (3.4), we may write

$$f(v) = \sum_{i=1}^{N} \int_R \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx + \int_R uv\, dx \tag{3.7}$$

which defines a homomorphism between $(\mathring{W}_p^1(R))'$ and $\mathring{W}_q^1(R)$.

In fact, (3.7) defines a one-to-one correspondence, for if $\omega \in \mathring{W}_q^1(R)$ is such that

$$\sum_{i=1}^{N} \int_R \frac{\partial \psi}{\partial x_i} \frac{\partial \omega}{\partial x_i} dx + \int_R \psi\omega\, dx = 0, \qquad \psi \in C_0^\infty(R),$$

then integration by parts and Weyl's lemma yield that $\omega \equiv 0$.

We end this section by observing that the same arguments will prove the following theorem.

THEOREM 3.2. *Theorem* 3.1 *is valid with the duality given by the form*

$$\langle u, v\rangle_{W_p^1, W_q^1} = \int_R (\nabla u \cdot \nabla v)\, dx + \int_R auv\, dx,$$

where $a \in C_0^\infty(R)$ *and* $a \geqq 0$.

4. An associated linear problem. Our analysis of the mildly nonlinear problem will entail a linearization process, and consequently, we must first investigate a particular class of linear problems.

To be precise, we consider the following problem.

PROBLEM (D). Let R be a bounded domain with ∂R of class C^1. Given a function $F(x) \in L_1(R)$, find $u \in \mathring{W}_p^1(R)$, $1 \leqq p < N/(N-1)$, such that

$$\int_R (\nabla u \cdot \nabla \phi)\, dx + \int_R au\phi\, dx = \int_R \phi F\, dx \tag{4.1}$$

for all $\phi \in \mathring{W}_q^1$, $q > N$, where $a \in L_{\tilde{q}}(R)$, $\tilde{q} > N/2$, and $a \geqq 0$.

Remark 1. Our use of the duality results of the previous section necessitates the restriction on the boundary of R.

Remark 2. The choice of $a \in L_{\tilde{q}}(R)$, $\tilde{q} > N/2$, assures us that the second term on the left side of (4.1) is defined.

We shall obtain the solution of (D) in two steps; first we consider the special case $a \in C_0^\infty(R)$, which we denote as Problem D*).

THEOREM 4.1. *Suppose* $F(x)$ *is a given* $L_1(R)$ *function. Then there exists one and only one solution of Problem* (D*), *that is,* (D) *under the additional hypothesis that* $a(x) \in C_0^\infty(R)$.

Proof. Let T be a linear functional defined on $\mathring{W}_q^1(R)$ by $T(\phi) = \int_R F(x)\phi(x)\, dx$. As a consequence of Sobolev's inequality, T is bounded since

$$|T(\phi)| \leqq \|F\|_{L_1} \cdot \|\phi\|_{L_\infty} \leqq c\|\phi\|_{W_q^1}.$$

From Theorem 3.2 we conclude that there exists a $u \in \mathring{W}_p^1(R)$, such that

$$T(\phi) = \langle u, \phi\rangle_{W_p^1, W_q^1} = \int_R (\nabla u \cdot \nabla \phi)\, dx + \int_R au\phi\, dx$$

for all $\phi \in \mathring{W}_q^1(R)$. Thus u is a solution of (4.1). If u_1 and u_2 are both solutions of (4.1), then $\omega = u_1 - u_2$ satisfies

$$\int_R (\nabla\omega \cdot \nabla\phi)\,dx + \int_R a\omega\phi\,dx = 0 \quad \text{for all } \phi \in \mathring{W}_q^1(R).$$

Employing once again the duality obtained in Theorem 3.2, we see that $\omega \equiv 0$.

The extension of this theorem to the case $a \in L_{\tilde{q}}(R)$ will require the following preliminary results. The existence of a solution to Problem (D) will follow from an a priori inequality which we prove now.

LEMMA 4.1. *If u is the solution of Problem* (D*), *then u satisfies*

$$\|u\|_{W_p^1(R)} \leqq C(\|a\|_{L_{\tilde{q}}(R)})\|F\|_{L_1(R)}. \tag{4.2}$$

Proof. First, we may write

$$u(x) = \int_R G^L(x, y)F(y)\,dy.$$

Then

$$|u(x)|^p \leqq \int_R G(x, y)|F(y)|\,|u(x)|^{p-1}\,dy$$

since G, the Green's function for Δ, dominates G^L, by the maximum principle [17].

It follows immediately that

$$\|u\|_{L_p(R)} \leqq C\|F\|_{L_1(R)}.$$

Now we may estimate

$$\|u\|_{W_p^1(R)} = \sup_{\phi \in W_q^1(R)} \frac{|\int_R (\nabla u \cdot \nabla\phi)\,dx|}{\|\phi\|_{W_q^1(R)}}.$$

By Hölder's inequality and (4.1) we obtain

$$\|u\|_{W_p^1(R)} \leqq \sup_{\phi \in W_q^1(R)} \frac{|\int_R F\phi\,dx|}{\|\phi\|_{W_q^1(R)}} + \sup_{\phi \in W_q^1(R)} \frac{|\int_R au\phi\,dx|}{\|\phi\|_{W_q^1(R)}},$$

and by Sobolev's inequality,

$$\begin{aligned} \|u\|_{W_p^1(R)} &\leqq C\|F\|_{L_1} + C\|a\|_{L_{\tilde{q}}(R)}\|F\|_{L_1(R)} \\ &\leqq C(\|a\|_{L_{\tilde{q}}(R)})\,\|F\|_{L_1(R)}. \end{aligned}$$

The uniqueness of solutions of Problem (D) will be more difficult to obtain. In fact, we must consider the existence of a solution to a problem which is "dual" to (D*). Theorem 3.2 will tie together existence in the "dual" problem and uniqueness in (D). Moreover, Theorem 3.2 will be of central importance in obtaining the proof of existence.

First we shall consider a problem which is "dual" to (D*) and then extend that result to the case $a \in L_{\tilde{q}}(R)$.

LEMMA 4.2. *For every* $v \in L_q(R)$, $q > N$, *there exists a unique* $\phi \in \mathring{W}_q^1(R)$ $(= (\mathring{W}_p^1(R))')$ *such that*

$$\int_R (\nabla\phi \cdot \nabla\psi)\, dx + \int_R a\phi\psi\, dx = \int_R v\psi\, dx \tag{4.3}$$

for all $\psi \in \mathring{W}_p^1(R)$, $1/p + 1/q = 1$ *and* $a \in C_0^\infty(R)$, $a \geqq 0$.

Proof. For $\psi \in \mathring{W}_p^1(R)$, $T(\psi) = \int_R v(x)\psi(x)\, dx$ defines a bounded linear functional on $\mathring{W}_p^1(R)$ because

$$|T(\psi)| \leqq C\|v\|_{L_q}\|\psi\|_{W_p^1}.$$

With the argument of Theorem 4.1, the lemma follows from Theorem 3.2.

The next result is necessary to extend the previous lemma to the case of $a \in L_{\tilde{q}}(R)$.

LEMMA 4.3. *If* ϕ *is a solution of* (4.3), *then*

$$\|\phi\|_{W_q^1(R)} \leqq C(\|a\|_{L_{\tilde{q}}(R)})\|v\|_{L_q(R)}, \qquad q > N.$$

Proof. As above we may obtain the estimate

$$\|\phi\|_{W_q^1(R)} \leqq C \sup_{\psi \in \mathring{W}_p^1(R)} \left[\frac{|\int_R (\nabla\phi \cdot \nabla\psi)\, dx + \int_R a\phi\psi\, dx|}{\|\psi\|_{W_p^1(R)}} + \frac{|\int_R a\phi\psi\, dx|}{\|\psi\|_{W_p^1(R)}}\right].$$

But ϕ is a solution of (4.3), so

$$\begin{aligned}\|\phi\|_{\mathring{W}_q^1(R)} &\leqq C \sup_{\psi \in \mathring{W}_p^1(R)} \left[\frac{|\int_R v\psi\, dx|}{\|\psi\|_{W_p^1(R)}} + \frac{|\int_R a\phi\psi\, dx|}{\|\psi\|_{W_p^1(R)}}\right]\\ &\leqq C(\|a\|_{L_{\tilde{q}}(R)})(\|v\|_{L_q} + \|\phi\|_{L_\infty}).\end{aligned}$$

However,

$$|\phi(x)| \leqq \int_R G^L(x, y)|v(y)|\, dy \leqq C\|v\|_{L_q(R)},$$

and the lemma is proved.

LEMMA 4.4. *Lemma* 4.2 *is valid for* $a \in L_{\tilde{q}}(R)$.

Proof. Let $\{a_n\}$ be a sequence of nonnegative, $C_0^\infty(R)$ functions which converge in $L_{\tilde{q}}(R)$ to a. For each n there exists, by Lemma 4.2, a solution in $\mathring{W}_q^1(R)$ of

$$\int_R (\nabla\phi^n \cdot \nabla\psi)\, dx + \int_R a_n\phi^n\psi\, dx = \int_R v\psi\, dx \tag{4.4}$$

for all $\psi \in \mathring{W}_p^1(R)$ and any $v \in L_q(R)$; and by the preceding lemma,

$$\|\phi^n\|_{W_q^1(R)} \leqq C(\|a_n\|_{L_{\tilde{q}}(R)})\|v\|_{L_q(R)}.$$

However, $a_n \to a$ in $L_{\tilde{q}}(R)$ and therefore $\|a_n\|_{L_{\tilde{q}}(R)}$, and hence $C(\|a_n\|_{L_{\tilde{q}}(R)})$, are bounded independently of n. Thus, there exists a subsequence, that we again write as $\{\phi^n\}$, which converges weakly to a function $\phi \in \mathring{W}_q^1(R)$. In addition,

$$\left|\int_R (a_n\phi^n\psi - a\phi\psi)\, dx\right| \leqq \left|\int_R (a_n - a)\phi^n\psi\, dx\right| + \left|\int_R a(\phi^n - \phi)\psi\, dx\right|$$

tends to zero as $n \to \infty$. Consequently, allowing n to tend to infinity in (4.4), we obtain

$$\int_R (\nabla\phi \cdot \nabla\psi)\,dx + \int_R a\phi\psi\,dx = \int_R v\psi\,dx$$

for all $\psi \in \mathring{W}_p^1(R)$ and any $v \in L_q(R)$.

Uniqueness follows from the duality.

We are now prepared to consider Problem (D).

THEOREM 4.2. *There is one and only one solution of Problem* (D).

Proof. Choose $\{a_n\}$ as in the previous lemma; i.e., $a_n \in C_0^\infty(R)$, $a_n \geqq 0$, for every n and $a_n \to a \in L_{\tilde{q}}(R)$. For each n, Theorem 4.1 guarantees the existence of a function $u^n \in \mathring{W}_p^1(R)$ such that

$$\int_R (\nabla u^n \cdot \nabla\phi)\,dx + \int_R a_n u^n \phi\,dx = \int_R F\phi\,dx \tag{4.5}$$

for all $\phi \in \mathring{W}_q^1(R)$. By Lemma 4.1,

$$\|u^n\|_{W_p^1(R)} \leqq C\|F\|_{L_1},$$

where (since $a_n \to a$ in $L_{\tilde{q}}(R)$) C is independent of n. By reflexivity, there exists a subsequence of $\{u^n\}$, we write $\{u^n\}$, which converges weakly to some $u \in \mathring{W}_p^1(R)$. We claim that this u is a solution of (D). For, clearly,

$$\left| \int_R (\nabla u^n \cdot \nabla\phi)\,dx + \int_R a_n u^n \phi\,dx - \int_R (\nabla u \cdot \nabla\phi)\,dx - \int_R au\phi\,dx \right|$$
$$\leqq C(\|u^n - u\|_{W_p^1(R)} + \|a_n - a\|_{L_{\tilde{q}}(R)}),$$

C independent of n. Then, since (4.5) holds for each n, we conclude that by taking the limit above as $n \to \infty$, we obtain

$$\int_R (\nabla u \cdot \nabla\phi)\,dx + \int_R au\phi\,dx = \int_R F\phi\,dx$$

for all $\phi \in \mathring{W}_q^1(R)$.

We now shall use our preliminary lemmas to show that the solution just obtained is the only solution of (D). Suppose, to the contrary, that u_1 and u_2 are both solutions of (D) in $\mathring{W}_p^1(R)$, $1 \leqq p < N/(N-1)$. Then $\omega = u_1 - u_2$ is a solution of

$$\int_R (\nabla\omega \cdot \nabla\phi)\,dx + \int_R a\omega\phi\,dx = 0$$

for all $\phi \in \mathring{W}_q^1(R)$, $q > N$. It is sufficient to show that for any $v \in L_q$, there is a $\phi \in \mathring{W}_q^1(R)$, so that

$$\int_R (\nabla\phi \cdot \nabla\psi)\,dx + \int_R a\phi\psi\,dx = \int_R v\psi\,dx \tag{4.6}$$

for all $\psi \in \mathring{W}_p^1(R)$. For, in particular, $\omega \in \mathring{W}_p^1(R)$ and we would have for every $v \in L_q(R)$ that $\int_R \omega v\,dx = 0$, i.e., $\omega \equiv 0$. Therefore, by Lemma 4.4, we conclude that the solution of (D) is unique.

Remark 3. It follows from Lemma 4.1, and we shall use in the next section, that

$$\|u\|_{L_p(R)} \leqq C\|F\|_{L_1}, \tag{4.7}$$

where C is independent of u and a.

5. A mildly nonlinear problem. We are now prepared to investigate the questions of existence and uniqueness of solutions of the following Dirichlet problem.

PROBLEM (M). Suppose $F(x, v)$ is a given function of $N + 1$ variables, such that $F(x, v(x))$ is in $L_1(R)$ whenever $x \in R$ and $v(x) \in L_p(R)$, $1 \leqq p < N/(N - 1)$. Find a function $u \in \mathring{W}_p^1(R)$ that satisfies

$$-\int_R (\nabla u \cdot \nabla \phi)(x)\,dx = \int_R \phi(x)F(x, u(x))\,dx \tag{5.1}$$

for all $\phi \in \mathring{W}_q^1(R)$, $q > N$.

We noted earlier that we shall linearize (5.1) to fit the results of the previous section. Consequently, we must restrict R to be bounded with ∂R of class C^1.

To accomplish the desired linearization, we introduce (see Lees [12]) the function $p(x; v)$ defined on $R \times E^1$ by

$$p(x; v) = \int_0^1 F_u(x, tv)\,dt, \tag{5.2}$$

where F_u denotes the derivative of F with respect to its last variable.

With the function p, we may define for every $v \in L_p(R)$, a linearized equation associated with (5.1); namely,

$$\int_R (\nabla u \cdot \nabla \phi)(x)\,dx + \int_R p(x; v(x))u(x)\phi(x)\,dx = -\int_R F(x, 0)\phi(x)\,dx, \tag{5.3}$$

which holds for all $\phi \in \mathring{W}_q^1(R)$, $q > N$. Note that we obtain (5.1) from (5.3) by setting $v = u$. In addition, if (5.3) defines an operator, $Tv = u$, from $L_p(R)$ into $\mathring{W}_p^1(R)$, existence of a solution of (M) will follow if we are able to show that T has a fixed point. Our plan is to restrict $p(x; v)$ (i.e., $F(x, v)$), so that T will satisfy the Schauder–Tikhonov theorem.

First, Theorem 4.2 applied to (5.3) yields that T is well-defined if $p(x; v) \geqq 0$ and $p(x; v(x)) \in L_{\tilde{q}}(R)$ for $v(x) \in L_p(R)$ and $\tilde{q} > N/2$. If we assume that (a) $F(x, v)$ is increasing in the last variable, then $p \geqq 0$. It may be seen that (b) $F_u(x, \cdot)$ continuous, (c) $\int_R |F_u(x, v(x))|^{\tilde{q}}\,dx < \infty$ for each $v \in L_p(R)$, and the intermediate value theorem imply $p(x; v) \in L_{\tilde{q}}(R)$.

From Theorem 4.2, we have the following lemma.

LEMMA 5.1. *For each* $v \in L_p(R)$, $1 \leqq p < N/(N - 1)$, *there exists a* $u \in \mathring{W}_p^1(R)$ *such that* $Tv = u$ *if* F *satisfies conditions* (a), (b) *and* (c).

Next if (d) $F_u(x, v)$ is continuous when considered as an operator from $L_p(R)$ to $L_{\tilde{q}}(R)$, T is continuous.

LEMMA 5.2. *Suppose $F(x, v)$ satisfies conditions* (a)–(d). *Then T is continuous on any compact subset of $L_p(R)$.*

Proof. Assume K is a compact subset of $L_p(R)$ and $v_0 \in K$. Let $u_0 = Tv_0$. Since F_u is continuous from L_p to $L_{\tilde{q}}$, then F_u is uniformly continuous on K. Thus for a given $\varepsilon > 0$, there is a $\delta > 0$ so that for any $v_1, v_2 \in L_p(R)$ with $\|v_1 - v_2\|_{L_p(R)} < \delta$, we have $\|F_u(x, v_1) - F_u(x, v_2)\|_{L_{\tilde{q}}} < \varepsilon$. Given $\varepsilon > 0$ and $0 \leqq \xi \leqq 1$, we may choose v so that for $\delta > \|v - v_0\|_{L_p(R)} \geqq \|\xi v - \xi v_0\|_{L_p(R)}$, we have $\|F_u(x, \xi v) - F_u(x, \xi v_0)\|_{L_{\tilde{q}}(R)} < \varepsilon$, and consequently $\|p(x; v) - p(x; v_0)\|_{L_{\tilde{q}}(R)} < \varepsilon$. From this last estimate, we shall show that $\|u - u_0\|_{W_p^1(R)} < \varepsilon$ also holds. If $\omega = u - u_0$, then from (5.3),

$$\int_R (\nabla\omega \cdot \nabla\phi)(x)\,dx + \int_R p(x; v_0)\omega(x)\phi(x)\,dx = \int_R f(x)\phi(x)\,dx,$$

where $f(x) = (p(x; v) - p(x; v_0))u(x)$. Then from Theorem 4.2 we may obtain the estimate

$$\|\omega\|_{W_p^1(R)} \leqq C\|f\|_{L_1(R)} \leqq C\|p(\cdot, v) - p(\cdot, v_0)\|_{L_{\tilde{q}}(R)},$$

$\tilde{q} > N/2$. Therefore, for $v \in L_p(R)$ such that $\|v - v_0\|_{L_p(R)} < \delta$,

$$\|Tv_0 - Tv\|_{W_p^1(R)} = \|u_0 - u\|_{W_p^1(R)} < \varepsilon;$$

i.e., T is continuous.

We have now assembled the necessary tools to prove the following theorem.

THEOREM 5.1. *Assume that R is a bounded domain in E^N with ∂R of class C^1. Suppose that for some p, $1 \leqq p < N/(N-1)$, the function $F(x, v)$ satisfies the following conditions*:

(i) *for $v(x) \in L_p(R)$, $F(x, v(x)) \in L_1(R)$*;
(ii) *$F(x, v)$ is increasing in the last variable*;
(iii) *$F_u(x, v)$ is continuous with respect to the last variable*;
(iv) *for $v(x) \in L_p(R)$, $F_u(x, v(x))$ is a continuous operator from $L_p(R)$ to $L_{\tilde{q}}(R)$, $\tilde{q} > N/2$.*

Then there is one and only one solution for Problem (M).

Proof. Let T be the operator defined by (5.3), Lemmas 5.1 and 5.2 show that T is continuous from a compact set in $L_p(R)$ to $\mathring{W}_p^1(R)$, $1 \leqq p < N/(N-1)$.

To use the fixed-point theorem, we shall need to show that T is a compact mapping. Consider the injection mapping of $\mathring{W}_p^1(R)$ into $L_p(R)$ denoted by J. Then by Rellich's theorem, J is compact. If we consider the composition of T followed by J, the resulting mapping $J \circ T$ is a continuous compact operator from a compact set in $L_p(R)$ to $L_p(R)$. We shall again call the composite map T since $(J \circ T)v = u = Tv$.

Finally, we claim that T, in fact, maps a bounded subset of $L_p(R)$ into itself. For if $Tv = u$, then u is a solution of (5.3), and hence, by Theorem 4.2 (equation (4.7)), u satisfies the a priori estimate

$$\|u\|_{L_p(R)} \leqq C\|F(x, 0)\|_{L_1(R)},$$

where C is independent of u and $p(x; v)$. In other words, if

$$\Sigma = \{v \in L_p(R) | \, \|v\|_{L_p(R)} < k, \text{ where } k \leqq C\|F(x, 0)\|_{L_1(R)}\},$$

then $T(\Sigma) \subset \Sigma$.

As a consequence of the Schauder-Tikhonov theorem, we obtain the existence of a fixed point of T. Evaluating $Tu = u$ by (5.3), we see that the fixed point is a solution of (M).

If the solution is not unique, say u_1 and u_2 are both solutions of (M), then by (5.3), for all $\phi \in \mathring{W}_q^1(R)$,

$$-\int_R [\nabla(u_1 - u_2)\cdot\nabla\phi](x)\,dx = \int_R [F(x, u_1(x)) - F(x, u_2(x))]\phi(x)\,dx.$$

We may rewrite this equation to obtain

$$\int_R [\nabla(u_1 - u_2)\cdot\nabla\phi](x)\,dx + \int_R F_u(x, \omega(x))(u_1(x) - u_2(x))\phi(x)\,dx = 0 \tag{5.4}$$

for all $\phi \in \mathring{W}_q^1(R)$ and where $\omega(x) = \theta u_1(x) + (1 - \theta)u_2(x)$, $0 < \theta < 1$. By hypotheses (ii) and (iv), it may be seen that $F_u(x, \omega) \geqq 0$ and $F_u(x, \omega) \in L_{\tilde{q}}(R)$, and so we may apply Theorem 4.2 to (5.4) to conclude that the only possible solution of (5.4) is $u_1 = u_2$.

To illustrate the last theorem, we present the following example: Define the function F by

$$F(x, t) = c(x)|t|^p \operatorname{sgn}(t) + d(x)t + e(x),$$

where $1 \leqq p < N/(N - 1)$, $c(x) \in L_\infty(R)$, $d(x) \in L_{\tilde{q}}(R)$, $e(x) \in L_1(R)$ and $e(x)$, $d(x)$ are nonnegative. We have constructed F so that conditions (i) and (ii) are satisfied, and since

$$F_u(x, t) = pc(x)|t|^{p-1} + d(x),$$

(iii) and (iv) are also satisfied. We may conclude, therefore, that there exists one and only one solution $u \in \mathring{W}_p^1(R)$ of

$$-\int_R (\nabla u\cdot\nabla\phi)(x)\,dx = \int_R [c(x)|u(x)|^p \operatorname{sgn}(u(x)) + d(x)u(x) + e(x)]\phi(x)\,dx$$

for all $\phi \in \mathring{W}_q^1(R)$.

We see by this example, that the function F on the right side of (5.1) may have a polynomial growth of order p. We may compare the above example with a more general result of Ladyzhenskaya and Ural'tseva [11]. If we consider their result applied to Problem (M), we see that Ladyzhenskaya and Ural'tseva impose on F the condition

$$|F(x, u)| \leqq (1 + |u|^\alpha)\phi(x),$$

where $\phi \in L_s$, $s > N/p$ and $0 \leqq \alpha < (Np/(N - p))(1 - 1/s) - 1$. Thus to obtain ϕ in the least restrictive space, we must have $0 \leqq \alpha < p - 1$; that is, F satisfies a polynomial growth condition of order $p - 1$.

Acknowledgment. The author gratefully acknowledges the assistance and encouragement of Professor R. B. Kellogg and Professor J. H. Bramble in the preparation of this paper.

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SIAM J. MATH. ANAL.
Vol. 2, No. 4, November 1971

A RECURRENCE CONCERNING RAYLEIGH FUNCTIONS*

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Abstract. L. Carlitz has suggested the problem of evaluating $a(n, k) = \sum_{r=1}^{n-1} r^k \sigma_r(v)\sigma_{n-r}(v)$, where $\sigma_r(v)$ are the Rayleigh functions. The special cases $k = 0, 1, 2, 3$ were given by N. Kishore.

Both N. Kishore and L. Carlitz gave recurrence relations for $a(n, 2k + 1, v)$ which involve $a(n, l, v)$, $l = 0, 1, 2, \cdots, 2k$. For $a(n, 2k, v)$ their formulas lead to nothing new, and therefore are not sufficient to evaluate $a(n, k, v)$. In this paper we give a recurrence relation for $b_n(z, v) = \sum_{r=1}^{n-1} \sigma_r(v)\sigma_{n-r}(v)e^{rz}$, which leads immediately to a recurrence relation for $a(n, k, v)$. The recurrence relation is valid for all k and n.

1. Introduction. The Rayleigh function $\sigma_n(v)$, defined by

$$\sigma_n(v) = \sum_{m=1}^{\infty} (j_{v,m})^{-2n}, \qquad n = 1, 2, \cdots, \tag{1.1}$$

where $j_{v,m}$ is the mth positive zero of $z^{-v}J_v(z)$, has been investigated by N. Kishore and L. Carlitz. N. Kishore [2], [3] obtained recurrences satisfied by $\sigma_n(v)$, and in particular, expressions for

$$a(n, k) = a(n, k, v) = \sum_{s=1}^{n-1} s^k \sigma_s(v)\sigma_{n-s}(v), \tag{1.2}$$

for the cases $k = 0, 1, 2, 3$. L. Carlitz [1] suggested the problem of evaluating $a(n, k)$ for all k and proved the following formula [1, (1.10)]:

$$k! \sum_{s=1}^{n-1} s^k \sigma_s(v)\sigma_{n-s}(v) = \sum_{j=0}^{k+1} \sum_{2s \leqq j} \alpha_{k,j,s}(v)(n - s)^{k-j+1}\sigma_{n-s}(v), \tag{1.3}$$

where the coefficients $\alpha_{k,j,s}(v)$ are the coefficients of the polynomials $A_{k,j}(x)$, which satisfy the formula [1, (4.4)]

$$k!u(x)(xD)^k u(x) = \sum_{j=0}^{k+1} A_{k,j}(x)(xD)^{k-j+1}u(x),$$

but are not determined explicitly. Here $D \equiv d/dx$, and

$$u(x) = u(x, v) = \sum_{n=1}^{\infty} \sigma_n(v)x^n. \tag{1.4}$$

We develop a recurrence relation for the functions

$$b_n(z) = b_n(z, v) = \sum_{s=1}^{n-1} \sigma_s(v)\sigma_{n-s}(v)e^{sz}, \qquad n = 2, 3, \cdots, \tag{1.5}$$

from which we immediately get a recurrence relation for $a(n, k)$ via

$$a(n, k) = \frac{d^k}{dz^k} b_n(z)|_{z=0}, \qquad n = 2, 3, \cdots, \quad k = 0, 1, 2, \cdots. \tag{1.6}$$

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* Received by the editors October 12, 1970, and in revised form February 22, 1971.

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2. The recurrence relation. Define $\sigma_0(\nu) = 0$, and omit all ν-scripts, with the understanding that these sums are functions of ν. We get

$$u(x) = \sum_{n=0}^{\infty} \sigma_n x^n, \qquad \sigma_0 = 0, \tag{2.1}$$

$$b_n(z) = \sum_{s=0}^{n} \sigma_s \sigma_{n-s} e^{sz}, \qquad b_0(z) = b_1(z) = 0, \tag{2.2}$$

$$a(n, k) = \sum_{s=0}^{n} s^k \sigma_s \sigma_{n-s}, \qquad a(0, k) = a(1, k) = 0. \tag{2.3}$$

From (2.1), we get

$$u(x)u(xe^z) = \sum_{n=0}^{\infty} \sigma_n x^n \sum_{l=0}^{\infty} \sigma_l x^l e^{lz} = \sum_{n=0}^{\infty} x^n \sum_{l=0}^{n} \sigma_l \sigma_{n-l} e^{lz} = \sum_{n=0}^{\infty} b_n(z) x^n. \tag{2.4}$$

On the other hand we have,[1]

$$u(x) = \tfrac{1}{2} x^{1/2} \frac{J_{\nu+1}(x^{1/2})}{J_\nu(x^{1/2})}. \tag{2.5}$$

Insert (2.5) in (2.4) to get

$$\tfrac{1}{4} x e^{z/2} J_{\nu+1}(x^{1/2}) J_{\nu+1}(x^{1/2} e^{z/2}) = J_\nu(x^{1/2}) J_\nu(x^{1/2} e^{z/2}) \sum_{n=0}^{\infty} b_n(z) x^n. \tag{2.6}$$

The expansion of $J_\nu(t)$ yields

$$J_\mu(x^{1/2}) J_\mu(x^{1/2} e^{z/2}) = (\tfrac{1}{4})^\mu x^\mu e^{z\mu/2} \sum_{r=0}^{\infty} C_r(z, \mu) x^r, \tag{2.7}$$

where

$$C_r(z, \mu) = \left(-\frac{1}{4}\right)^r \frac{1}{r!} \sum_{l=0}^{r} \binom{r}{l} \frac{e^{lz}}{\Gamma(\mu + 1 + l)\Gamma(\mu + 1 + r - l)}. \tag{2.8}$$

Substitute (2.7) into (2.6), and equate coefficients of x^n, to get

$$\tfrac{1}{16} e^z C_r(z, \nu + 1) = \sum_{k=2}^{r+2} b_k(z) C_{r+2-k}(z, \nu), \qquad r = 0, 1, 2, \cdots. \tag{2.9}$$

From (2.8), we have

$$C_0(z, \mu) = 1/\Gamma^2(\mu + 1), \tag{2.10}$$

which we substitute into (2.9) to get

$$b_{n+2}(z) = \Gamma^2(\nu + 1)\left\{\frac{1}{16} e^z C_n(z, \nu + 1) - \sum_{k=2}^{n+1} b_k(z) C_{n+2-k}(z, \nu)\right\}, \tag{2.11}$$

$n = 1, 2, \cdots$, which is the desired recurrence formula.

It is obvious that a recurrence formula for $a(n, k)$ could also be derived from (2.11), by means of (1.6), but we derive a simpler formula below. As special cases,

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[1] N. Kishore [2], [3].

we get

$$b_2(z) = \frac{1}{4^2(\nu+1)^2} e^z,$$

$$\text{(2.12)} \quad b_3(z) = \frac{1}{4^3(\nu+1)^3(\nu+2)}[e^z + e^{2z}],$$

$$b_4(z) = \frac{1}{4^4(\nu+1)^4(\nu+2)^2(\nu+3)}[2(\nu+2)(e^z + e^{3z}) + (\nu+3)e^{2z}].$$

Formula (1.6), together with (2.12), yields

$$a(2,k) = \frac{1}{4^2(\nu+1)^2} \qquad (=\sigma_1^2),$$

$$\text{(2.13)} \qquad a(3,k) = \frac{1+2^k}{4^3(\nu+1)^3(\nu+2)} \qquad (=(1+2^k)\sigma_1\sigma_2),$$

$$a(4,k) = \frac{2(\nu+2)(1+3^k)+(\nu+3)2^k}{4^4(\nu+1)^4(\nu+2)^2(\nu+3)}, \quad k = 0,1,2,\cdots.$$

These results agree with the results given by N. Kishore, where they overlap.

3. A simpler formula. Equation (2.11) gives a recurrence formula for $b_n(z)$, and therefore for $a(n,k)$, assuming we do not know the values of σ_n, $n = 1, 2, \cdots$. If we assume these values to be known, we can derive a much simpler recurrence formula for $b_n(z)$ as follows: equation (2.6) is rewritten as

$$\text{(3.1)} \qquad \tfrac{1}{2}x^{1/2}J_{\nu+1}(x^{1/2})u(xe^z) = J_\nu(x^{1/2})\sum_{n=0}^{\infty} b_n(z)x^n,$$

and we substitute the expansions of $J_\nu(x^{1/2})$, $J_{\nu+1}(x^{1/2})$ and $u(xe^z)$ in (3.1), which yields

$$\text{(3.2)} \qquad \frac{1}{4}x\sum_{n=0}^{\infty} x^n \sum_{r=0}^{n} \frac{(-\frac{1}{4})^r\sigma_{n-r}e^{(n-r)z}}{\Gamma(\nu+r+2)r!} = \sum_{n=0}^{\infty} x^n \sum_{r=0}^{n} \frac{(-\frac{1}{4})^r b_{n-r}(z)}{r!\Gamma(\nu+r+1)}.$$

Now equate coefficients of x^{n+1} to get

$$\text{(3.3)} \qquad \frac{1}{4}\sum_{r=0}^{n} \frac{(-\frac{1}{4})^r\sigma_{n-r}e^{(n-r)z}}{r!\Gamma(\nu+r+2)} = \sum_{r=0}^{n+1} \frac{(-\frac{1}{4})^r b_{n+1-r}(z)}{r!\Gamma(\nu+r+1)}.$$

Or, since $b_0(z) = b_1(z) = \sigma_0 = 0$,

$$\text{(3.4)} \qquad \begin{aligned} b_{n+1}(z) = \Gamma(\nu+1)\Bigg\{&\sum_{r=1}^{n-1} \frac{(-\frac{1}{4})^r}{r!\Gamma(\nu+r+2)}\left[\frac{1}{4}\sigma_{n-r}e^{(n-r)z} - (\nu+r+1)b_{n+1-r}(z)\right] \\ &+ \frac{1}{4}\frac{\sigma_n e^{nz}}{\Gamma(\nu+2)}\Bigg\}, \qquad n = 1,2,3,\cdots. \end{aligned}$$

Insert $\Gamma(\nu + 1)$ inside the brackets, and use (1.6) to obtain the recurrence

$$a(n+1,k) = \frac{1}{\nu+1}\left\{\sum_{r=1}^{n-1} \frac{(-\frac{1}{4})^r}{r!(\nu+2)_r}\left[\frac{1}{4}(n-r)^k\sigma_{n-r} - (\nu+r+1)a(n+1-r,k)\right] + \frac{1}{4}n^k\sigma_n\right\}, \quad n = 1,2,3,\cdots. \tag{3.5}$$

It should be noted that this recurrence is on n, and holds for all k, $k = 0, 1, 2, \cdots$. Both N. Kishore [3] and L. Carlitz [1] found recurrences on *odd* k only, for all n, which contain also all the unknowns $a(n, l)$, for l even, $l \leqq k$. Their recurrences fail to give anything new, when k is even, owing to the fact that

$$a(n,k) = \sum_{s=0}^{n} s^k\sigma_s\sigma_{n-s} = \sum_{s=0}^{n}(n-s)^k\sigma_s\sigma_{n-s} = \sum_{l=0}^{k}(-1)^{k-l}\binom{k}{l}n^l a(n,k-l),$$

and the coefficient of $a(n, k)$ on the right-hand side of the preceding equation is $(-1)^k$. Thus, $a(n, k)$ cancels on both sides, when k is even, and their recurrence formulas are incomplete.

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SIAM J. MATH. ANAL.
Vol. 2, No. 4, November 1971

A VARIATION OF HADAMARD'S FINITE PART INTEGRALS*

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Abstract. A variation of Hadamard's finite part integrals is described which can be used to solve a class of problems in a direct manner, eliminating the "ascent-descent" technique which Hadamard used. It is shown that the Green's formula derived from the standard singular solution for the scalar wave equation yields the Kirchhoff formulas, if, when the space dimension is odd, one retains the logarithmically infinite part of the principal value integral.

In the course of his researches concerning partial differential equations, Hadamard [1] devised the concept of "the finite part" of certain divergent integrals. The purpose was, apparently, to allow one to apply the "fundamental formula," as he called it, to certain rather natural regions for which the resulting integrals do not necessarily converge.

In the case of the wave equation, for example, if u is a solution of the inhomogeneous equation

$$\Box u = f \tag{1}$$

and v is a solution of the homogeneous equation

$$\Box v = 0, \tag{2}$$

then the fundamental formula takes the form [1', p. 63]

$$\int_D fv\,dV + \int_{\partial D}\left(v\frac{du}{d\nu} - u\frac{dv}{d\nu}\right)dS = 0, \tag{F}$$

where D is any (sufficiently nice) region, ∂D is its boundary, dV and dS represent volume and surface area elements, respectively, and $d/d\nu$ is the directional derivative (the "transversal" derivative) associated with the direction of the normal to the surface.

Of course for (F) to make sense in the usual way the integrals appearing there must converge, which hardly ever happens in the cases treated by Hadamard, the reason, of course, being due to the fact that for v one naturally chooses an "elementary solution" of (2)—which means that v necessarily has some sort of singularity—and at the same time wishes to choose for D a region for which v is singular along some portion of its boundary ∂D.

Hadamard [1] showed how such problems could be treated, in fairly general circumstances, by splitting the divergent integrals appearing in (F) into two parts, the "finite part" and the "infinite part." Since the sum of all the parts on the left-hand side of (F) vanishes, it must be that the sum of the finite parts and the sum of the infinite parts vanish separately (because of the systematic way in which the integrals were split). Thus, if each of the integrals in (F) is replaced by its finite part, then—in the circumstances described by Hadamard—the resulting equation will be valid.

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* Received by the editors December 22, 1970, and in revised form February 26, 1971.

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Of course the whole point of this procedure of Hadamard's was not just that the resulting equation (F) be true, but rather that it be of such a form as to yield the solution u of (1) in terms of the forcing term f and given initial data. In other words, Hadamard showed how the solution u of a Cauchy problem for (1) could be obtained directly from the fundamental formula (F) with v an elementary solution of (2) and D a suitably chosen region. (For a more modern treatment of such problems by means of generalized functions see Gel'fand and Shilov [2].)

There is, however, the curious fact that in order to treat problems in spaces of even dimension n, Hadamard resorted to the device of "ascent-descent." That is, increase the dimension of the space from n to $n + 1$ by adding a dummy $(n + 1)$th coordinate, treat the new problem in this space of odd dimension, employing finite part integrals, and then descend to the original n-dimensional space by integrating out the extra coordinate. A consequence of this procedure, it should be noted, is that the elementary solutions for the even-dimensional spaces are never put to use.

However, if one believes that the fundamental formula (F) is exactly what its name implies, then it is not at all difficult to see how one can modify Hadamard's method of finite part to yield the solution to Cauchy's problem in spaces of even dimension. Namely, if (F) somehow contains within it the solution of the given problem, and yet the equation obtained from it by retaining only the finite part does not, then the solution must have been thrown away in the process. In the case of four dimensions, for example, it is the *infinite* part which ought to be kept.

To see how this comes about, and incidentally to indicate how one can determine in any given case which "part" of (F) should be kept, let us consider the classical Cauchy problem in four dimensions of finding the solution $u(P, t)$ of (1) subject to the initial conditions (for simplicity)

$$u(P, 0) = 0 = \frac{\partial u}{\partial t}(P, 0), \tag{3}$$

where P denotes an arbitrary point of ordinary 3-dimensional Euclidean space. Of course the solution to this problem is well known [1, p. 239] being

$$\begin{aligned} u(P, t) = -\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\int_0^{t}\frac{1}{R}f(&R\cos\theta\cos\varphi, R\cos\theta\sin\varphi, \\ &R\sin\theta, t - R)\sin\theta\, dR\, d\theta\, d\varphi. \end{aligned} \tag{4}$$

We shall solve this problem in two ways: once by Hadamard's ascent-descent procedure combined with finite part integrals, and once without the ascent-descent but keeping the infinite part of (F).

Hadamard's procedure begins by rewriting the problem as

$$\left(\nabla^2 + \frac{\partial^2}{\partial w^2} - \frac{\partial^2}{\partial t^2}\right)\bar{u}(P, w, t) = \bar{f}(P, w, t), \tag{1'}$$

$$\bar{u}(P, w, 0) = 0 = \frac{\partial \bar{u}}{\partial t}(P, w, 0), \tag{3'}$$

where

$$\bar{f}(P, w, t) \equiv f(P, t) \quad \text{for all } w.$$

This problem is one in 5-dimensional space-time, and the corresponding elementary solution $\bar{v}$ of the homogeneous equation (2′), say, is

$$\bar{v} \equiv [(t - T)^2 - \bar{R}^2]^{-3/2},$$

where

$$\bar{R}^2 = R^2 + (w - W)^2,$$

$$R^2 = (x - X)^2 + (y - Y)^2 + (z - Z)^2.$$

Hence, the fundamental formula (F′) holds with $\bar{u}$, $\bar{v}$ in place of u, v, and with D any (nice enough) region not containing any singularities of $\bar{v}$ either inside or on the boundary. In particular, D may be any region lying inside (but not meeting) the cone $T = t - \bar{R}$.

As usual, however, it is precisely the region bounded by this cone (together with the plane $T = 0$) that we wish to use for D, the only thing preventing us being the fact that the resulting integrals would diverge. Therefore, we do the next best thing, employing for D the family of all neighboring regions instead. Or, more precisely, *a* family D_ε which converges to D as $\varepsilon \to 0+$. It really does not matter *which* family is chosen. We can suit our own convenience.

For instance, we could use the family of surfaces $T = t - \sqrt{R^2 + \varepsilon}$ (together with the surface $T = 0$, of course), or even $T = t - \bar{R} - \varepsilon$. (This has one singular point—the vertex—but is otherwise a nice enough surface.) Either way one finds that the volume integral

$$\int_{D_\varepsilon} \bar{f}\bar{v}\, dV = V_{-1}\varepsilon^{-1/2} + V_0 + o(1) \quad \text{as } \varepsilon \to 0,$$

where V_{-1}, V_0 are independent of ε, and also the surface integral

$$\int_{\partial D_\varepsilon} \left(\bar{v}\frac{d\bar{u}}{dv} - \bar{u}\frac{d\bar{v}}{dv}\right) dS = S_{-1}\varepsilon^{-1/2} + S_0 + o(1) \quad \text{as } \varepsilon \to 0.$$

(V_0 and S_0 are, of course, the "finite parts" of the volume and surface integrals, respectively.) It follows from what has been said above that

$$V_{-1} + S_{-1} = 0,$$

$$V_0 + S_0 = 0, \quad \text{etc.}$$

However, the most useful of these equations turns out to be (in this case) the second one, since S_0 is just $4\pi^2$ times the value of $\bar{u}$ at the vertex of the cone $T = t - \bar{R}$, i.e.,

$$S_0 = 4\pi^2\bar{u}(P, w, t) = 4\pi^2 u(P, t)$$

(since $\bar{u}$ is necessarily independent of w, because f is) and

$$V_0 = \pi \int_{0 \leq R \leq t} \frac{1}{R} f(Q, t - R)\, dQ,$$

which together give (4), if a change to polar coordinates is made in the integral.

On the other hand, staying in the original 4-dimensional space-time, where the elementary solution is

$$v = [(t - T)^2 - R^2]^{-1},$$

if D in (F) is replaced by D_ε, where D_ε is the region bounded by $T = t - R - \varepsilon$ (and the plane $T = 0$), then the volume integral becomes

$$\int_{D_\varepsilon} fv\, dV = V_{-1} \ln \varepsilon + V_0 + o(1) \quad \text{as } \varepsilon \to 0$$

and

$$\int_{\partial D_\varepsilon} \left(v \frac{du}{dv} - u \frac{dv}{dv} \right) dS = S_{-1} \ln \varepsilon + S_0 + o(1) \quad \text{as } \varepsilon \to 0.$$

But this time it is S_{-1} which is just a multiple of u evaluated at the vertex of $T = t - R$, namely,

$$S_{-1} = 2\pi u(P, t)$$

and

$$V_{-1} = \frac{1}{2} \int_{0 \leqq R \leqq t} \frac{1}{R} f(Q, t - R)\, dQ,$$

which together give (4) again, since $V_{-1} + S_{-1} = 0$.

The reason either of these two procedures works is, of course, due to the intimate relation between the elementary solution v and the boundary of D, which is the surface (characteristic surface of the wave operator) $T = t - R$. In the expansion of the surface integral (an asymptotic expansion as a function of ε, the "size" of the deleted neighborhood of the singularity) it will always be the case that one of the coefficients is just a multiple of the value of u at the vertex. When the dimension n is odd, this is the "finite part," but when the dimension is even, it is an "infinite part." In either case though, it is the equation resulting from keeping this term which solves the Cauchy problem.

For example, in the case of the n-dimensional wave equation (1) when n is even and greater than two (and with zero initial conditions), one finds that (again taking D_ε to be the region bounded by the cone $T = t - R - \varepsilon$ together with the plane $T = 0$)

$$\int_{\partial D_\varepsilon} \left(v \frac{du}{dv} - u \frac{dv}{dv} \right) dS = S_{1-n/2}\varepsilon^{2-n/2} + S_{-n/2}\varepsilon^{1-n/2} + \cdots + S_{-2}\varepsilon^{-1} + S_{-1} \ln \varepsilon$$
$$+ S_0 + o(1) \quad \text{as } \varepsilon \to 0+,$$

where only the coefficient

$$S_{-1} = (-)^{n/2}(n - 2)2^{2-n} \binom{n-3}{n/2 - 2} \Omega_{n-2} u(P, t)$$

is independent of values of u away from the vertex (P, t) of the cone. (Ω_{n-1} is the surface area of the unit sphere in n-dimensional space.) Thus, the equation which solves the problem is that obtained by equating to zero the coefficient of $\ln \varepsilon$ in equation (F), namely,

$$V_{-1} + S_{-1} = 0.$$

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SIAM J. MATH. ANAL.
Vol. 2, No. 4, November 1971

EXISTENCE THEOREM AND CONVERGENCE OF MINIMIZING SEQUENCES IN EXTREMUM PROBLEMS*

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Abstract. In the paper, a necessary and sufficient condition is given for the strong convergence of minimizing sequences for a differentiable convex functional in a reflexive Banach space.

In the following, let E denote a real reflexive Banach space and f a real functional on E. We say that f is *convex* on E if for every $x, y \in E$ and $\alpha \in (0, 1)$ the inequality

$$f(\alpha x + (1 - \alpha)y) \leqq \alpha f(x) + (1 - \alpha)f(y)$$

holds, and f is strictly convex if for $x \neq y$ the sharp inequality holds. A sequence (x_n), $x_n \in E$ $(n = 1, 2, \cdots)$, is said to be a *minimizing sequence* for f if $f(x_n) \to \inf_{x \in E} f(x)$ $(n \to \infty)$. If f is differentiable at a point $x \in E$, we shall denote by $Df(x, h)$ $(h \in E)$ and $D^2f(x, h, k)$ $(h, k \in E)$ its first and second order linear Gateaux differential at x, respectively. If E^* denotes the dual of E, let $(x, y) = x(y)$, where $x \in E^*$, $y \in E$. We define the *gradient* of a differentiable functional, $F = \operatorname{grad} f$, by the equation $Df(x, h) = (F(x), h)$. We shall use the following representation of a differentiable functional:

$$f(x) = f(x_0) + \int_0^1 (F(x_0 + t(x - x_0)), x - x_0)\, dt. \tag{1}$$

Here, $(F(x_0 + t(x - x_0)), x - x_0)$ is usually supposed to be continuous in t, $t \in [0, 1]$, for $x, x_0 \in E$. (For the definition and basic properties of differentials of functionals see, e.g., the book of M. M. Vainberg [6].)

One can see without difficulty that in the finite-dimensional space the "growth in each direction" of a real convex functional f, i.e., the condition

$$\lim_{\beta \to +\infty} f(\beta x) = +\infty \tag{2}$$

for arbitrary $x \in E$, $x \neq 0$ and β real, already guarantees the existence of an absolute minimum of f in E, and if, further, f is strictly convex, then each minimizing sequence converges in the norm of E. Convexity implies namely continuity of f (see, e.g., [3]) and from (2) follows $\lim_{|x| \to \infty} f(x) = +\infty$.

In an infinite-dimensional space the situation is completely different: a convex functional which satisfies (2) need not even be bounded from below, as the following example shows.

Example 1. Let $E = l_2$ and for $x = \{x_1, x_2, \cdots\} \in l_2$ define

$$f(x) = \sum_{n=1}^{\infty} \frac{x_n^2 - 2n^2 x_n}{n^3}.$$

Then $|f(x)| < +\infty$, f is strictly convex and continuous on l_2 and for arbitrary $x \in l_2$, (2) is satisfied, yet for $x = \{\underbrace{0, \cdots, 0, n^2}_{n}, 0, \cdots\}$ we have $f(x) = -n$.

* Received by the editors August 17, 1970, and in revised form February 8, 1971.

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The following example shows that in an infinite-dimensional space a convex functional which satisfies (2) does not have to assume its minimum even if it is bounded from below.

Example 2. Let $E = l_2$, $x \in l_2$ and

$$f(x) = \sum_{n=1}^{\infty} \frac{(x_n - 1)^2}{n^2}.$$

Then $f(x) \geqq 0$, f is continuous and strictly convex on l_2 and for $x^k = \{\underbrace{1, \cdots, 1}_{k}, 0, \cdots\}$ $(k = 1, 2, \cdots)$ we have $f(x^k) = \sum_{n=k+1}^{\infty} 1/n^2 \to 0$ $(k \to +\infty)$. Evidently, (x^k) is a minimizing sequence for f but f does not attain its minimum in l_2.

Consequently, if we want to study the question of existence of a minimum of a convex functional f and the behavior of minimizing sequences in infinite-dimensional spaces, we have to work with such assumptions which give us more information about the global behavior of f.

One such assumption requires the existence of the second order Gateaux differential which satisfies, e.g., the inequality

$$D^2 f(x, h, h) \geqq \gamma(|h|)|h| \tag{3}$$

or

$$D^2 f(x, h, h) \geqq \gamma(|x|)|h| \tag{3'}$$

for each $x \in E$, where γ is a continuous nonnegative function defined on $[0, +\infty)$ and satisfies some appropriate growth condition (see, e.g., [4], [5], [6], [7]). Such conditions are quite strong: if we assume, for instance, that $\gamma(t) = ct\,(c > 0)$ and the second differential of f satisfies the following continuity condition: $D^2 f(x_0 + a_1 h_1 + a_2 h_2, h_1, h_2) \to D^2 f(x_0, h_1, h_2)$ as $a_1 \to 0$, $a_2 \to 0$ (a_1, a_2 real) for arbitrary $h_1, h_2 \in E$ and some $x_0 \in E$, then the space E already has a "Hilbert" structure (compare also [5]).

On the other hand, very often we know only that the functional f has only the first order Gateaux differential. In this case, as the proof of Theorem 5 in [4] suggests, it is possible to replace the condition (3) by a slightly weaker one:

$$Df(x, x - y) - Df(y, x - y) = (F(x) - F(y), x - y) \geqq \gamma(|x - y|) \cdot |x - y|,$$

where γ again satisfies an appropriate growth requirement. (See also [1].)

The existence of the minimum of f can be established under still weaker conditions and the question of strong convergence of each minimizing sequence can definitely be answered, as our next result shows.

THEOREM 1. *Let f be a differentiable functional on E such that its gradient F is a monotone operator*, *i.e.*,

$$(F(x) - F(y), x - y) \geqq 0 \tag{4}$$

holds for arbitrary $x, y \in E$. Let $(F(x + ty), y)$ be continuous in $t \in [0, 1]$ for any fixed $x, y \in E$. Let λ be a real-valued function defined on $[0, +\infty)$ such that

$$(F(x), x) \geqq \lambda(|x|). \tag{5}$$

If, for a certain R_0,

$$\int_0^{R_0} \frac{|\lambda(t)|}{t}\, dt < +\infty$$

(*in the sense of Lebesgue*) *and*

$$\int_0^{R_0} \frac{\lambda(t)}{t}\, dt > 0,$$

then there exists $x_0 \in E$ ($|x_0| < R_0$) *such that* $f(x_0) = \inf_{x\in E} f(x) > -\infty$. *If* F *is strictly monotone* (*i.e., if the sharp inequality in the definition of monotonicity* (4) *holds for* $x \neq y$), *then* f *attains its minimum at only one point.*

THEOREM 2. *Let* f *be a differentiable functional on* E *such that its gradient* $F: E \to E^*$ *is a monotone operator and let* $(F(x + ty), y)$ *be continuous in* $t \in [0, 1]$ *for arbitrary fixed* $x, y \in E$. *If there is a point* $x_0 \in E$ *for which* $f(x_0) = \inf_{x\in E} f(x) > -\infty$ (*e.g., if the assumptions of the preceding theorem are satisfied*), *then each minimizing sequence converges strongly to* x_0 *if and only if there exists a monotone* (*nondecreasing*) *function* γ *defined on* $[0, +\infty)$, *positive on* $(0, +\infty)$, *such that*

$$(F(y), y - x_0) \geqq \gamma(|y - x_0|) \tag{6}$$

holds for any $y \in E$.

Proof of Theorem 1. Let $x_1, x_2 \in E$. From (1), (4) it follows that

$$\begin{aligned} f(x_1) - f(x_2) &= \int_0^1 (F(x_2 + t(x_1 - x_2)) - F(x_2), x_1 - x_2)\, dt + (F(x_2), x_1 - x_2) \\ &\geqq (F(x_2), x_1 - x_2); \end{aligned}$$

hence f is weakly lower semicontinuous on E. Using a similar argument we can easily see that monotonicity of F guarantees convexity of f (see, e.g., [2]). Further, using (1), (5) and the integrability of $\lambda(t)/t$ over $(0, R_0)$ we obtain the following inequality for arbitrary x; $|x| \leqq R_0$:

$$\begin{aligned} f(x) &= f(0) + \int_0^1 (F(tx), tx)\frac{dt}{t} > f(0) + \int_0^1 \frac{\lambda(t|x|)}{t|x|}|x|\, dt \\ &= f(0) + \int_0^{|x|} \frac{\lambda(t)}{t}\, dt. \end{aligned}$$

On the set $\{x : |x| = R_0\}$ we have clearly

$$f(x) > f(0) + c, \qquad c > 0. \tag{7}$$

We shall prove next that $f(x) > f(0)$ for $x \in E$, $|x| \geqq R_0$. If $f(x) \leqq f(0)$ were to hold for some $x \in E$, $|x| > R_0$, then by convexity of f we would have for any $\alpha \in (0, 1)$, $f(\alpha x) = f(\alpha x + (1 - \alpha)0) \leqq \alpha f(x) + (1 - \alpha)f(0) \leqq f(0)$ which contradicts (7) if $\alpha = R_0/|x| < 1$. Hence, $f(x) > f(0)$ for any $|x| \geqq R_0$ and consequently the functional attains its absolute minimum in the sphere $\{x : |x| < R_0\}$. We also see that the strict monotonicity of F implies the existence of only one point at which the minimum is attained. (Notice that $(F(x_0), h) = 0$ for any $h \in E$ if $f(x_0) = \min_{x\in E} f(x)$.)

Proof of Theorem 2. *Sufficiency.* Let (x_n) be a minimizing sequence. We have then $f(x_n) \to f(x_0) = \min_{x \in E} f(x)$. By (4) and (6), we have

$$\begin{aligned}\int_0^1 \gamma(t|x_n - x_0|)\, dt &\leqq \int_0^1 (F(x_0 + t(x_n - x_0)), t(x_n - x_0))\, dt \\ &\leqq \int_0^1 (F(x_0 + t(x_n - x_0)), x_n - x_0)\, dt \\ &= f(x_n) - f(x_0).\end{aligned}$$

Hence,

$$(*) \qquad \int_0^1 \gamma(t|x_n - x_0|)\, dt \to 0, \quad n \to +\infty,$$

since (x_n) is a minimizing sequence. From the above inequality we also see that (x_n) is a bounded sequence in E. (Notice that $\gamma(t) > 0$ on $(0, +\infty)$.) Suppose now that the sequence (x_n) does not converge strongly to x_0. Then we can find δ, M such that $0 < \delta \leqq M < +\infty$ and a subsequence (x_{n_j}) such that $\delta \leqq |x_{n_j} - x_0| \leqq M$ $(j = 1, 2, \cdots)$. We then have

$$\int_0^1 \gamma(t|x_{n_j} - x_0|)\, dt = \frac{1}{|x_{n_j} - x_0|} \int_0^{|x_{n_j} - x_0|} \gamma(t)\, dt \geqq \frac{1}{M} \int_0^\delta \gamma(t)\, dt > 0$$

for $j = 1, 2, \cdots$. But this is impossible in view of $(*)$; hence the result follows.

Necessity. Suppose that each minimizing sequence converges strongly to x_0. Define $\gamma(r) = \inf\{(F(y), y - x_0) : |y - x_0| = r\}$. Clearly, $\gamma(r) \geqq 0$ since by monotonicity of F we have the inequality

$$(F(y), y - x_0) \geqq (F(x_0), y - x_0) = 0.$$

We shall prove first that

$$(**) \qquad \gamma(r) > 0 \quad \text{for } r > 0.$$

Suppose that $(**)$ is not true. Then we can find $r_0 > 0$ such that $\gamma(r_0) = 0$ and thus a sequence (y_n) such that $|y_n - x_0| = r_0$ and

$$(8) \qquad 0 \leqq (F(y_n), y_n - x_0) \to 0, \quad n \to +\infty.$$

Next we shall prove that (y_n) is a minimizing sequence.

Let g_n be defined on $[0, 1]$ by

$$(9) \qquad g_n(t) = (F(x_0 + t(y_n - x_0)), y_n - x_0).$$

We clearly have $g_n(t) \geqq 0$ for $t \in [0, 1]$. Using the monotonicity of F we obtain for arbitrary t, $0 < t < 1$, the following inequality:

$$(F(x_0 + t(y_n - x_0)), (1 - t)(y_n - x_0)) \leqq (F(x_0 + (y_n - x_0)), (1 - t)(y_n - x_0)),$$

whence for $0 < t < 1$,

$$g_n(t) \leqq (F(y_n), y_n - x_0) := \gamma_n.$$

We then have

$$f(y_n) - f(x_0) = \int_0^1 g_n(t)\,dt \leqq \gamma_n \to 0, \quad n \to +\infty.$$

Hence, (y_n) is a minimizing sequence. By hypothesis, $|y_n - x_0| \to 0$ $(n \to +\infty)$. However, this is impossible, because $|y_n - x_0| = r_0$ $(n = 1, 2, \cdots)$. We have proved that $\gamma(r) > 0$ for $r > 0$.

Now we show that γ is a nondecreasing function. Let $y \in E$, $y \neq x_0$ and let t, τ be real numbers such that $0 < \tau < t$. Then, by monotonicity of F we obtain the following estimate:

$$\begin{aligned}
&(F(x_0 + t(y - x_0)), t(y - x_0)) - (F(x_0 + \tau(y - x_0)), \tau(y - x_0)) \\
&\quad = (F(x_0 + t(y - x_0)), (t - \tau)(y - x_0)) - (F(x_0 + \tau(y - x_0)), (t - \tau)(y - x_0)) \\
&\qquad + (F(x_0 + t(y - x_0)), \tau(y - x_0)) - 2(F(x_0 + \tau(y - x_0)), \tau(y - x_0)) \\
&\qquad + (F(x_0 + \tau(y - x_0)), t(y - x_0)) \\
&\quad \geqq (F(x_0 + t(y - x_0)) - F(x_0 + \tau(y - x_0)), \tau(y - x_0)) \\
&\qquad + (F(x_0 + \tau(y - x_0)), (t - \tau)(y - x_0)) := I_1 + I_2.
\end{aligned}$$

Now,

$$I_1 = \frac{\tau}{t - \tau}(F(x_0 + t(y - x_0)) - F(x_0 + \tau(y - x_0)), (t - \tau)(y - x_0)) \geqq 0$$

since $0 < \tau < t$. Likewise,

$$I_2 = \frac{t - \tau}{\tau}(F(x_0 + \tau(y - x_0)), \tau(y - x_0)) \geqq 0.$$

Further, let $r > \rho > 0$ and $\tau = \rho/r < 1$, $t = 1$. Then, using the inequality just proved, we finally obtain the result:

$$\begin{aligned}
\gamma(r) &= \inf_{|y - x_0| = r} (F(x_0 + (y - x_0)), y - x_0) \\
&\geqq \inf_{|y - x_0| = r} (F(x_0 + \tau(y - x_0)), \tau(y - x_0)) \\
&= \inf_{|z - x_0| = \tau r = \rho} (F(z), z - x_0) = \gamma(\rho),
\end{aligned}$$

and the theorem is proved.

Remark 1. Notice, that under the assumptions of Theorem 1, each minimizing sequence converges weakly to x_0. (See, e.g., [4].)

Remark 2. Notice that if each minimizing sequence of a convex functional f which attains its minimum at only one point x_0, converges strongly to x_0, then already $\lim_{|x| \to +\infty} f(x) = +\infty$.

Proof. Suppose the contrary: then for a certain sequence (x_n), $x_n \in E$, $|x_n| \to +\infty$ $(n \to +\infty)$, we must have $f(x_n) \to K > f(x_0)$, $K < +\infty$. Now consider

$$L = \inf\{K : K = \lim_{n \to +\infty} f(x_n), |x_n| \to +\infty\}.$$

Then again $L > f(x_0)$. Obviously it is possible to find a sequence (x_n), $|x_n| \to +\infty$, and such that $f(x_n) \to L$. Let $y_n = (x_0 + x_n)/2$. Using convexity of f we get $f(y_n) < \frac{1}{2}f(x_0) + \frac{1}{2}f(x_n)$. Take such a small $\varepsilon_0 > 0$ that $f(x_0) < L - \varepsilon_0$. Take further any ε, $0 < \varepsilon < \varepsilon_0$. Then there exists an integer n_0 such that for $n \geqq n_0$, we obtain

$$f(y_n) < \tfrac{1}{2}(L - \varepsilon_0) + \tfrac{1}{2}(L + \varepsilon) = L - (\varepsilon_0 - \varepsilon)/2.$$

Clearly, $|y_n| \to +\infty$ but a subsequence of $(f(y_n))$ must tend to some $K < L$, which contradicts the definition of L.

Remark 3. On the other hand, if we know that $\lim_{|x| \to +\infty} f(x) = +\infty$, where f is a convex functional which attains its minimum at only one point x_0, then, of course, not each minimizing sequence needs to tend strongly to x_0. For instance, if we define a functional f in l_2 as follows:

$$f(x) = \begin{cases} \displaystyle\sum_{n=1}^{\infty} x_n^2/n^2 & \text{for } |x| \leqq 1, \\ \displaystyle\sum_{n=1}^{\infty} x_n^2/n^2 + (|x| - 1)^2 & \text{for all other } x \end{cases}$$

($x = \{x_1, x_2, \cdots\}$), then clearly $f(x) \to +\infty$ as $|x| \to +\infty$ and the sequence $x^k = \{\underbrace{0, \cdots, 0, 1}_{k}, 0, \cdots\}$ is minimizing.

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SIAM J. MATH. ANAL.
Vol. 2, No. 4, November 1971

BOUNDED SOLUTIONS FOR A SECOND ORDER NONLINEAR EQUATION*

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Abstract. The boundedness of all solutions for the second order nonlinear equation

$$[p(t)u']' + \sum_{k=1}^{N} a_k(t) f_k(u) = e(t)$$

is studied. It is assumed that the product $p(t)a_k(t)$ is, for each k, a locally integrable perturbation of a continuous function, is locally of bounded variation and has a small negative variation. By appealing to a Stieltjes version of Gronwall's inequality, bounds are obtained for the energy integral associated with a particular solution, and consequently for the solution.

1. Introduction. In this paper we shall obtain sufficient conditions for the boundedness of all solutions satisfying the differential equation

$$[p(t)u']' + a(t)u = 0, \tag{1}$$

$$[p(t)u']' + a(t)\,\mathrm{sgn}\,(u)|u|^{\gamma} = e(t), \tag{2}$$

where $\gamma > 0$, or

$$[p(t)u']' + \sum_{k=1}^{N} a_k(t) f_k(u) = e(t). \tag{3}$$

We shall assume throughout that the coefficients $p(t)$ and $a_k(t)$, $k = 1, 2, \cdots, n$, are positive, continuous, real-valued and locally of bounded variation on the interval $[0, \infty)$, or some locally integrable perturbation thereof. We shall require $e(t)$ to be locally Lebesgue integrable on $[0, \infty)$. A solution $u(t)$ is interpreted as an absolutely continuous pair $u(t)$ and $p(t)u'(t)$ which satisfies the differential equation almost everywhere.

W. Leighton [8] has considered equation (1), where $p(t)$, $a(t)$ and $[p(t)a(t)]'$ are continuous, and has shown that if $[pa]' \geqq 0$ then every solution is bounded. This paper contains several generalizations of W. Leighton's result for solutions of equations (2) and (3).

2. Boundedness results for equation (2). Consider equation (2), where $p(t)$, $a(t)$ and $e(t)$ satisfy the following conditions:

(i) $a(t) = \{q(t) + \psi(t)\}$;

(ii) $p(t)$ and $q(t)$ are positive, continuous and locally of bounded variation on $[0, \infty)$;

(iii) $\psi(t)$ and $e(t)$ are locally Lebesgue integrable on $[0, \infty)$.

For the special case when $\gamma = 1$ we are assured of the existence and uniqueness of a fundamental system of solutions on the interval $[0, \infty)$ (see P. Hartman [5, p. 322]). We shall implicitly assume the existence of solutions for the more general equation (2).

* Received by the editors July 9, 1970, and in revised form March 1, 1971.

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Let $u(t)$ by any solution of (2) and define $E^2(t)$ as

$$E^2(t) = \frac{[p(t)u'(t)]^2}{2p(t)q(t)} + \frac{|u(t)|^{\gamma+1}}{\gamma+1}.$$

The function $p(t)q(t)E^2(t)$ represents a modified energy integral for the solution $u(t)$.

THEOREM 1. *Let* $0 < \gamma \leqq 1$. *If the inequalities*

$$\int_0^\infty (p(s)q(s))^{-1}\, d(pq)(s) < \infty, \tag{4}$$

$$\int_0^\infty |\psi(s)|(p(s)q(s))^{-1/2}\, ds < \infty, \quad \textit{and} \tag{5}$$

$$\int_0^\infty |e(s)|(p(s)q(s))^{-1/2}\, ds < \infty \tag{6}$$

hold, then $E^2(t)$ *is of bounded variation on* $[0, \infty)$. *In particular,* $E^2(t)$ *is bounded and* $\lim_{t\to\infty} E^2(t)$ *exists.*

Before giving a proof of Theorem 1, we make some necessary observations and remarks. Theorem 1 is of particular interest if the product pq becomes unbounded. In this case we are able to assume something less than integrability over $[0, \infty)$ for the functions $\psi(t)$ and $e(t)$. Theorem 1 extends a Lemma of H. E. Gollwitzer [3] in which he considers the special case where $p(t) \equiv 1, \gamma = 1$, and $e(t) \equiv 0$.

Throughout this paper we will let $p(t) = p_+(t) - p_-(t)$ represent the Jordan decomposition of a function $p(t)$, locally of bounded variation, where $p_+(t)$ and $p_-(t)$ are the positive and negative variations of $p(t)$, respectively.

From condition (4) we have that

$$\log p(t)q(t) = \log p(0)q(0) + \int_0^t (pq)^{-1}\, d(pq)_+(s) - \int_0^t (pq)^{-1}\, d(pq)_-(s)$$

is bounded away from $-\infty$, and hence $p(t)q(t)$ is bounded away from zero. Since $u(t)$ and $p(t)u'(t)$ are absolutely continuous functions and $p(t)q(t)$ is bounded away from zero, $E^2(t)$ is continuous and locally of bounded variation.

In the statement of Theorem 1, inequality (4) cannot be replaced by the statement "$p(s)q(s)$ is bounded below by a positive number." It is well established that the boundedness of solutions for (1) depends upon an appropriate boundedness condition for the negative variation of the coefficient $a(t)$ (see, for example, R. Bellman [1, pp. 111–113]).

The following Stieltjes version of Gronwall's inequality, which is a special case of a result given by W. Schmaedeke and G. Sell [9], will be used in the proof of Theorems 1 and 3.

LEMMA 1. *Let* g *be a continuous nondecreasing function of bounded variation on* $[0, \infty)$, *let* f *be a nonnegative continuous function locally of bounded variation on* $[0, \infty)$, *and let* $\varepsilon \geqq 0$. *If*

$$f(t) \leqq \varepsilon + \int_0^t f(s)\, dg(s), \qquad 0 \leqq t < \infty,$$

then there exists a positive constant K, *depending on* g *but not on* f, *such that*

$$f(t) \leqq K\varepsilon, \qquad 0 \leqq t < \infty.$$

Proof of Theorem 1. To simplify notation, let $u^\gamma = \operatorname{sgn}(u)|u|^\gamma$. Multiplying (2) by $u'(t)/q(t)$ and integrating by parts we obtain

$$\begin{aligned} E^2(t) = E^2(0) &- \frac{1}{2}\int_0^t [pu']^2(pq)^{-2}\,d(pq)(s) - \int_0^t \psi(pq)^{-1}pu'u^\gamma\,ds \\ &+ \int_0^t e(pq)^{-1}pu'\,ds. \end{aligned} \tag{7}$$

Let $\mathscr{E}(t) = E^2(t) + 1$. Since $(pq)_+$ and $(pq)_-$ are positive, continuous and nondecreasing, and since

$$0 \leqq [pu']^2/(2pq) \leqq E^2(s) + 1,$$

$$|pu'|(2pq)^{-1/2} \leqq E(s) < E^2(s) + 1 \text{ and}$$

$$|pu'u^\gamma|(2pq)^{-1/2} \leqq E^2(s) + 1$$

it follows that

$$\begin{aligned} \mathscr{E}(t) \leqq \mathscr{E}(0) &+ \int_0^t \mathscr{E}(pq)^{-1}\,d(pq)_-(s) + \int_0^t \mathscr{E}|\psi|(pq)^{-1/2}\,ds \\ &+ \sqrt{2}\int_0^t \mathscr{E}|e|(pq)^{-1/2}\,ds. \end{aligned} \tag{8}$$

We can express (8) as a Volterra integral inequality

$$\mathscr{E}(t) \leqq \mathscr{E}(0) + \int_0^t \mathscr{E}(s)\,dQ(s),$$

where

$$\begin{aligned} Q(t) = \int_0^t (pq)^{-1}\,d(pq)_-(s) &+ \int_0^t |\psi|(pq)^{-1/2}\,ds \\ &+ \sqrt{2}\int_0^t |e|(pq)^{-1/2}\,ds. \end{aligned}$$

Lemma 1 and the hypotheses (4), (5) and (6) imply that $\mathscr{E}(t)$, and hence $E^2(t)$, is bounded on $[0, \infty)$. Since $E^2(t)$ is bounded, the integrals in (7) are convergent. From this it follows that $E^2(t)$ is of bounded variation on $[0, \infty)$.

The following question immediately presents itself: Can Theorem 1 be established without the restriction $\gamma \leqq 1$? S. P. Hastings [6, Appendix] and C. V. Coffman and D. F. Ullrich [2] have given examples to show that Theorem 1 cannot be extended to include the case $\gamma > 1$. In particular, they establish the existence of positive continuous functions $a(t)$ satisfying $a(t) \to 0$ as $t \to \infty$ such that at least one solution of $u'' + (1 + a(t))u^3 = 0$ has finite escape time.

Consider now (2), where $p(t)$, $a(t)$ and $e(t)$ satisfy the following conditions:

(xi) $a(t) = \{q(t) + \psi(t)\}$;

(xii) $p(t)$, $q(t)$ and $\psi(t)$ are positive absolutely continuous functions on $[0, \infty)$, and

(xiii) $e(t)$ is locally Lebesgue integrable on $[0, \infty)$.

We now give a result which is valid for all positive γ.

THEOREM 2. *If the inequalities*

$$\int_0^\infty (p(s)q(s))^{-1}(pq)'_-(s)\, ds < \infty, \tag{9}$$

where $(pq)'_- = \max\{0, -(pq)'\}$,

$$\int_0^\infty |(\psi(s)/q(s))'|\, ds < \infty, \tag{10}$$

$$\limsup_{t\to\infty} |\psi(t)/q(t)| < 1, \quad \textit{and} \tag{11}$$

$$\int_0^\infty |e(s)|(p(s)q(s))^{-1/2}\, ds < \infty \tag{12}$$

hold, then $E^2(t)$ is of bounded variation on $[0, \infty)$. In particular, $E^2(t)$ is bounded and $\lim_{t\to\infty} E^2(t)$ exists.

This theorem is, in many ways, similar to results by R. Bellman [1, p. 112] and J. S. W. Wong [12].

Proof of Theorem 2. Let us again use the notation $u^\gamma = \operatorname{sgn}(u)|u|^\gamma$. As was observed in Theorem 1,

$$\begin{aligned} E^2(t) = E^2(0) &- \frac{1}{2}\int_0^t [pu'](pq)^{-2}(pq)'\, ds - \int_0^t \psi(pq)^{-1}pu'u^\gamma\, ds \\ &+ \int_0^t e(pq)^{-1}pu'\, ds. \end{aligned} \tag{13}$$

Integrating the second integral by parts, we obtain

$$\begin{aligned} E^2(t) \leqq K_1 + (1 - \varepsilon^{-1})\frac{|u|^{\gamma+1}}{\gamma+1} &- \frac{1}{2}\int_0^t [pu'](pq)^{-2}(pq)'\, ds \\ &+ \int_0^t \left(\frac{\psi}{q}\right)'\left[\frac{|u|^{\gamma+1}}{\gamma+1}\right] ds + \int_0^t e(pq)^{-1}pu'\, ds \end{aligned}$$

for some ε^{-1}, $0 < \varepsilon^{-1} < 1$, some constant K_1, and sufficiently large values of t. Hence

$$\begin{aligned} \varepsilon^{-1}E^2(t) \leqq K_2 + \int_0^t E^2(pq)^{-1}(pq)'_-\, ds &+ \int_0^t E^2\left|\left(\frac{\psi}{q}\right)'\right| ds \\ &+ \sqrt{2}\int_0^t E|e|(pq)^{-1/2}\, ds. \end{aligned}$$

Letting $\mathscr{E}(t) = E^2(t) + 1$, we have

$$\mathscr{E}(t) \leqq \varepsilon K_2 + \int_0^t \mathscr{E}(pq)^{-1}(pq)'_- \, ds + \varepsilon \int_0^t \mathscr{E} \left| \left(\frac{\psi}{q} \right)' \right| ds \tag{14}$$
$$+ \varepsilon\sqrt{2} \int_0^t \mathscr{E}|e|(pq)^{-1/2} \, ds.$$

We can express (14) as a Volterra integral inequality

$$\mathscr{E}(t) \leqq \varepsilon K_2 + \int_0^t \mathscr{E}(s) \, dQ(s),$$

where

$$Q(s) = \varepsilon \int_0^t (pq)^{-1}(pq)'_- \, ds + \varepsilon \int_0^t \left| \left(\frac{\psi}{q} \right)' \right| ds + \varepsilon\sqrt{2} \int_0^t |e|(pq)^{-1/2} \, ds.$$

Gronwall's inequality [4, p. 36] and the hypotheses (9), (10) and (12) together imply that $\mathscr{E}(t)$, and hence $E^2(t)$, is bounded on $[0, \infty)$. Since $E^2(t)$ is bounded, the integrals in (13) are convergent. From this it follows that $E^2(t)$ is of bounded variation on $[0, \infty)$.

3. Boundedness results for equation (3). Consider equation (3), where $p(t)$, $a_k(t)$, $f_k(u)$ and $e(t)$, $k = 1, 2, 3, \cdots, N$, satisfy the following conditions:

(xxi) $a_k(t)$, $k = 1, 2, \cdots, N$, and $p(t)$ are positive continuous functions, locally of bounded variation on the interval $[0, \infty)$;

(xxii) $e(t)$ is locally Lebesgue integrable; and

(xxiii) the functions $f_k(u)$, $k = 1, 2, \cdots, N$, are continuous on $(-\infty, \infty)$ and satisfy $uf_k(u) > 0$ for $u \neq 0$. The indefinite integral of the function $f_k(u)$ will be denoted by $F_k(u)$, that is, $F_k(u) = \int_0^u f_k(x) \, dx$.

Equation (3) and the corresponding conditions which we have imposed on the coefficients represents a generalization of the differential equation used to describe the angular displacement of a pendulum, namely,

$$\ddot{\varphi} + (g/1) \sin \varphi = 0, \qquad -\pi/2 < \varphi < \pi/2.$$

C. T. Taam [10] has established the existence and uniqueness of solutions to (3) for the particular case when $f_k(u) = u^{2k-1}$ and $e(t) \equiv 0$. We shall implicitly assume the existence of solutions for the more general situation.

Let $b(t)$ be a positive continuous function which is locally of bounded variation on $[0, \infty)$. We shall call $b(t)$ a *bounding function for the coefficients* $a_1, a_2, \cdots, a_M$, $M \leqq N$, if the conditions

$$\int_0^\infty b(s)a_k^{-1}(s) \, d(a_k/b)_+(s) < \infty, \qquad k \leqq M, \tag{15}$$

$$\liminf_{t \to \infty} \left(\sum_{k=1}^M a_k(t) \right) \Big/ b(t) > 0 \tag{16}$$

are satisfied.

Let $u(t)$ be a given solution of (3). Let $E^2(t)$ be defined by

$$E^2(t) = \frac{[p(t)u'(t)]^2}{2p(t)b(t)} + \sum_{k=1}^{N} \frac{p(t)a_k(t)F_k(u)}{p(t)b(t)}.$$

THEOREM 3. *Let $b(t)$ be a bounding function for the coefficients $a_1, a_2, \cdots, a_M$, $M \leqq N$. If the inequalities*

$$\int_0^\infty (p(s)b(s))^{-1}\, d(pb)_-(s) < \infty, \tag{17}$$

$$\int_0^\infty (p(s)a_k(s))^{-1}\, d(pa_k)_+(s) < \infty \quad \textit{for } M < k, \tag{18}$$

$$F_k(u) \to \infty \quad \textit{as } |u| \to \infty \quad \textit{for } 1 \leqq k \leqq M, \quad \textit{and} \tag{19}$$

$$\int_0^\infty |e(s)|(p(s)b(s))^{-1/2}\, ds < \infty \tag{20}$$

hold, then $E^2(t)$ is of bounded variation on $[0, \infty)$ and every solution of (3) is bounded over the interval $[0, \infty)$.

Proof of Theorem 3. Condition (17) implies that the product $p(t)b(t)$ is bounded away from zero. Using this fact, we have that $E^2(t)$ is positive, continuous and locally of bounded variation. We also have that $a_k(t)/b(t)$ is, for each k, locally of bounded variation.

Multiply (3) by $u'(t)/b(t)$ and integrate by parts to obtain

$$\begin{aligned} E^2(t) &= E^2(0) - \frac{1}{2}\int_0^t [pu']^2(pb)^{-2}\, d(pb)(s) \\ &\quad + \sum_{k=1}^{N} \int_0^t F_k(u)\, d(a_k/b)(s) + \int_0^t e(pb)^{-1}pu'\, ds \\ &= E^2(0) - \frac{1}{2}\int_0^t [pu']^2(pb)^{-2}\, d(pb)(s) \\ &\quad + \sum_{k=1}^{M} \int_0^t F_k(u)\, d(a_k/b)(s) + \sum_{k=M+1}^{N} \int_0^t F_k(u)(pb)^{-1}\, d(pa_k) \\ &\quad - \sum_{k=M+1}^{N} \int_0^t F_k(u)(pa_k)(pb)^{-2}\, d(pb)(s) + \int_0^t e(pb)^{-1}pu'\, ds. \end{aligned} \tag{21}$$

Letting $\mathscr{E}(t) = E^2(t) + 1$, we obtain the inequality

$$\begin{aligned} \mathscr{E}(t) \leqq \mathscr{E}(0) &+ (1 + N - M)\int_0^t \mathscr{E}(pb)^{-1}\, d(pb)_-(s) \\ &+ \sum_{k=1}^{M} \int_0^t \mathscr{E}(b/a_k)\, d(a_k/b)_+(s) \\ &+ \sum_{k=M+1}^{N} \int_0^t \mathscr{E}(pa_k)^{-1}\, d(pa_k)_+(s) \\ &+ \int_0^t \mathscr{E}|e|(pb)^{-1/2}\, ds. \end{aligned} \tag{22}$$

We can express (22) as a Volterra integral inequality

$$\mathscr{E}(t) \leqq \mathscr{E}(0) + \int_0^t \mathscr{E}(s)\, dQ(s),$$

where

$$Q(t) = (1 + N - M)\int_0^t (pb)^{-1}\, d(pb)_-(s) + \sum_{k=1}^{M} \int_0^t (b/a_k)\, d(a_k/b)_+(s)$$
$$+ \sum_{k=M+1}^{N} \int_0^t (pa_k)^{-1}\, d(pa_k)_+(s) + \int_0^t |e|\,(pb)^{-1/2}\, ds.$$

Lemma 1 and the hypotheses (15), (17), (18) and (20) imply that $\mathscr{E}(t)$ and hence $E^2(t)$ is bounded on $[0, \infty)$ and that the integrals in (21) are convergent. From this, it follows that $E^2(t)$ is of bounded variation on $[0, \infty)$.

Making use of condition (16), we have that

$$\min_{1 \leqq k \leqq M} F_k(u) \leqq \delta E^2(t)$$

for some positive number δ. Since $\min_{1 \leqq k \leqq M} F_k(u)$ is bounded on $[0, \infty)$ and each $F_k(u)$, $1 \leqq k \leqq M$, satisfies (19), it follows that $u(t)$ must also be bounded on this interval.

4. Applicability of Theorem 3. In the statement of Theorem 3, we assumed the existence of a bounding function $b(t)$ for the coefficients $a_1, a_2, \cdots, a_M$. A practical question immediately arises: Given an equation of the form (3) is it possible to find a bounding function $b(t)$ and, if so, what conditions must it necessarily satisfy?

If a bounding function for the coefficients exists, condition (17) implies that the product $p(t)b(t)$ is bounded away from zero. Condition (18) implies that the functions $p(t)a_k(t)$, $k > M$, are bounded above. By a similar argument, we obtain from (15) that $a_k(t)/b(t) \leqq \beta_k$, $1 \leqq k \leqq M$, for some positive β_k. From (16), we have that $\sum_{k=1}^{M} a_k(t) \geqq \beta_0 b(t)$ for some positive β_0 and for sufficiently large values of t. A combination of the last two estimates shows that $b(t)$ necessarily satisfies the inequalities

$$\beta_0 b(t) \leqq \sum_{k=1}^{M} a_k(t) \leqq \left(\sum_{k=1}^{M} \beta_k\right) b(t) \tag{23}$$

for large values of t. The fact that condition (23) is not a sufficient condition for $b(t)$ to be a bounding function will be established.

We make note of the fact that if the functions $b(t)$, $p(t)$ and $a_k(t)$ are positive, absolutely continuous and satisfy

$$(pa_k)^{-1}(pa_k)' \leqq (pb)^{-1}(pb)' \quad \text{a.e. for } 1 \leqq k \leqq M,$$

then it will follow that the function $a_k(t)/b(t)$ is nonincreasing and inequality (15) is valid. In particular, if the function

$$c(t) = \exp\left\{\int_0^t \max_{1 \leqq i \leqq M} \left(\frac{a_i'(\tau)}{a_i(\tau)}\right)_+ d\tau\right\}$$

satisfies

$$\liminf_{t\to\infty}\left(\sum_{k=1}^{M} a_k(t)\right)/c(t) > 0,$$

then $c(t)$ is a bounding function for the coefficients $a_1, a_2, \cdots, a_M$.

Theorem 3 extends a result by C. T. Taam [11] in which he has considered (3), where $e(t) \equiv 0$ and the coefficients $p(t)$ and $a_k(t)$ are positive, absolutely continuous on $[a, \infty)$ and satisfy

$$(pa_K)^{-1}(pa_K)'_- \quad \text{is integrable over } [a, \infty) \text{ for some } K,$$

and

$$(pa_k)^{-1}(pa_k)'_+ \quad \text{is integrable over } [a, \infty) \text{ for } k \neq K.$$

Theorem 3 can be applied, for example, to the equation $u'' + tu + t^2u^3 = 0$ by choosing $b(t) = t^2$, whereas the cited result of C. T. Taam cannot be applied.

C. T. Taam [10] has also considered (3), where $e(t) \equiv 0$ and where $p^{-1}(t)$ and $a_k(t)$, $1 \leqq k \leqq N$, are nonnegative, nondecreasing, Lebesgue-measurable functions on $[T, \infty)$. He has shown that these conditions are sufficient to imply that all solutions of (3) oscillate and that the amplitudes are monotonically nonincreasing. We see, however, that Theorem 3 of this paper can be applied to equation $u'' + (1 + t + \sin t)u + t^2u^3 = 0$, whereas neither of the results by C. T. Taam can be applied.

We now give an example to show that C. T. Taam's result [10, Theorem 2] is independent of Theorem 3 and that condition (23) is not a sufficient condition for a function $b(t)$ to be a bounding function for the coefficients. We note that it is sufficient to construct two nonincreasing functions $a_1(t), a_2(t) \in C'[0, \infty)$ satisfying $\frac{1}{4}t \leqq a_1(t) + a_2(t) \leqq t$, $t \geqq 4$, and for which no bounding function $b(t)$ exists. Let the functions $a_1(t)$ and $a_2(t)$ be defined by:

$$a_1(t) = \sum_{n=0}^{\infty} \chi[2n, 2n+2)(t)\{(n+1) + S(t-2n)\},$$

$$a_2(t) = \sum_{n=0}^{\infty} \chi[2n, 2n+2)(t)\{(n+1) + S(t-2n-1)\},$$

where $\chi[\mathrm{I}]$ is the characteristic function for the interval I and

$$S(t) = \begin{cases} 0, & t \leqq 0, \\ \frac{1}{2}[1 - \cos \pi t], & 0 \leqq t \leqq 1, \\ 1, & 1 \leqq t. \end{cases}$$

Define a function $c(t)$ by

$$c(t) = \exp\left\{\int_0^t \max_i \frac{a_i'(\tau)}{a_i(\tau)}\, d\tau\right\}.$$

We see that if there exists a bounding function $b(t)$ for the coefficients $a_1(t)$ and $a_2(t)$, then $b(t)$ satisfies

$$\int_0^\infty b(s)c^{-1}(s)\,d(c/b)_+(s) = \int_0^\infty b(s)a_1^{-1}(s)\,d(a_1/b)_+(s) + \int_0^\infty b(s)a_2^{-1}(s)\,d(a_2/b)_+(s) < \infty.$$

Hence $c(t)/b(t)$ is bounded.

We now evaluate $c(t)$ at the even integers:

$$\begin{aligned} c(2m) &= \exp\left\{\sum_{n=0}^{m-1}\int_{2n}^{2n+1}(a_1'/a_1)\,d\tau + \sum_{n=0}^{m-1}\int_{2n+1}^{2n+2}(a_2'/a_2)\,d\tau\right\} \\ &= \prod_{n=0}^{m-1}\frac{a_1(2n+1)}{a_1(2n)}\prod_{n=0}^{m-1}\frac{a_2(2n+2)}{a_2(2n+1)} \\ &= (m+1)^2. \end{aligned}$$

We observe that $a_1(t) + a_2(t) \leqq 3 + t$ and that

$$\frac{a_1(t) + a_2(t)}{b(t)} \leqq \frac{3+t}{Kc(t)} \tag{24}$$

for some K and sufficiently large values of t. For $t = 2m$, the right-hand side of (24) approaches zero as $m \to \infty$. This cannot happen, however, if $b(t)$ is to satisfy inequality (23). Therefore no such bounding function $b(t)$ exists.

Acknowledgments. The author wishes to thank the referee for his valuable criticisms and for his suggestions on improving the presentation of this paper.

Theorem 3 and the example of § 4 represent extensions of results contained in the author's dissertation, which was written at the University of California at Riverside under the guidance of Professor Frederic T. Metcalf.

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SIAM J. MATH. ANAL.
Vol. 2, No. 4, November 1971

POINTWISE BOUNDS ON DERIVATIVES OF SOLUTIONS TO ORDINARY DIFFERENTIAL EQUATIONS*

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Abstract. Various pointwise estimates on the derivatives of solutions to nth order linear differential equations in terms of an associated Taylor polynomial are derived. These estimates are used to obtain a necessary and sufficient condition for a function which satisfies a sequence of linear differential equations on an interval to be regular on that inverval.

In this paper, pointwise estimates on the derivatives of a solution to the nth order linear differential equation

$$L(y(x)) \equiv y^{(n)}(x) - \sum_{j=0}^{n-1} \sigma_j(x)y^{(j)}(x) - \phi(x) = 0, \tag{1}$$

in terms of an associated Taylor polynomial, are obtained. We shall always assume that (1) holds on some compact interval on which all of the coefficients are Lebesgue integrable. Thus we are concerned with finding bounds for the error function and its derivatives obtained on approximating the solution to an nth order initial value problem by a polynomial satisfying the same initial values. Theorem 1 provides a sharper estimate than that obtained by Hornich [3] and is free of a certain norm restriction needed in his paper.

In Theorem 2 we use a comparison-type of estimate to a solution of (1) in terms of the solution of a linear differential equation with constant coefficients to obtain a necessary and sufficient condition for a function which satisfies a sequence of linear differential equations on an interval to be regular on that interval.

Theorem 2 is applied in Theorem 3 to obtain a condition for a C^∞-function of two variables x, y to be a polynomial in $z = x + iy$.

Finally, different methods than those of Theorem 1 are used to obtain sharper estimates in the case that certain of the coefficients in (1) are zero.

We introduce the following notation: Let $\Phi(x)$ be a real-valued function on $[0, h]$. Then $\hat{\Phi}(x)$ will denote the extended real-valued function $\sup_{0 \leqq s \leqq x} |\Phi(s)|$ on $[0, h]$.

THEOREM 1. *Let $f \in C^n[0, h]$, and let*

$$P_{n-1}(x) = \sum_{j=0}^{n-1} \frac{f^{(j)}(0)x^j}{j!}.$$

Assume that f satisfies (1) on $[0, h]$. Then for $j = 0, 1, \cdots, n-1$ and for $0 \leqq x \leqq h$,

$$|f^{(j)}(x) - P_{n-1}^{(j)}(x)| \leqq \frac{\hat{L}(P_{n-1}(x))x^{n-j} \exp S(x)}{(n-j)!},$$

* Received by the editors October 14, 1970, and in revised form February 16, 1971. The research of the first author was supported by a Post-Doctoral Fellowship of the University of Alberta. The research of the second author was supported by the National Research Council of Canada under Grant A-5210.

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where

$$S(x) = \sum_{j=0}^{n-1} \frac{\hat{\sigma}_j(x)x^{n-j}}{(n-j)!}.$$

Proof. Following [3], we obtain, on differentiating Taylor's formula with the integral form of the remainder,

$$f^{(j)}(x) - P_{n-1}^{(j)}(x) = \frac{1}{(n-j-1)!}\int_0^x (x-t)^{n-j-1} f^{(n)}(t)\,dt, \qquad j = 0, 1, \cdots, n-1. \tag{2}$$

Then by the triangle inequality and (1) and (2), we have, for $0 \leqq x \leqq h$,

$$\begin{aligned} |f^{(n)}(x)| &\leqq \left|\phi(x) + \sum_{j=0}^{n-1} \sigma_j(x)P_{n-1}^{(j)}(x)\right| + \left|\sum_{j=0}^{n-1} \sigma_j(x)(f^{(j)}(x) - P_{n-1}^{(j)}(x))\right| \\ &\leqq |L(P_{n-1}(x))| + \int_0^x \left(\sum_{j=0}^{n-1} \frac{|\sigma_j(x)|(x-t)^{n-j-1}}{(n-j-1)!}\right)|f^{(n)}(t)|\,dt. \end{aligned} \tag{3}$$

Denoting

$$\sum_{j=0}^{n-1} \frac{\hat{\sigma}_j(x)(x-t)^{n-j-1}}{(n-j-1)!}$$

by $S(x, t)$, we have that

$$|f^{(n)}(x)| \leqq |L(P_{n-1}(x))| + \int_0^x S(x,t)|f^{(n)}(t)|\,dt, \qquad 0 \leqq x \leqq h. \tag{4}$$

Since $S(x, t)$ is nondecreasing in x, by an easy modification of Gronwall's lemma, there follows

$$\begin{aligned} |f^{(n)}(x)| &\leqq |L(P_{n-1}(x))| \\ &\quad + \int_0^x |L(P_{n-1}(t))| S(x,t) \exp\left(\int_t^x S(x,s)\,ds\right) dt, \qquad 0 \leqq x \leqq h. \end{aligned} \tag{5}$$

Inequality (5) implies that

$$|f^{(n)}(x)| \leqq \hat{L}(P_{n-1}(x)) \exp S(x), \qquad 0 \leqq x \leqq h, \tag{6}$$

and now (2) and (6) imply that

$$\begin{aligned} |f^{(j)}(x) - P_{n-1}^{(j)}(x)| &\leqq \hat{L}(P_{n-1}(x)) \int_0^x \frac{(x-t)^{n-j-1}}{(n-j-1)!} \exp S(t)\,dt \\ &\leqq \hat{L}(P_{n-1}(x)) \frac{x^{n-j}}{(n-j)!} \exp S(x), \end{aligned}$$

which completes the proof of Theorem 1.

Remark. Inequality (6) may also be obtained from (4) using a result of [1].

THEOREM 2. *Let $f \in C^\infty[0, h]$ and suppose that*

$$\limsup_{j \to \infty} \left(\frac{|f^{(j)}(0)|}{j!} \right)^{1/j} = \frac{1}{\beta}, \qquad 0 \leqq \beta \leqq h. \tag{7}$$

Denote the set of natural numbers by Z. Let $0 \leqq \alpha \leqq \min(\beta, h)$. Then for f to be regular on $[0, \alpha)$ (that is, $f(x) \equiv \sum_{j=0}^{\infty} (f^{(j)}(0)x^j/j!)$ on $[0, \alpha)$), it is necessary and sufficient that the following conditions should be satisfied:

(A) *There exists a positive-valued function $G(n, \varepsilon)$ defined on the set $Z \times (0, \infty)$ such that*

$$\limsup_{n \to \infty} \left(\frac{G(n, \varepsilon)}{n!} \right)^{1/n} = 1. \tag{8}$$

(B) *There exists an unbounded subset I of Z, such that for each $n \in I$, functions $\phi_n(x)$, $\sigma_{n,j}(x)$, $j = 0, 1, \cdots, n-1$, are defined, which are Lebesgue integrable on $[0, \alpha)$ and such that*

$$\sum_{j=0}^{n-1} j!\hat{\sigma}_{n,j}(x)(\alpha - x)^{n-j} \leqq n!, \tag{9a}$$

and for each $\varepsilon > 0$,

$$|\phi_n(x)| \left(\frac{\alpha - (1+\varepsilon)x}{1+\varepsilon} \right)^n \leqq G(n, \varepsilon), \qquad 0 \leqq x \leqq \frac{\alpha}{1+\varepsilon}. \tag{9b}$$

(C) *f satisfies the sequence of differential equations on $[0, \alpha)$:*

$$y^{(n)}(x) = \sum_{j=0}^{n-1} \sigma_{n,j}(x) y^{(j)}(x) + \phi_n(x), \qquad n \in I. \tag{10}$$

Before proving the theorem, we require two lemmas. For the sake of brevity we omit the proofs.

LEMMA 1. *Let $f, g \in C^n[0, h]$, let f satisfy* (1) *and let g satisfy*

$$M(z(x)) \equiv z^{(n)}(x) - \sum_{j=0}^{n-1} s_j(x) z^{(j)}(x) - s(x) = 0, \qquad 0 \leqq x \leqq h, \tag{11}$$

with $|\phi(x)| \leqq s(x)$, $|\sigma_j(x)| \leqq s_j(x)$, $j = 0, 1, \cdots, n-1$, $0 \leqq x \leqq h$, the s_j's and s being Lebesgue integrable on $[0, h]$. Then if $|f^{(j)}(0)| = g^{(j)}(0)$, $j = 0, 1, \cdots, n-1$, and defining

$$P_{n-1}(x) = \sum_{j=0}^{n-1} \frac{f^{(j)}(0)x^j}{j!}, \qquad Q_{n-1}(x) = \sum_{j=0}^{n-1} \frac{g^{(j)}(0)x^j}{j!},$$

we have

$$|f^{(j)}(x) - P_{n-1}^{(j)}(x)| \leqq |g^{(j)}(x) - Q_{n-1}^{(j)}(x)|, \qquad j = 0, 1, \cdots, n-1, \quad 0 \leqq x \leqq h.$$

LEMMA 2. *Let $M, \alpha, \varepsilon, \delta$ be positive numbers with $\delta < \varepsilon < \alpha$, and let $p(n)$ be a positive-valued function defined on the natural numbers such that*

$$\limsup_{n \to \infty} (p(n))^{1/n} = 1.$$

Let $\theta_{n,j}$, $j = 0, 1, \cdots, n-1$, $n = 1, 2, \cdots$, *be nonnegative numbers satisfying* $\sum_{j=0}^{n-1} \theta_{n,j} \leqq 1$, *and let* ψ_n, $n = 1, 2, \cdots$, *be nonnegative numbers satisfying* $\psi_n \leqq p(n)$.

Let $g_n(x)$ *be a solution of the equation*

$$z^{(n)}(x) = \sum_{j=0}^{n-1} \frac{n!}{j!} \frac{\theta_{n,j}}{(\alpha - x)^{n-j}} z^{(j)}(x) + \frac{n!\psi_n}{(\alpha - x)^{n+1}} \quad \text{on } [0, \alpha - \varepsilon], \tag{12}$$

with $|g_n^{(j)}(0)| \leqq Mj!/(\alpha - \delta)^j$, $j = 0, 1, \cdots, n-1$. *Then if*

$$Q_{n-1}(x) = \sum_{j=0}^{n-1} \frac{g_n^{(j)}(0)x^j}{j!},$$

the sequence $\{g_n(x) - Q_{n-1}(x)\}$ *converges uniformly to zero on* $[0, \alpha - \varepsilon]$.

Proof of Theorem 2. Suppose that f satisfies (7) and conditions (A), (B) and (C). Let $\varepsilon > 0$ be arbitrarily chosen. For $0 \leqq x \leqq \alpha/(1 + \varepsilon)$, write

$$\sigma_{n,j}(x) = \frac{\theta_{n,j}(x)}{(\alpha(1+\varepsilon)^{-1} - x)^{n-j}} \frac{n!}{j!}, \qquad j = 0, 1, \cdots, n-1,$$

$$Q_n(x) = \frac{\psi_n(x)}{(\alpha(1+\varepsilon)^{-1} - x)^{n-j}}.$$

Then on $[0, \alpha/(1 + \varepsilon)]$, f satisfies the equation

$$y^{(n)}(x) = \sum_{j=0}^{n-1} \frac{n!}{j!} \frac{\theta_{n,j}(x) y^{(j)}(x)}{(\alpha(1+\varepsilon)^{-1} - x)^{n-j}} + \frac{n!\psi_n(x)}{(\alpha(1+\varepsilon)^{-1} - x)^{n-1}},$$

and condition (B) implies that for $0 \leqq x \leqq \alpha/(1 + \varepsilon)$,

$$\sum_{j=0}^{n-1} \hat{\theta}_{n,j}(x) \leqq 1, \qquad |\psi_n(x)| \leqq \frac{G(n, \varepsilon)}{n!}.$$

Then Lemma 1 implies that for $0 \leqq x \leqq \alpha/(1 + \varepsilon)$, $n \in I$,

$$|f(x) - P_{n-1}(x)| \leqq |g_n(x) - Q_{n-1}(x)|, \tag{13}$$

where $g_n(x)$ solves the equation

$$z^{(n)}(x) = \sum_{j=0}^{n-1} \frac{n!}{j!} \frac{\hat{\theta}_{n,j}(\alpha/(1+\varepsilon))}{(\alpha(1+\varepsilon)^{-1} - x)^{n-j}} z^{(j)}(x) + \frac{G(n)}{(\alpha(1+\varepsilon)^{-1} - x)^{n+1}}, \tag{14}$$
$$0 \leqq x \leqq \alpha/(1 + \varepsilon),$$

with $g_n^{(j)}(0) = |f^{(j)}(0)|$, $j = 0, 1, \cdots, n-1$, and $P_{n-1}(x)$, $Q_{n-1}(x)$ are the Taylor polynomials associated with f, g_n, respectively.

Now (7) implies that for some constant M and for all nonnegative integers j,

$$|f^{(j)}(0)| \leqq \frac{Mj!}{(\alpha(1 + 2\varepsilon)^{-1})^j} \qquad (\text{since } \alpha \leqq \beta). \tag{15}$$

By (14) and (15), the conditions of Lemma 2 apply to the sequence of functions $\{g_n(x)\}$, $n \in I$, and so the sequence

$$\{g_n(x) - Q_{n-1}(x)\} \tag{16}$$

tends to zero uniformly on $[0, \alpha(1 + 3\alpha)^{-1}]$. Inequality (13), condition (16) and the arbitrary choice of $\varepsilon > 0$ imply that f is regular on $[0, \alpha)$.

To prove that the conditions of the theorem are necessary for regularity on $[0, \alpha)$, assume that f satisfies (8) and is regular on $[0, \alpha)$, with $0 \leqq \alpha \leqq \beta$.

Then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!} \quad \text{for } 0 \leqq x < \alpha.$$

Let $\varepsilon > 0$. Then (7) implies that $|f^{(n)}(0)| \leqq N((1 + \varepsilon)/\alpha)^n n!$ for some constant $N = N(\varepsilon)$, and for all nonnegative integers n. It follows that

$$|f(x)| \leqq N \sum_{n=0}^{\infty} \left[x\left(\frac{1 + \varepsilon}{\alpha}\right)\right]^n = \frac{N\alpha}{\alpha - x(1 + \varepsilon)}, \qquad 0 \leqq x < \frac{\alpha}{1 + \varepsilon},$$

and term-by-term comparison of the two power series shows that

$$|f^{(n)}(x)| \leqq \frac{\alpha N n!(1 + \varepsilon)^{n+1}}{(\alpha - x(1 + \varepsilon))^{n+1}}, \qquad n = 0, 1, 2, \cdots.$$

Conditions (A), (B) and (C) will now follow with $I = Z$, $G(n, \varepsilon) = \alpha N(\varepsilon)n!$,

$$\sigma_{n,j}(x) \equiv 0, \qquad j = 0, 1, \cdots, n - 1,$$

$$\phi_n(x) \equiv f^{(n)}(x), \qquad n = 0, 1, \cdots,$$

and this completes the proof of the theorem.

Remark. It is not hard to show that the conclusion of the theorem is still valid if f and ϕ_n are allowed to be complex-valued functions.

As a consequence of Theorem 2, we have the following corollaries.

COROLLARY 1. *Let $f \in C^\infty[0, h]$ with $\limsup_{n\to\infty} |f^{(n)}(0)|/n! < \infty$. Then if f is not regular at $x = 0$ (that is, f is not regular on $[0, \delta)$ for any $\delta > 0$), there must exist for each $\varepsilon > 0$ a natural number $N(\varepsilon)$ and a sequence of positive numbers $x_n \to 0$ as $n \to \infty$ such that for $n > N$,*

$$|f^{(n)}(x_n)| > n!/\varepsilon^n.$$

COROLLARY 2. *Let f satisfy the hypotheses of Corollary 1. Then for each $k > 0$, there exists an $n(k)$ such that for $n > n(k)$, f cannot be a solution of an equation of the type (1) with*

$$|\sigma_j(x)| < \frac{n!}{j!}k^{n-j}, \quad j = 0, 1, \cdots, n - 1, \quad |\phi(x)| \leqq n!k^n \quad \textit{for } 0 \leqq x \leqq h.$$

Corollary 2 is a sharpening of a similar result in [4].

The following is a corollary of Lemma 1.

COROLLARY 3. *Let $f \in C^\infty[0, h)$, let $\phi_n(x)$, $\sigma_{n,j}(x)$ be Lebesgue integrable on $[0, h)$ and let $\psi_n(x)$, $\theta_{n,j}(x)$ be analytic on $[0, h)$ with*

$$|\sigma_{n,j}(x)| \leqq \theta_{n,j}(x), \qquad j = 0, 1, \cdots, n - 1,$$

$$|\phi_n(x)| \leqq \psi_n(x).$$

Let f satisfy the sequence of equations

$$y^{(n)}(x) = \sum_{j=0}^{n-1} \sigma_{n,j}(x)y^{(j)}(x) + \phi_n(x),$$

and let $g_n(x)$ be a solution of the equation

$$z^{(n)}(x) = \sum_{j=0}^{n-1} \theta_{n,j}(x)z^{(j)}(x) + \psi_n(x).$$

Suppose further that $\{g_n(x) - Q_{n-1}(x)\}$ tends to 0 uniformly on $[0, h - \varepsilon)$ for all ε, where

$$Q_{n-1}(x) = \sum_{j=0}^{n-1} \frac{g_n^{(j)}(0)x^j}{j!}.$$

Then f is analytic on $[0, h)$.

As a simple application of Theorem 2, if m_i, n_i are sequences of natural numbers tending to infinity with $m_i < n_i$ and f is a function satisfying the inequalities

$$|y^{(n_i)}| \leqq \frac{n_i!}{m_i!} \frac{|y^{(m_i)}|}{(1 - x)^{n_i - m_i}}, \qquad 0 \leqq x < 1,$$

then f is analytic on $[0, 1)$ (for example, $f = 1/(1 + x)$). This conclusion is false if the right-hand sides of these inequalities are multiplied by any constant greater than 1 (for example, $f = 1/(1 - \varepsilon + x)$).

We may also use Theorem 2 to characterize the behavior of functions of two variables in terms of inequalities involving their partial derivatives. For an example of this we give the following theorem.

THEOREM 3. *Let $f(x, y)$ be a complex-valued C^∞-function of $(x, y) \in R^2$, analytic as a function of $z = x + iy$ in some domain D, and without a finite accumulation point of zeros. Suppose further than the inequality*

$$|f(x, y) f_{x^j y^{n-j}}(x, y)| \leqq 1, \qquad (x, y) \in R^2,$$

holds for all mixed partials $f_{x^j y^{n-j}}$ of order n in some sequence I of natural numbers n. Then $f(z)$ is a polynomial in z.

Proof. Without loss of generality, we assume that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for $|z| \leqq \delta$, $\delta > 0$. Since f has no accumulation points of zeros, there exists a set $\mathscr{L}$ of lines through the point $z = 0$ whose union is everywhere dense in the plane, on which f does not vanish. (We assume without loss that $f(0) \neq 0$.)

Let $l_\theta = \{z | z = re^{i\theta}, r \text{ real}\}$ be a line of $\mathscr{L}$.

Let $F(r) = f(r\cos\theta, r\sin\theta)$.

Then

$$F^{(n)}(r) = \sum_{j=0}^{n} \binom{n}{j} f_{x^j y^{n-j}} \cos^j \theta \sin^{n-j} \theta$$

(where we have omitted the arguments of the partial derivative). Hence for $n \in I$,

$$\begin{aligned}|F(r)F^{(n)}(r)| &\leqq |f(r\cos\theta, r\sin\theta)| \sum_{j=0}^{n} \binom{n}{j} |f_{x^j y^{n-j}} \cos^j\theta \sin^{n-j}\theta| \\ &= \sum_{j=0}^{n} \binom{n}{j} |\cos^j\theta \sin^{n-j}\theta| \, |f(r\cos\theta, r\sin\theta) f_{x^j y^{n-j}}| \\ &\leqq 2^{n/2}.\end{aligned}$$

Now for $-R \leqq r \leqq R$, $|F(r)| > \varepsilon$, $R > 0$, and so

$$|F^{(n)}(r)| \leqq 2^{n/2}/\varepsilon, \qquad n \in I.$$

Thus, by the remark following Theorem 2, $F(r)$ is analytic on $[-R, R]$ for arbitrary $R > 0$, and thus on $(-\infty, \infty)$.

Hence $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on each line $l \in \mathscr{L}$, and thus on a set which is dense in the plane. Thus f is an entire function.

But $|f(z)f^{(n)}(z)| \leqq 1$, $n \in I$, where $f^{(n)}(z)$ now denotes total differentiation with respect to z. Since $f(z)f^{(n)}(z)$ is an entire function, it follows that $f(z)f^{(n)}(z) = c_n$, a constant, for $n \in I$.

Suppose that $c_n \neq 0$, $n \in I$. Then if, n, $n + m \in I$, we have $kf^{(n)}(z) = f^{(n+m)}(z)$ for some constant k. Putting $\phi(z) = f^{(n)}(z)$, we have

$$\phi^{(m)}(z) = k\phi(z).$$

Thus $\phi(z) = \sum_{j=1}^{m} d_j e^{\lambda_j z}$, where the λ_j are mth complex roots of k. Thus $f(z) = P_{n-1}(z) + \sum_{j=1}^{m} d_j \lambda_j^n e^{\lambda_j z}$, where $P_{n-1}(z)$ is a polynomial of degree $n - 1$.

Thus

$$f(z)f^{(n)}(z) = \left(P_{n-1}(z) + \sum_{j=1}^{m} d_j \lambda_j^n e^{\lambda_j z}\right) \sum_{j=1}^{m} d_j e^{\lambda_j z}.$$

It is easy to see that this implies that $d_j = 0$, $j = 1, \cdots, m$, and so $f(z) = P_{n-1}(z)$. If $c_n = 0$, on the other hand, we have $f(z)f^{(n)}(z) \equiv 0$ and either $f \equiv 0$ or again $f(z)$ is a polynomial of degree (at most) $n - 1$. This completes the proof of Theorem 3.

Remark. Without the assumption of local analyticity, the theorem fails as, for example, with $f(x, y) = \frac{1}{2}(\sin x + \sin y)$.

We proceed to state a sharpening of Theorem 1 for certain special cases.

THEOREM 4. *Let $f \in C^n[0, h]$, and suppose that f satisfies* (1) *on* $[0, h]$. *Assume, in addition, that $\sigma_{n-1}(x) = \cdots = \sigma_{n-q}(x) \equiv 0$ on $[0, h]$, where $q \geqq 1$. Then if*

$$P_{n-1}(x) = \sum_{j=0}^{n-1} \frac{f^{(j)}(0)}{j!} x^j,$$

we have

$$|f^{(j)}(x) - P_{n-1}^{(j)}(x)| \leqq \frac{x^{n-j}}{(n-j)!} \hat{L}(P_{n-1}(x)) \left(\frac{q(q+1)}{2} F\left(\frac{2S(x)}{q(q+1)}\right)\right),$$

where

$$F(\theta) = \sum_{m=0}^{\infty} \frac{\theta^m}{m!(m+1)!}.$$

Outline of proof. Beginning with the inequality (4) of Theorem 1, we apply a lemma due to Chu and Metcalf [2] to obtain an explicit estimate of $|f^n(x)|$. The remainder of the proof consists in majorizing a particular resolvent kernel which appears in the Chu–Metcalf lemma.

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SIAM J. MATH. ANAL.
Vol. 2, No. 4, November 1971

TRIGONOMETRIC APPROXIMATION IN THE SOBOLEV SPACES $W^{r,2}[-\pi, \pi]$ WITH CONSTANT WEIGHTS*

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Abstract. It is shown that, although sines and cosines are not, in general, complete in $W^{r,2}[-\pi, \pi]$, they are complete in the subspace of those functions whose first $r-1$ derivatives are periodic of period 2π. Also, the sequence of sines and cosines is extended to a complete sequence.

1. Introduction. In this paper, two results on trigonometric approximation in the Sobolev spaces $W^{r,2}[-\pi, \pi]$ with constant weights are presented. The first result is a negative one, namely, that, although the sequence $x^r, x^{r-1}, \cdots, x, 1$, $\sin x, \cos x, \cdots, \sin kx, \cos kx, \cdots$ is complete in $W^{r,2}[-\pi, \pi]$, it is no longer complete when one removes the highest power x^r from the sequence. The second is that, whenever the function and its first $r-1$ derivatives are periodic of period 2π, there is a Parseval relation in $W^{r,2}[-\pi, \pi]$ with respect to the trigonometric functions alone. In addition, there is a certain delineation of structure of the Sobolev spaces. The Sobolev space $W^{r,2}[-\pi, \pi]$ is the space of all functions f on $[-\pi, \pi]$ whose $(r-1)$st derivative is absolutely continuous, whose rth derivative is in $L^2[-\pi, \pi]$, and whose norm is given by

$$\|f\|^2 = \sum_{k=0}^{r} \lambda_k \int_{-\pi}^{\pi} [f^{(k)}(x)]^2 \, dx, \tag{1}$$

where λ_k is a given positive constant, $0 \leqq k \leqq r$, and $f^{(k)}(x)$ denotes the kth derivative of $f(x)$.

Agmon [1, pp. 27–28] has done related work in the context of several variables. This paper is restricted to a more extensive treatment of the single variable case.

2. Analysis. Let us show, first of all, that the sines and cosines do not form a complete set in $W^{1,2}[-\pi, \pi]$. To do this, we shall demonstrate, in $W^{1,2}[-\pi, \pi]$ with $\lambda_0 = \lambda_1 = 1$, that the function $f = x$ is not in the closure of the linear span of sines and cosines alone. It is sufficient to prove that Parseval's relation [3, p. 191] does not hold. Now the sequence $1, \sin x, \cos x, \cdots, \sin kx, \cos kx, \cdots$ is already orthogonal in $W^{1,2}[-\pi, \pi]$. Also,

$$\|1\|^2 = \int_{-\pi}^{\pi} dx = 2\pi,$$

$$\|\sin kx\|^2 = \int_{-\pi}^{\pi} \sin^2 kx \, dx + \int_{-\pi}^{\pi} k^2 \cos^2 kx \, dx = (1 + k^2)\pi,$$

$$\|\cos kx\|^2 = \int_{-\pi}^{\pi} \cos^2 kx \, dx + \int_{-\pi}^{\pi} k^2 \sin^2 kx \, dx = (1 + k^2)\pi.$$

* Received by the editors December 3, 1970, and in revised form March 24, 1971.

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Therefore, the sequence

$$T_0(x) = (2\pi)^{-1/2}, \quad T_1(x) = (2\pi)^{-1/2} \sin x, \quad T_2(x) = (2\pi)^{-1/2} \cos x, \cdots,$$
$$T_{2k-1}(x) = (1 + k^2)^{-1/2}\pi^{-1/2} \sin kx, \qquad T_{2k}(x) = (1 + k^2)^{-1/2}\pi^{-1/2} \cos kx, \cdots$$

is an orthonormal sequence. Define

$$(f, g) \equiv \int_{-\pi}^{\pi} fg\, dx + \int_{-\pi}^{\pi} f'g'\, dx. \tag{2}$$

Then one finds that

$$\begin{aligned} &(x, T_0) = 0, \qquad (x, T_{2k}) = 0, && k \geqq 1, \\ &(x, T_{2k-1}) = 2(-1)^{k+1}\pi^{1/2}(1 + k^2)^{-1/2}k^{-1}, && k \geqq 1. \end{aligned} \tag{3}$$

Also,

$$\|x\|^2 = 2\pi(1 + \pi^2/3). \tag{4}$$

Now, in any inner product space, Bessel's inequality [3, p. 172] is valid, namely,

$$\sum_{k=0}^{\infty} (f, p_k)^2 \leqq \|f\|^2, \tag{5}$$

where $\{p_k\}$ is any orthonormal sequence. Note that Parseval's relation is just (5) replaced by equality. Using (3) in the left-hand side of (5) and (4) in the right-hand side, we find immediately that

$$2 \sum_{k=1}^{\infty} k^{-2}(1 + k^2)^{-1} \leqq 1 + \pi^2/3. \tag{6}$$

But

$$2 \sum_{k=1}^{\infty} k^{-2}(1 + k^2)^{-1} < 2 \sum_{k=1}^{\infty} k^{-2} = \pi^2/3 < 1 + \pi^2/3.$$

Therefore, Parseval's relation does not hold, and the sines and cosines do not form a complete sequence.

There is another way to see that $f(x) = x$ is not in the closed linear span of the sines and cosines. We can do this by comparing the classical Fourier series for f with a modified series in $W^{1,2}[-\pi, \pi]$ defined formally by

$$a_0^* T_0(x) + \sum_{k=1}^{\infty} [a_k^* T_{2k}(x) + b_k^* T_{2k-1}(x)],$$

where, using (2),

$$a_k^* \equiv (f, T_{2k}) \quad \text{and} \quad b_k^* \equiv (f, T_{2k-1}).$$

Since $f(x) = x$ is an odd function on $[-\pi, \pi]$, its classical Fourier series and the modified series will both consist of sines alone. The classical Fourier coefficients with respect to an orthonormal set of sines and cosines in $L^2[-\pi, \pi]$ are defined by

$$
\begin{aligned}
&a_0 \equiv (2\pi)^{-1/2} \int_{-\pi}^{\pi} f(x)\,dx, \qquad a_k \equiv \pi^{-1/2} \int_{-\pi}^{\pi} f(x) \cos kx\,dx, \qquad k \geqq 1, \\
&b_k \equiv \pi^{-1/2} \int_{-\pi}^{\pi} f(x) \sin kx\,dx, \qquad k \geqq 1.
\end{aligned} \tag{7}
$$

Hence, for the classical Fourier series of $f(x) = x$, we obtain

$$x = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin nx}{n}, \qquad -\pi < x < \pi.$$

In contrast, using (3), we see that the modified series is given by

$$X(x) = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin nx}{n(1 + n^2)}. \tag{8}$$

The series (8), together with its derived series, is absolutely and uniformly convergent. It turns out that we can obtain $X(x)$ in closed form as follows. Note [2, p. 446] that

$$\cosh ax = \pi^{-1} \sinh a\pi \left[a^{-1} + 2a \sum_{n=1}^{\infty} (-1)^n (a^2 + n^2)^{-1} \cos nx \right], \tag{9}$$

$-\pi \leqq x \leqq \pi$. Letting $a = 1$ in (9), one sees from (8) that

$$\cosh x = \pi^{-1} \sinh \pi [1 - X'(x)].$$

Thus,

$$X'(x) = 1 - \pi \cosh x / \sinh \pi, \tag{10}$$

and so

$$X(x) = x - \pi \sinh x / \sinh \pi \neq x, \qquad x \neq 0. \tag{11}$$

Also, from the uniform convergence of the series for $X(x)$ and $X'(x)$, it is clear that the modified series converges in the mean to $X(x)$ as given by (11) instead of to x. Thus, the sequence $1, \sin x, \cdots, \sin kx, \cdots$ is not complete in $W^{1,2}[-\pi, \pi]$, but is complete in $L^2[-\pi, \pi]$.

However, it can be shown that the sequence $x, 1, \sin x, \cos x, \cdots, \sin kx, \cos kx, \cdots$ is a complete sequence in $W^{1,2}[-\pi, \pi]$ with constant positive weights. First of all, in $L^2[-\pi, \pi]$, given any $\varepsilon > 0$, there exists a trigonometric polynomial $s_n(x)$ such that

$$\int_{-\pi}^{\pi} [f'(x) - s_n(x)]^2\,dx < \varepsilon. \tag{12}$$

If we let

$$t_n(x) \equiv \int_{-\pi}^{x} s_n(t)\,dt,$$

we see that, in general, t_n contains an x term. Since $f(x)$ is absolutely continuous [4, p. 255], we have the following:

$$\int_{-\pi}^{\pi}\left[\int_{-\pi}^{x} f'(t)\,dt - t_n(x)\right]^2 dx = \int_{-\pi}^{\pi} [f(x) - f(-\pi) - t_n(x)]^2\,dx. \tag{13}$$

Therefore, letting $r_n(x) = f(-\pi) + t_n(x)$, one sees from (13), upon using the Cauchy–Schwarz inequality [3, p. 159], that

$$\int_{-\pi}^{\pi} [f(x) - r_n(x)]^2\,dx \leqq 4\pi^2\varepsilon.$$

Therefore, it is seen that

$$\|f - r_n\|^2 = \lambda_0 \int_{-\pi}^{\pi} (f - r_n)^2\,dx + \lambda_1 \int_{-\pi}^{\pi} (f' - s_n)^2\,dx \leqq (\lambda_1 + 4\pi^2\lambda_0)\varepsilon.$$

Since ε was arbitrary, it follows that the sequence x, 1, $\sin x$, $\cos x, \cdots, \sin kx$, $\cos kx, \cdots$ is complete in $W^{1,2}[-\pi, \pi]$. The results established so far in this section constitute a crucial step in the inductive proof of the following theorem.

THEOREM 1. *In the Sobolev space* $W^{r,2}[-\pi,\pi]$, $r \geqq 1$, *the sequence* x^{r-1}, $x^{r-2}, \cdots, 1, \sin x, \cos x, \cdots, \sin kx, \cos kx, \cdots$ *is not a complete sequence. However,* $x^r, x^{r-1}, \cdots, x, 1, \sin x, \cos x, \cdots, \sin kx, \cos kx, \cdots$ *is a complete sequence.*

Proof. We shall show, first of all, that the sequence $x^{r-1}, x^{r-2}, \cdots, x, 1$, $\sin x, \cos x, \cdots, \sin kx, \cos kx, \cdots$ is not complete in $W^{r,2}[-\pi,\pi]$. This we do by mathematical induction. Let us suppose that the result is correct for $W^{r-1,2}[-\pi,\pi]$; i.e., x^{r-1} cannot be approximated in $W^{r-1,2}[-\pi,\pi]$ to within arbitrary $\varepsilon > 0$ by finite linear combinations of $x^{r-2}, x^{r-3}, \cdots, 1$, $\sin x$, $\cos x, \cdots$ for any set $\{\lambda_k\}_{k=0}^{r-1}$ of positive weights. Assume that it is possible for some set $\{\lambda_k\}$, to approximate x^r in $W^{r,2}[-\pi,\pi]$ to within arbitrary $\varepsilon > 0$ by linear combinations of x^{r-1}, $x^{r-2}, \cdots, 1, \sin x, \cos x, \cdots$. Thus, given any positive ε, one can find some linear combination

$$R(x) \equiv \sum_{i=1}^{r-1} \beta_i x^{r-i} + T_n(x),$$

where $T_n(x)$ is a trigonometric polynomial, such that

$$\sum_{k=0}^{r} \lambda_k \int_{-\pi}^{\pi} [(x^r)^{(k)} - R^{(k)}(x)]^2\,dx < r^2\varepsilon.$$

It follows then that

$$\sum_{k=1}^{r} \lambda_k \int_{-\pi}^{\pi} [(x^r)^{(k)} - R^{(k)}(x)]^2\,dx < r^2\varepsilon.$$

After a short calculation, one finds that

$$\left[\sum_{i=1}^{r-1} \beta_i (r-i) x^{r-i-1} + T_n'(x)\right]\Big/ r$$

approximates x^{r-1} to within ε in $W^{r-1,2}[-\pi,\pi]$. This is a contradiction. Since we have already seen that the sequence 1, $\sin x$, $\cos x, \cdots, \sin kx$, $\cos kx, \cdots$ is not complete in $W^{1,2}[-\pi,\pi]$, the first half of the theorem is proved.

To show the second half, assume that $x^{r-1}, x^{r-2}, \cdots, x, 1, \sin x, \cos x, \cdots$ is complete in $W^{r-1,2}[-\pi,\pi]$ for any set $\{\lambda_k\}_{k=0}^{r-1}$ of positive weights. We want to show that $x^r, x^{r-1}, \cdots, x, 1, \sin x, \cos x, \cdots$ is complete in $W^{r,2}[-\pi,\pi]$ for any set $\{\lambda_k\}_{k=0}^{r}$ of positive weights. Suppose that $f \in W^{r,2}[-\pi,\pi]$. Then $f' \in W^{r-1,2}[-\pi,\pi]$. Therefore, by inductive hypothesis, given any $\varepsilon > 0$, there exists some finite linear combination

$$S(x) \equiv \sum_{i=1}^{r-1} \alpha_i x^{r-i} + T_n(x)$$

such that

$$\sum_{k=1}^{r} \lambda_k \int_{-\pi}^{\pi} [f^{(k)}(x) - I^{(k)}(x)]^2\, dx < \varepsilon,$$

where

$$I(x) \equiv \int_{-\pi}^{x} S(t)\, dt + f(-\pi).$$

Note that $I(x)$ is a linear combination of $x^r, x^{r-1}, \cdots, x, 1, \sin x, \cos x, \cdots, \sin nx, \cos nx$. From the above inequality, one sees that

$$\int_{-\pi}^{\pi} [f'(x) - I'(x)]^2\, dx = \int_{-\pi}^{\pi} [f'(x) - S(x)]^2\, dx < \varepsilon/\lambda_1.$$

Therefore, using the fact that

$$\int_{-\pi}^{\pi} [f(x) - I(x)]^2\, dx = \int_{-\pi}^{\pi} \left[\int_{-\pi}^{x} f'(t)\, dt - \int_{-\pi}^{x} S(t)\, dt \right]^2 dx$$

and the Cauchy–Schwarz inequality [3, p. 159], we have

$$\int_{-\pi}^{\pi} [f(x) - I(x)]^2\, dx \leqq 4\pi^2 \varepsilon/\lambda_1.$$

Hence,

$$\sum_{k=0}^{r} \lambda_k \int_{-\pi}^{\pi} [f^{(k)}(x) - I^{(k)}(x)]^2\, dx < (1 + 4\pi^2 \lambda_0/\lambda_1)\varepsilon.$$

Since ε was arbitrary and since the result has already been established in $W^{1,2}[-\pi,\pi]$, the induction is complete.

In $W^{r,2}[-\pi,\pi]$, the inner product is defined by

$$(f, g) \equiv \sum_{k=0}^{r} \lambda_k \int_{-\pi}^{\pi} f^{(k)}(x) g^{(k)}(x)\, dx. \tag{14}$$

Using (14), one finds that the orthonormal set of sines and cosines is given by

$$T_0(x) = (2\pi\lambda_0)^{-1/2}, \qquad T_1(x) = \left(\pi \sum_{i=0}^{r} \lambda_i\right)^{-1/2} \sin x,$$

$$T_2(x) = \left(\pi \sum_{i=0}^{r} \lambda_i\right)^{-1/2} \cos x, \quad \cdots, T_{2k-1}(x) = \left(\pi \sum_{i=0}^{r} \lambda_i k^{2i}\right)^{-1/2} \sin kx,$$

$$T_{2k}(x) = \left(\pi \sum_{i=0}^{r} \lambda_i k^{2i}\right)^{-1/2} \cos kx, \cdots.$$

Define $a_k^* \equiv (f, T_{2k})$ and $b_k^* \equiv (f, T_{2k-1})$. These are seen to be the coefficients for a modified series

$$(15) \qquad a_0^* T_0(x) + \sum_{k=1}^{\infty} [a_k^* T_{2k}(x) + b_k^* T_{2k-1}(x)].$$

Although the sines and cosines, in general, are not complete in $W^{r,2}[-\pi, \pi]$, they are complete in a certain subspace of this Sobolev space, namely, that of those functions which, together with their first, second, $\cdots$, $(r-1)$st derivatives, are periodic of period 2π. Furthermore, in this context, the classical Fourier series and the modified series are precisely the same. We have the following theorem.

THEOREM 2. *Consider the subspace of all f in $W^{r,2}[-\pi, \pi]$ with constant weights with the property that $f^{(k)}(-\pi) = f^{(k)}(\pi)$, $0 \leqq k \leqq r-1$. Then Parseval's relation relative to sines and cosines is satisfied for all such f, and the modified series and the classical Fourier series coincide.*

Proof. Applying (14), we see, after a short calculation, that the coefficients for the modified series (15) are given by

$$(16) \qquad \begin{aligned} a_0^* &= (\lambda_0/2\pi)^{1/2} \int_{-\pi}^{\pi} f\,dx, \\ a_k^* &= \pi^{-1/2}\left(\sum_{i=0}^{r} \lambda_i k^{2i}\right)^{1/2} \int_{-\pi}^{\pi} f(x)\cos kx\,dx, \qquad k \geqq 1, \\ b_k^* &= \pi^{-1/2}\left(\sum_{i=0}^{r} \lambda_i k^{2i}\right)^{1/2} \int_{-\pi}^{\pi} f(x)\sin kx\,dx, \qquad k \geqq 1. \end{aligned}$$

Under the periodicity hypotheses of the theorem, the classical Fourier coefficients for $f^{(k)}(x)$ are just $\pm n^k a_n$ and $\pm n^k b_n$, $0 \leqq k \leqq r$. Therefore, by Parseval's relation for the L^2-norm,

$$(17) \qquad \sum_{k=0}^{r} \lambda_k \int_{-\pi}^{\pi} [f^{(k)}(x)]^2\,dx = \lambda_0 a_0^2 + \sum_{n=1}^{\infty}\left(\sum_{i=0}^{r} \lambda_i n^{2i}\right)(a_n^2 + b_n^2).$$

Now, by comparing (7) and (16),

$$(a_0^*)^2 = \lambda_0 a_0^2, \quad (a_k^*)^2 = \left(\sum_{i=0}^{r} \lambda_i k^{2i}\right) a_k^2, \quad (b_k^*)^2 = \left(\sum_{i=0}^{r} \lambda_i k^{2i}\right) b_k^2.$$

So,

$$\begin{aligned} (a_0^*)^2 + \sum_{k=1}^{\infty} [(a_k^*)^2 + (b_k^*)^2] &= \lambda_0 a_0^2 + \sum_{k=1}^{\infty}\left(\sum_{i=0}^{r} \lambda_i k^{2i}\right)(a_k^2 + b_k^2) \\ &= \sum_{k=0}^{r} \lambda_k \int_{-\pi}^{\pi} [f^{(k)}(x)]^2\,dx \end{aligned}$$

by (17). It remains to show that the classical and the modified series are the same. The modified series (15) for $f(x)$ is

$$(18) \quad a_0^*(2\pi\lambda_0)^{-1/2} + \sum_{k=1}^{\infty}\left[a_k^*\left(\pi \sum_{i=0}^{r} \lambda_i k^{2i}\right)^{-1/2} \cos kx + b_k^*\left(\pi \sum_{i=0}^{r} \lambda_i k^{2i}\right)^{-1/2} \sin kx\right].$$

Applying (16), (18) becomes

$$(2\pi)^{-1}\int_{-\pi}^{\pi} f(x)\,dx + \sum_{k=1}^{\infty} \pi^{-1}\left[\left(\int_{-\pi}^{\pi} f(x)\cos kx\,dx\right)\cos kx + \left(\int_{-\pi}^{\pi} f(x)\sin kx\,dx\right)\sin kx\right]$$

$$= a_0 + \sum_{k=1}^{\infty} \pi^{-1/2}(a_k \cos kx + b_k \sin kx).$$

So the modified series and the classical series are the same.

COROLLARY 1. *Consider the subspace of all periodic functions f of period 2π in $W^{1,2}[-\pi,\pi]$. For all such f, Parseval's relation with respect to sines and cosines holds, and the series* (15) *and the ordinary Fourier series are the same.*

COROLLARY 2. *Parseval's relation with respect to cosines alone holds for any even function in $W^{1,2}[-\pi,\pi]$.*

Example. Let $f(x) = x^2/\pi^2 - 1$, so that $f(-\pi) = f(\pi) = 0$. Suppose $\lambda_0 = \lambda_1 = 1$. Since f is even, we need only apply (2) to find

$$(x^2/\pi^2 - 1, \quad (2\pi)^{-1/2}) = -2(2\pi)^{1/2}/3$$

for $k = 0$, and

$$(x^2/\pi^2 - 1, \quad (1 + k^2)^{-1/2}\pi^{-1/2}\cos kx) = 4(-1)^k(1 + k^2)^{1/2}\pi^{-3/2}k^{-2}$$

for $k \geqq 1$. Now

$$\|x^2/\pi^2 - 1\|^2 = 8(2\pi^2 + 5)/15\pi.$$

Hence we need to verify that

$$8(2\pi^2 + 5)/15\pi = 8\pi/9 + \sum_{k=1}^{\infty} [4(-1)^k(1 + k^2)^{1/2}\pi^{-3/2}k^{-2}]^2.$$

One sees that this reduces to showing that

$$\pi^4/90 + \pi^2/6 = \sum_{k=1}^{\infty} k^{-4} + \sum_{k=1}^{\infty} k^{-2},$$

a result that is well known [2, p. 446].

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SIAM J. MATH. ANAL.
Vol. 2, No. 4, November 1971

DISCONJUGACY TESTS FOR SINGULAR LINEAR DIFFERENTIAL EQUATIONS*

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Abstract. Special disconjugacy tests of the de la Vallée Poussin type for a closed interval $[\alpha, \beta]$, which need not be bounded, are derived for linear differential equations with continuous coefficients on the open interval (α, β). The method applies, in general, to linear perturbations of disconjugate linear equations. The results include a precise first-term asymptotic description at both α and β of a fundamental system of solutions.

1. Introduction. In this paper, the solutions of the linear differential equation

$$Ly \equiv y^{(n)} + p_1(t)y^{n-1} + \cdots + p_n(t)y = 0, \tag{1.1}$$

where $p_k \in C(\alpha, \beta)$, will be studied with respect to their zeros in $[\alpha, \beta]$, $-\infty \leqq \alpha < \beta \leqq \infty$. By a solution of (1.1), we shall mean a function in $C^n(\alpha, \beta)$ different from the identically zero function.

Special attention must be paid to the endpoints α and β since they may be singular points of the equation. By a neighborhood of α, we shall mean an open interval (α, a) with $\alpha < a \leqq \beta$, and by a neighborhood of β, we shall mean an open interval (b, β) with $\alpha \leqq b < \beta$. The following abbreviations will be used throughout the paper:

$$\lim_{t\to\alpha} \equiv \lim_{t\to\alpha+}, \quad \lim_{t\to\beta} \equiv \lim_{t\to\beta-}, \quad f(\alpha) = f(\alpha+), \quad f(\beta) = f(\beta-).$$

Finally, a continuous function f will be said to be integrable on (α, β) provided it is improperly integrable on (α, β); that is, the limit as $c \to \alpha+$ and $d \to \beta-$ of $\int_c^d f(t)\,dt$ exists. The corresponding limit will be simply denoted by $\int_\alpha^\beta f(t)\,dt$.

DEFINITION 1.1. The point β (α) is a *singular point* of (1.1) if $\beta = \infty$ $(\alpha = -\infty)$ or if one of the coefficients in the equation is not integrable in some neighborhood of β (α).

DEFINITION 1.2. An n-tuple $(u_1, \cdots, u_n)$ of solutions of (1.1) is a *principal system* at $b \in [\alpha, \beta]$ provided there exists a deleted neighborhood N of b such that:

(i) $u_k(t) > 0$, $t \in N$, $k = 1, \cdots, n$,

(ii) $\lim_{t\to b} u_k(t)/u_{k+1}(t) = 0$, $k = 1, \cdots, n-1$.

Clearly, if $(u_1, \cdots, u_n)$ is a principal system at $b \in [\alpha, \beta]$, then the set $\{u_1, \cdots, u_n\}$ is a fundamental system on (α, β) for (1.1). Hence for any solution ψ, there exist constants $c_1, \cdots, c_n$ such that

$$\psi = c_1u_1 + \cdots + c_nu_n.$$

Now, it is easy to show that if $(v_1, \cdots, v_n)$ is any other principal system at b and

$$\psi = b_1v_1 + \cdots + b_nv_n,$$

then for the pair b_k, c_j which are both different from zero and have maximum k and j, $k = j$. Hence, the following definition due to Levin [4] is well-posed.

* Received by the editors January 14, 1971, and in revised form April 17, 1971.

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DEFINITION 1.3. A solution ψ has a *zero of order* k, $0 \leqq k \leqq n-1$, at $b \in [\alpha, \beta]$ provided there exists a principal system $(u_1, \cdots, u_n)$ at b and constants $c_1, \cdots, c_{n-k}$ such that $c_{n-k} \neq 0$ and

$$\psi(t) = c_1 u_1(t) + \cdots + c_{n-k} u_{n-k}(t), \qquad \alpha < t < \beta.$$

A solution ψ has a zero of order k at b, if and only if $\lim \psi(t)/u_j(t) = 0$ for $j = n, \cdots, n-k+1$, and $\neq 0$ for $j = n-k$, as $t \to b$. For example, $\exp(t)$ has no zeros at ∞ as a solution of $(D^2 - 1)y = 0$, but has one zero at ∞ as a solution of $(D^2 - 3D + 2)y = 0$. Thus, zeros of solutions depend in general upon the operator. However, Definition 1.3 is equivalent to the usual definition of zeros at nonsingular points. Denote the number of zeros counted, as prescribed in Definition 1.3, of a solution ψ in an interval $I \subset [\alpha, \beta]$ by $Z_\psi I$, and let $Z_\psi b \equiv Z_\psi[b, b]$.

DEFINITION 1.4. Equation (1.1) is *disconjugate on* $I \subset [\alpha, \beta]$ if for any solution ψ, $Z_\psi I \leqq n - 1$. Equation (1.1) is *disconjugate at* $b \in [\alpha, \beta]$ if there exists a neighborhood N of b such that (1.1) is disconjugate on N.

It is well known that for any $b \in (\alpha, \beta)$, (1.1) is disconjugate at b because of the assumption of continuity on the coefficients. Actually, integrability at b is sufficient for this result as well as the other results stated in this paper. Levin [4] and Hartman [1] have shown that if $Ly = 0$ is disconjugate at an endpoint, say β, then $Ly = 0$ has a principal system at β. Hence, one can talk about the number of zeros $Z_\psi \beta$ of a solution ψ at a singular point β.

DEFINITION 1.5. An n-tuple $(u_1, \cdots, u_n)$ of solutions is a *fundamental principal system* on $[a, b] \subset [\alpha, \beta]$ provided $(u_1, \cdots, u_n)$ is a principal system at b and $(u_n, \cdots, u_1)$ is a principal system at a.

Definition 1.5 differs slightly from the corresponding concept used by us in [8], where we added the condition $u_k^{(k-1)}(a) = 1$ in order to obtain uniqueness. As defined above, the function u_k in a fundamental principal system is unique up to multiplication by a positive constant. Levin [4] first used the idea of a fundamental principal system in studying disconjugacy. The fundamental relationship in this regard is described by the following theorem, which is at least partially contained in the results of Levin.

THEOREM 1.1. *Equation* (1.1) *is disconjugate on* $[\alpha, \beta]$, *if and only if for any* $[\alpha, b] \subset [\alpha, \beta]$, $b > \alpha$, *there exists a fundamental principal system on* $[\alpha, b]$ *for* (1.1).

Of course, the relative roles of α and β in Theorem 1.1 can be interchanged. A short proof of Theorem 1.1 will be given at the beginning of § 2.

For very simple disconjugate equations, it is easy to show the existence of a fundamental principal system by simply getting the general solution of the equation; for example, a system on $[\alpha, \beta]$ for $L \equiv D^n$ is $(u_1, \cdots, u_n)$ with

$$u_k(t) = (\beta - t)^{n-k}(t - \alpha)^{k-1}$$

if $-\infty < \alpha < \beta < \infty$, and with

$$u_k(t) = (t - \alpha)^{k-1}$$

if $-\infty < \alpha < \beta = \infty$. Of course, $y^{(n)} = 0$ does not have a fundamental principal system on $[-\infty, \infty]$, because this equation is not disconjugate on $[-\infty, \infty]$.

Let

$$I(t,s;\xi_2,\cdots,\xi_k)=\int_s^t \xi_2(t_2)\int_s^{t_2}\cdots\int_s^{t_{k-1}}\xi_k(t_k)\,dt_k\,dt_{k-1}\cdots dt_2, \qquad k=2,\cdots,n. \tag{1.2}$$

We showed in [8] how to represent a fundamental principal system on $[\alpha,\beta]$ in terms of integrals of the form (1.2) with $s=\alpha$ provided α was not singular and $Ly=0$ was disconjugate on (α_0,β) for some $\alpha_0<\alpha$. This result can be extended to the following theorem (proof in § 2).

THEOREM 1.2. *Assume that* $-\infty \leqq \alpha < \beta \leqq \infty$ *and that* (1.1) *is disconjugate on* $[\alpha,\beta]$. *Then there exists* ξ_k, $k=1,\cdots,n$, *such that the following hold*:

(i) $\xi_k \in C^{n-k+1}(\alpha,\beta)$, $\xi_k>0$ *on* (α,β), $k=1,\cdots,n$;

(ii) ξ_k *is integrable on* $[\alpha,t]$ *for each* $\alpha<t<\beta$, *and*

$$\lim_{t\to\beta}\int_\alpha^t \xi_k(s)\,ds=\infty, \qquad k=2,\cdots,n; \tag{1.3}$$

(iii) *the fundamental principal system* $(u_1,\cdots,u_n)$ *on* $[\alpha,\beta]$ *of* (1.1) *exists, and*

$$u_1(t)=\xi_1(t), \qquad u_k(t)=\xi_1(t)I(t,\alpha;\xi_2,\cdots,\xi_k), \quad k=2,\cdots,n. \tag{1.4}$$

Once the representation (1.4) is obtained, the asymptotic theory in Willett [8] essentially carries over to the present context. In [9], we showed how to use this theory to obtain disconjugacy results for nonsingular equations. The main purpose of this paper is to extend that analysis to singular equations over closed intervals $[\alpha,\beta]$. The general result in this regard is Theorem 3.1. Examples of the type of specific disconjugacy tests implied by Theorem 3.1 are the following two results. In the statements of these results, we denote the greatest integer contained in a number x by $[x]$.

THEOREM 1.3. *If* $-\infty<\alpha<\beta<\infty$, $(t-\alpha)^{k-1}(\beta-t)^{k-1}p_k(t)$ *is integrable on* (α,β), $k=1,\cdots,n$, *and*

$$\sum_{k=1}^n A_k\int_\alpha^\beta |p_k(t)|(t-\alpha)^{k-1}(\beta-t)^{k-1}\,dt<(\beta-\alpha)^{k-1}, \tag{1.5}$$

where

$$A_1=2^{n-1}, \quad A_n=1\Big/\left[\frac{n-1}{2}\right]!\left[\frac{n}{2}\right]!, \quad A_k=\frac{2^{n-1}-1}{(k-1)!}, \qquad k=2,\cdots,n-1, \tag{1.6}$$

then (1.1) *is disconjugate on* $[\alpha,\beta]$. *Furthermore*, (1.1) *has a fundamental system* $\{w_1,\cdots,w_n\}$ *of solutions such that*

$$w_k(t)=\begin{cases}(\beta-t)^{n-k}(t-\alpha)^{k-1}[1+o(1)] & \text{as } t\to\beta,\\ c_k(\beta-t)^{n-k}(t-\alpha)^{k-1}[1+o(1)] & \text{as } t\to\alpha,\end{cases} \tag{1.7}$$

c_k, *nonzero constant*.

THEOREM 1.4. *If* $-\infty < \alpha < \infty, t^{k-1}p_k(t)$ *is integrable on* $(\alpha, \infty), k = 1, \cdots, n,$ *and*

$$\sum_{k=1}^{n} B_k \int_{\alpha}^{\infty} |p_k(t)|(t-\alpha)^{k-1}\, dt < 1, \tag{1.8}$$

where

$$B_1 = 2, \quad B_n = 1\Big/\left[\frac{n-1}{2}\right]!\left[\frac{n}{2}\right]!, \quad B_k = \frac{2^{k-1}}{(k-1)!}, \qquad k = 2, \cdots, n-1, \tag{1.9}$$

then (1.1) *is disconjugate on* $[\alpha, \infty]$. *Furthermore,* (1.1) *has a fundamental system* $\{w_1, \cdots, w_n\}$ *of solutions such that*

$$w_k(t) = \begin{cases} (t-\alpha)^{k-1}[1+o(1)] & \text{as } t \to \infty, \\ b_k(t-\alpha)^{k-1}[1+o(1)] & \text{as } t \to \alpha, \end{cases} \qquad b_k, \text{ nonzero constant.}$$

We had already shown in [9] that (1.5) was sufficient for disconjugacy on $[\alpha, \beta)$, provided $p_k \in C[\alpha, \beta)$. Hence, for a comparison of (1.5) with previous tests of a similar type as found, for example, in [2], [3], [4], [5], [9] and [10], which are often called de la Vallée Poussin-type tests [7], see [9].

2. Fundamental principal systems.

Proof of Theorem 1.1. Assume that (1.1) is disconjugate on $[\alpha, \beta]$ and that $\alpha < b \leqq \beta$. Definition 1.4 implies that (1.1) is disconjugate at α and b. Hence, from the results in either [1], [4], or [8], there exists a principal system $(u_1, \cdots, u_n)$ at b. Since $Z_{u_1}b = n-1$ by definition and (1.1) is disconjugate on $[\alpha, b]$, $Z_{u_1}[\alpha, b) = 0$; in particular, $Z_{u_1}\alpha = 0$. Thus, u_1 is positive in a neighborhood of α and $\lim_{t \to \alpha} v(t)/u_1(t)$ exists as a finite number for each solution v. Hence,

$$c_{21} = -\lim_{t \to \alpha} u_2(t)/u_1(t)$$

exists; and if

$$\varphi_1 = u_1, \qquad \varphi_2 = u_2 + c_{21}u_1,$$

then

$$\lim_{t \to \alpha} \varphi_2(t)/\varphi_1(t) = 0;$$

hence, $Z_{\varphi_2}\alpha \geqq 1$ by the comment following Definition 1.3. But $Z_{\varphi_2}\alpha > 1$ and $Z_{\varphi_2}b = n-2$, which follows from the definition of φ_2, imply $Z_{\varphi_2}[\alpha, b] \geqq n$, which contradicts the disconjugacy of (1.1) on $[\alpha, b]$. Hence, $Z_{\varphi_2}\alpha = 1$.

Suppose now that functions

$$\varphi_k = u_k + c_{k,k-1}u_{k-1} + \cdots + c_{k1}u_1, \qquad k = 2, \cdots, j-1 < n, \tag{2.1}$$

have been determined so that the c_{kj} are constants and $Z_{\varphi_k}\alpha = k-1$, $k = 1, \cdots, j-1$. Let

$$\varphi_j = u_j + b_{j,j-1}\varphi_{j-1} + \cdots + b_{j1}\varphi_1, \tag{2.2}$$

where $b_{j1} = -\lim_{t\to\alpha} u_j(t)/\varphi_1(t)$ and

$$b_{jr} = -\lim_{t\to\alpha} \frac{u_j(t) + b_{j,r-1}\varphi_{r-1}(t) + \cdots + b_{j1}\varphi_1(t)}{\varphi_r(t)}, \quad r = 2, \cdots, j-1,$$

are determined inductively. The limit defining b_{jr} necessarily exists because $Z_{\varphi_r}\alpha = r-1$ and for $v = u_j + b_{j,r-1}\varphi_{r-1} + \cdots + b_{j1}\varphi_1$, $Z_v\alpha \geqq r-1$ by the way $b_{j,r-1}, \cdots, b_{j1}$ are determined. The existence of constants $c_{j,j-1}, \cdots, c_{j1}$ so that (2.1) holds for $k = j$ follows by substituting for $\varphi_1, \cdots, \varphi_{j-1}$ in (2.2) from (2.1). Thus, the principle of finite induction implies the existence of solutions $\varphi_1, \cdots, \varphi_n$ satisfying (2.2) such that $Z_{\varphi_k}\alpha = k-1$, $k = 1, \cdots, n$. From the form of (2.2), it is clear that $Z_{\varphi_k}b = n-k$, $k = 1, \cdots, n$; hence, $(\varphi_1, \cdots, \varphi_n)$ is a fundamental principal system on $[\alpha, b]$.

Next, consider the converse and suppose $Ly = 0$ is not disconjugate on $[\alpha, \beta]$. Since $Ly = 0$ has a fundamental principal system on $[\alpha, \beta]$, $Ly = 0$ is disconjugate at both α and β. So, by a result of Levin [4, Theorem 3.3], there exists a solution ψ, integer k, and point $b \in (\alpha, \beta]$ such that $Z_\psi\alpha = n-k$, $Z_\psi b \geqq k$. Let $(u_1, \cdots, u_n)$ be a fundamental principal system for $[\alpha, b]$; hence, there exist constants $c_1, \cdots, c_n$ such that

$$\psi = c_1u_1 + \cdots + c_nu_n.$$

But $Z_\psi\alpha = n-k$ implies $c_1 = \cdots = c_{n-k} = 0$, and $Z_\psi b \geqq k$ implies $c_n = \cdots = c_{n-k+1} = 0$, which implies $\psi \equiv 0$. This contradiction implies $Ly = 0$ is disconjugate on $[\alpha, \beta]$.

Proof of Theorem 1.2. Let $(y_1, \cdots, y_n)$ be a fundamental principal system on $[\alpha, \beta]$; such a system exists by Theorem 1.1. Denote the Wronskian of k functions $u_1, \cdots, u_k$ in general by $W_k(u_1, \cdots, u_k)$; in particular, let

$$W_k = W_k(y_1, \cdots, y_k).$$

Then, Theorem 2.1 of Levin [4] implies the following:

$$W_k > 0 \quad \text{on } (\alpha, \beta), \qquad k = 1, \cdots, n, \tag{2.3}$$

$$\lim_{t\to\alpha} W_{k-1}(y_1, \cdots, y_{k-2}, y_k)/W_{k-1} = 0, \qquad k = 2, \cdots, n, \tag{2.4}$$

$$\lim_{t\to\beta} W_{k-1}(y_1, \cdots, y_{k-2}, y_k)/W_{k-1} = \infty, \qquad k = 2, \cdots, n. \tag{2.5}$$

Let $W_{-1} \equiv W_0 \equiv 1$, and define

$$\xi_k(t) = W_{k-2}W_k/W_{k-1}^2, \qquad k = 1, \cdots, n.$$

Then (cf., e.g., Pólya and Szegö [6, p. 113]),

$$\xi_k(t) = \frac{d}{dt}\frac{W_{k-1}(y_1, \cdots, y_{k-2}, y_k)}{W_{k-1}(y_1, \cdots, y_{k-1})}, \qquad k = 2, \cdots, n.$$

Hence, (2.4) implies ξ_k integrable on $[\alpha, t]$ for all $\alpha < t < \beta$, and (2.5) implies (1.3).

Since $W_k \neq 0$ on (α, β), L can be factorized as follows:

$$\begin{aligned} Ly &= \frac{W_n}{W_{n-1}} D \frac{W_{n-1}^2}{W_{n-2}W_n} \cdots D \frac{W_1^2}{W_0 W_2} D \frac{y}{W_1} \\ &= \frac{W_n}{W_{n-1}} D \frac{1}{\xi_n} \cdots D \frac{1}{\xi_2} D \frac{y}{\xi_1} . \end{aligned} \tag{2.6}$$

Let u_k, $k = 1, \cdots, n$, be defined by (1.4). Then, (2.6) implies $Lu_k = 0$, and from (1.3) and (1.4), it is clear that $(u_1, \cdots, u_n)$ is a fundamental principal system on $[\alpha, \beta]$.

THEOREM 2.1. *Assume* (1.1) *is disconjugate on* $[\alpha, \beta]$ *and let* $(u_1, \cdots, u_n)$ *be a fundamental principal system represented as in Theorem* 1.2. *Then, the Cauchy function* $g(t, s)$ *of* (1.1) *is given by*

$$\begin{aligned} g(t, s) &= u_n(t)v_n(s) - \cdots + (-1)^{n-1}u_1(t)v_1(s) \\ &= \xi_1(t)I(t, s; \xi_2, \cdots, \xi_n)v_n(s) \\ &= (-1)^{n-1}\xi_1(t)I(s, t; \xi_n, \cdots, \xi_2)v_n(s) , \end{aligned}$$

where

$$v_n(t) = [\xi_1(t) \cdots \xi_n(t)]^{-1}, \qquad v_k(t) = v_n(t)I(t, \alpha; \xi_n, \cdots, \xi_{k+1}), \qquad k = 1, \cdots, n-1.$$

Furthermore, $(v_n, \cdots, v_1)$ *is a fundamental principal system on* $[\alpha, \beta]$ *for the formal adjoint equation to* (1.1).

Proof. The proof is identical to the corresponding part of the proofs of Theorems 1.1 and 1.2 in Willett [8], where α was assumed to be a nonsingular point. All that was required in those proofs was the appropriate representation (1.4) of a fundamental system of solutions, which Theorem 1.2 gives us in the present situation.

Now let u_k and v_k be defined as in Theorem 2.1, and let j be a fixed integer such that $1 \leqq j \leqq n$. Define

$$h_j(t, s) = \begin{cases} \displaystyle\sum_{k=1}^{j-1} (-1)^{n-k}u_k(t)(v_k(s)/v_j(s))', & \alpha < s < t, \\ \displaystyle\sum_{k=j+1}^{n} (-1)^{n-k+1}u_k(t)(v_k(s)/v_j(s))', & t \leqq s < \beta, \end{cases} \tag{2.7}$$

where $\sum_{k=1}^{j-1} \equiv 0$ if $j = 1$ and $\sum_{k=j+1}^{n} \equiv 0$ if $j = n$.

THEOREM 2.2. *As a function of* s, $h_j(t, s)$ *is integrable on* (α, β) *and*

$$(-1)^{n-j-1}h_j(t, s) > 0, \qquad s \neq t, \tag{2.8}$$

$$u_j(t) = (-1)^{n-j-1} \int_\alpha^\beta h_j(t, s)\, ds. \tag{2.9}$$

Furthermore, if $v_j f$ is integrable on (α, β) and

$$w_j(t) = \int_\alpha^\beta h_j(t, s)\left(\int_s^\beta v_j(\tau)f(\tau)\,d\tau\right) ds, \tag{2.10}$$

then

$$Lw_j = f \quad on\ (\alpha, \beta), \tag{2.11}$$

$$\lim_{t\to\beta} w_j(t)/u_j(t) = 0, \tag{2.12}$$

$$\lim_{t\to\alpha} w_j(t)/u_j(t) = (-1)^{n-j-1}\int_\alpha^\beta v_j(s)f(s)\,ds. \tag{2.13}$$

Proof. From the properties of the functions $\xi_1, \cdots, \xi_n$ given in Theorem 1.2 and the definition of $v_1, \cdots, v_n$ given in Theorem 2.1, it is clear that $(v_k/v_j)'$ is integrable on (α, t) when $k < j$ and is integrable on (t, β) when $k > j$, for all $\alpha < t < \beta$. Hence, $h_j(t, s)$ is an integrable function of s on (α, β). The proofs of (2.8)–(2.12) are identical to the proofs of the corresponding parts of Theorems 3.1 and 3.2 in Willett [8]. To obtain (2.13), note that

$$w_j(t) = \int_\alpha^\beta h_j(t, s)\,ds\int_\alpha^\beta v_j(\tau)f(\tau)\,d\tau - \int_\alpha^\beta h_j(t, s)\left(\int_\alpha^s v_j(\tau)f(\tau)\,d\tau\right) ds; \tag{2.14}$$

and since (2.12) with α and β interchanged is just

$$\int_\alpha^\beta h_j(t, s)\left(\int_\alpha^s v_j(\tau)f(\tau)\,d\tau\right) ds = o(u_j(t)) \quad \text{as } t \to \alpha,$$

(2.9) and (2.14) imply (2.13).

Consider now the perturbed linear equation

$$Ly = f[y(t)] \equiv r_1(t)y^{(n-1)}(t) + \cdots + r_n(t)y(t), \tag{2.15}$$

where $r_k \in C(\alpha, \beta)$, $k = 1, \cdots, n$, and (1.1), that is $Ly = 0$, is disconjugate on $[\alpha, \beta]$. Theorem 2.2 implies that any solution $y \in C^n(\alpha, \beta)$ of the integro-differential equation

$$y(t) = u_j(t) + \int_\alpha^\beta h_j(t, s)\left(\int_s^\beta v_j(\tau)f[y(\tau)]\,d\tau\right) ds \tag{2.16}$$

is a solution of the differential equation (2.15).

THEOREM 2.3. *Assume* (1.1) *is disconjugate on* $[\alpha, \beta]$. *If for each* j, $j = 1, \cdots, n$, *there exists a solution* $y_j \in C^n(\alpha, \beta)$ *of* (2.16) *such that*

$$c_j = \int_\alpha^\beta v_j(\tau)f[y_j(\tau)]\,d\tau \tag{2.17}$$

exists (*is a finite number*), *then* $(z_1, \cdots, z_n)$, *where*

$$z_j = y_j + |c_j|u_j, \qquad j = 1, \cdots, n,$$

is a fundamental principal system on $[\alpha, \beta]$ *for* (2.15).

Proof. Theorem 2.2 implies

$$z_j(t) = u_j(t) + o(u_j(t)) + |c_j|u_j(t) \quad \text{as } t \to \beta;$$

hence, $z_j(t)$ is positive in some neighborhood of β and

$$\lim_{t\to\beta} z_j(t)/z_{j+1}(t) = 0, \qquad j = 1, \cdots, n-1;$$

that is, $(z_1, \cdots, z_n)$ is a principal system at β. Theorem 2.2 also implies

$$\lim_{t\to\alpha} z_j(t)/u_j(t) = 1 + |c_j| + (-1)^{n-j-1}c_j \geqq 1;$$

hence, $z_j(t)$ is positive in some neighborhood of α and

$$\lim_{t\to\alpha} \frac{z_{j+1}(t)}{z_j(t)} = \lim_{t\to\alpha} \frac{z_{j+1}(t)}{u_{j+1}(t)} \lim_{t\to\alpha} \frac{u_j(t)}{z_j(t)} \lim_{t\to\alpha} \frac{u_{j+1}(t)}{u_j(t)} = 0;$$

that is, $(z_n, \cdots, z_1)$ is a principal system at α.

3. Disconjugacy tests for perturbations of disconjugate equations. Theorems 3.1 and 2.3 reduce the question of disconjugacy for the perturbed equation (2.15) to finding conditions under which the integro-differential equations

$$y_j(t,b) = u_j(t,b) + \int_\alpha^b h_j(t,s,b)\left(\int_s^b v_j(\tau,b)f[y_j(\tau,b)]\,d\tau\right) ds, \tag{3.1}$$

$j = 1, \cdots, n$, $\alpha < b \leqq \beta$, have solutions. One should not forget that (3.1) is in general a singular equation and the existence of a solution is a nontrivial question. However, since $f[y] \equiv r_1(t)y^{(n-1)} + \cdots + r_n(t)y$ is linear in $y, y', \cdots, y^{(n-1)}$, one can quite naturally apply successive approximations or the principle of contraction mappings to (3.1).

Let

$$\begin{aligned} \rho_1(t) &= v_j(t,b)u_j(t,b), \\ \rho_n(t) &= 1 + v_j(t,b)\int_\alpha^b \left|\frac{\partial^{n-1}h_j}{\partial t^{n-1}}(t,s,b)\right| ds, \\ \rho_k(t) &= v_j(t,b)\int_\alpha^b \left|\frac{\partial^{k-1}h_j}{\partial t^{n-1}}(t,s,b)\right| ds, \qquad k = 2, \cdots, n-1. \end{aligned} \tag{3.2}$$

Actually, $\rho_k(t) \equiv \rho_k(t,j,b)$; hence, let

$$\sigma_k(t) = \max_{j=1,\cdots,n}\ \sup_{\alpha<b\leqq\beta} \rho_k(t,j,b), \qquad k = 1, \cdots, n. \tag{3.3}$$

THEOREM 3.1. *If* (1.1) *is disconjugate on* $[\alpha, \beta]$ *and*

$$v \equiv \sum_{k=1}^n \int_\alpha^\beta |r_k(t)|\sigma_{n-k+1}(t)\,dt < 1, \tag{3.4}$$

then (2.15) *is disconjugate on* $[\alpha, \beta]$. *Furthermore,* (2.15) *has a fundamental system* $\{w_1, \cdots, w_n\}$ *of solutions such that*

$$w_j(t) = u_j(t,\beta)[1 + o(1)] \quad \text{as } t \to \beta, \tag{3.5}$$

$$w_j(t) = b_ju_j(t,\beta)[1 + o(1)] \quad \text{as } t \to \alpha, \quad b_j, \text{ nonzero constant}. \tag{3.6}$$

Proof. Let b, $a < b \leqq \beta$, and j, $1 \leqq j \leqq n$, be given and fixed in what follows. For $z \in C^n(\alpha, b)$, define

$$\|z\| = \max_{k=1,\cdots,n} \sup_{\alpha < t < b} \frac{z^{(k-1)}(t) v_j(t, b)}{\rho_k(t, j, b)}. \tag{3.7}$$

Let

$$B = \{z \in C^n(\alpha, b) : \|z\| < \infty\}$$

so that B with norm defined by (3.7) is a Banach space. Furthermore, $\|u_j\| \leqq 1$ by (2.9), and so $u_j \in B$. For $z \in B$, let

$$Tz = u_j(t, b) + \int_\alpha^b h_j(t, s, b)\left(\int_s^b v_j(\tau, b) f[z(\tau)]\, d\tau\right) ds.$$

Then for $y, z \in B$,

$$\|Tz - Ty\| \leqq \|z - y\| \sum_{k=1}^{n} \int_\alpha^b |r_k(t)| \rho_{n-k+1}(t)\, dt \leqq \nu \|z - y\|;$$

hence, for $y \in B$,

$$\|Ty\| \leqq \|Ty - T0\| + \|u_j\| \leqq \|u_j\| + \nu\|y\| < \infty,$$

and so $TB \subset B$. Since $\nu < 1$ by assumption, the contraction mapping principle implies that there exists a unique $y_j \in B$ such that $Ty_j = y_j$; that is, y_j is a solution of (3.1).

To conclude the proof, let $b = \beta$ and

$$w_j(t) = [y_j(t, \beta) + |c_j| u_j(t, \beta)]/(1 + |c_j|),$$

where c_j is defined as in (2.17).

We note that the roles of α and β can be interchanged in the above development; that is, we can let $(u_1, \cdots, u_n)$ with $u_j = u_j(t, a)$, $\alpha \leqq a < \beta$, be the fundamental principal system on $[a, \beta]$. Accordingly,

$$\sigma_k(t) = \max_{j=1,\cdots,n} \sup_{\alpha \leqq a < \beta} \rho_k(t, a, j), \tag{3.8}$$

and Theorem 3.1 holds as stated.

Proof of Theorems 1.3 *and* 1.4. These two theorems are essentially special cases of Theorem 3.1. In these cases, $L = D^n$ and special estimates are made for σ_k, $k = 1, \cdots, n$. To see how the estimates are made in the case of Theorem 1.3, see the proof of Theorem 1.1 in Willett [9].

In the case of Theorem 1.4, the version of Theorem 3.1 needed is the one with $\beta \equiv \infty$ fixed and σ_k defined by (3.8) with $\alpha \leqq a < \infty$. In this case, a fundamental principal system for $D^n y = 0$ on $[a, \infty]$ is $(u_1, \cdots, u_n)$, where

$$u_k(t, a) = (t - a)^{k-1}/k - 1!, \qquad k = 1, \cdots, n.$$

It follows that $v_k = u_{n-k+1}$, $k = 1, \cdots, n$; hence

$$\rho_k(t) \leqq (t - a)^{n-k} B_{n-k+1} \leqq (t - \alpha)^{n-k} B_{n-k+1}, \qquad k = 1, \cdots, n.$$

Thus, (1.8) implies (3.4), and the conclusion follows.

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SIAM J. MATH. ANAL.
Vol. 2, No. 4, November 1971

THE ASYMPTOTIC DISTRIBUTION OF EIGENVALUES FOR THE BOUNDARY VALUE PROBLEM $y''(x) - \lambda^2 p(x)y(x) = 0, y \in L_2(-\infty, +\infty)$*

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Abstract. In this paper we study the problem $y''(x) - \lambda^2 p(x)y(x) = 0$, $y(x) \in L_2(-\infty, +\infty)$. Sufficient conditions are given on $p(x)$ to insure the existence of eigenvalues and to enable one to compute the asymptotic distribution of all large positive eigenvalues. The equation considered is of a form of interest in wave mechanics and particle scattering.

1. Introduction. We are concerned with the eigenvalue problem

$$y''(x) - \lambda^2 p(x)y(x) = 0, \tag{1.1}$$

$$y \in L_2(-\infty, +\infty), \tag{1.2}$$

where $p(x)$ is a real-valued function of the real variable x and $y \in L_2(a, b)$ means $\int_a^b |y(x)|^2 \, dx < \infty$. In particular we are concerned with the asymptotic distribution of large positive eigenvalues for this problem. We shall show that for $p(x)$ in a certain class of functions, there corresponds to each negative minimum of $p(x)$ an increasing sequence $\lambda_{k,l}$ of eigenvalues for $k = 1, \cdots, m$ such that for each k the eigenvalue sequence has an asymptotic expansion in powers of $1/l$ as $l \to +\infty$.

The asymptotic relation is denoted by "$\sim$", and by

$$f(x, \omega) \sim \sum_{n=0}^{\infty} a_n(x)\omega^n$$

we mean

$$\lim_{|\omega| \to \infty} \omega^{-m}\left[f(x, \omega) - \sum_{n=0}^{m} a_n(x)\omega^n\right] = 0$$

for ω in some sector and x in some set and $m = 0, 1, 2, \cdots$. By means of the Liouville transformation, (1.1) can be reduced to the form $y''(x) + (\lambda^2 + Q(x))y(x) = 0$.

The boundary condition, $y \in L_2(-\infty, +\infty)$, imposes certain restrictions on $p(x)$. Let $p(x) > 0$ and let $A(x) = +\sqrt{p(x)}$. Then, if

$$\int_{-\infty}^{+\infty} \left| \frac{A''(x)}{2A^2(x)} - \frac{3(A'(x))^2}{4A^3(x)} \right| dx < \infty,$$

the differential equation (1.1) possesses two solutions $y_+(x)$ and $y_-(x)$ which have the WKBJ representations

$$y_\pm(x) = \left[|A(x)|^{-1/2} \exp\left(\pm\lambda \int^x A(\tau)\, d\tau\right)\right][1 + O(\lambda^{-1})] \tag{1.3}$$

* Received by the editors June 11, 1970, and in final revised form March 25, 1971.

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valid for $x \in (-\infty, +\infty)$; see, for example, Langer [6, p. 550]. If $p(x)$ only has polynomial-like behavior as $|x| \to \infty$, then neither solution will be square-integrable on $(-\infty, +\infty)$; however, $y_+(x)$ ($y_-(x)$) tends to zero exponentially as $x \to -\infty$ ($x \to +\infty$). Hence $p(x)$ must possess at least one zero for the existence of eigenvalues of the problem (1.1), (1.2). We note, however, that while the WKBJ representation (1.3) has a singularity at the zeros of $p(x)$, actual solutions of the differential equation (1.1) are analytic for all real x. Under appropriate assumptions there exist solutions having the representations (1.3) on intervals not containing zeros of $p(x)$, but these solutions will, in general, differ from interval to interval.

The zeros of $p(x)$ are referred to as turning points or transition points, and the problem of representation of solutions at such points is very difficult and has not yet been fully solved.

If $p(x)$ has polynomial-like behavior as $|x| \to \infty$ and if $p(x) > 0$ for $|x|$ sufficiently large and $x > 0$ ($x < 0$), then the representations (1.3) show the existence of a solution $y_-(x)$ ($y_+(x)$) which approaches zero exponentially as $x \to +\infty$ ($x \to -\infty$) and $y_-(x) \in L_2(a, +\infty)$ ($y_+(x) \in L_2(-\infty, a)$) for any finite a. However, if $p(x) < 0$ for $|x|$ sufficiently large and $x > 0$ ($x < 0$), then the representations (1.3) exhibit oscillatory behavior and the question of square integrability is more delicate and will not be treated here. Our techniques require the zeros of $p(x)$ to be simple, and hence we shall assume $p(x)$ satisfies the following hypotheses for the remainder of this paper:

(H.1) *$p(x)$ is a real analytic function for $x \in (-\infty, +\infty)$;*

(H.2) *$p(x)$ possesses a finite, even number of simple real zeros a_i, $i = 1, \cdots, 2m$, with $a_{2m} < a_{2m-1} < \cdots < a_1$;*

(H.3) *$p(x) = c_0 x^l[1 + q(x)]$ with $c_0 > 0$, $l \geqq 0$ an integer, and*

$$\lim_{|x| \to \infty} q(x) = 0.$$

We may consider $p(x)$ as a complex analytic function of a complex variable x in a region (by a region we mean an open connected set together with all or part of its boundary) $\tilde{\mathscr{R}}$ containing the real axis and such that $p(x) \neq 0$ in $\tilde{\mathscr{R}}$ if $x \neq a_i$, $i = 1, \cdots, 2m$. Denoting by I_k the interval $a_{k+1} \leqq x \leqq a_k$, we can define a single-valued analytic branch of $p^{1/2}(x)$, denoted by $A(x)$, in $\tilde{\mathscr{R}} - \bigcup_{k=1}^{m} I_{2k-1}$ such that $A(x) > 0$ for $x > a_1$. For τ real and $\tau \neq a_i$, $i = 1, \cdots, 2m$, we define $A(x)$ on the upper and lower cuts by $A(\tau+) = \lim_{t \to 0^+} A(\tau + it)$ and $A(\tau-) = \lim_{t \to 0^-} A(\tau + it)$ respectively.

Solutions of (1.1) can be written in the form

$$y(x) = \exp\left(\int^x u(s)\, ds\right), \tag{1.4}$$

where $u(x)$ satisfies the Riccati differential equation

$$u'(x) + u^2(x) = \lambda^2 p(x). \tag{1.5}$$

The Riccati differential equation (1.5) possesses two formal solutions $u_+(x, \lambda)$ and $u_-(x, \lambda)$,

$$u_+(x, \lambda) = \lambda A(x) - \frac{1}{4}\frac{p'(x)}{p(x)} + P(x, \lambda) \tag{1.6}$$

and

$$u_-(x, \lambda) = -\lambda A(x) - \frac{1}{4}\frac{p'(x)}{p(x)} + P(x, -\lambda), \tag{1.7}$$

where

$$P(x, \lambda) = \sum_{n=1}^{\infty} (\lambda A(x))^{-n} P_n(x) \tag{1.8}$$

and the $P_n(x)$ are determined recursively.

We now state Lemma 1.1; the proof will be given in §§ 2 and 3.

LEMMA 1.1. *For all $X > a_1$ $(X < a_{2m})$ there exists a $\lambda_0 > 0$ such that the differential equation* (1.5) *possesses a solution $u_0(x, \lambda)(u_{2m}(x, \lambda))$ with the properties*:

(i)
$$u_0(x, \lambda) + \lambda A(x) + \frac{1}{4}\frac{p'(x)}{p(x)} \sim P(x, -\lambda)$$

$$\left(u_{2m}(x, \lambda) - (-1)^m \lambda A(x) + \frac{1}{4}\frac{p'(x)}{p(x)} \sim P(x, (-1)^m \lambda)\right),$$

as $|\lambda A(x)| \to \infty$ *for* $x \in [X, +\infty)$ $(x \in (-\infty, X])$ *and* $\lambda \in S_0$, *where* $S_0 = \{\lambda | \mathrm{Re}\,\lambda \geqq \lambda_0, |\mathrm{Im}\,\lambda| \leqq \lambda_1\}$.

(ii)
$$y_0(x, \lambda) = \exp\left(\int^x u_0(s, \lambda)\, ds\right)$$

$$\left(y_{2m}(x, \lambda) = \exp\left(\int^x u_{2m}(x, \lambda)\, ds\right)\right)$$

is the solution of the differential equation (1.1) *satisfying* $y_0 \in L_2[X, +\infty)$ $(y_{2m} \in L_2(-\infty, X))$.

(iii) $y_0(x, \lambda)$ $(y_{2m}(x, \lambda))$ *is analytic in* λ *for* $\lambda \in S_0$.

By analogy with the terminology when $p(x)$ is a polynomial, we shall refer to $y_0(x, \lambda)(y_{2m}(x, \lambda))$ as the subdominant solution as $x \to +\infty$ $(x \to -\infty)$. Moreover, we know $y_0(x, \lambda)$ and $y_{2m}(x, \lambda)$ are solutions of the differential equation (1.1) for all real x. Hence the problem of existence of eigenvalues can be phrased in terms of the linear dependence of $y_0(x, \lambda)$ and $y_{2m}(x, \lambda)$ by using the Wronskian determinant:

$$\begin{vmatrix} y_0(x, \lambda) & y_{2m}(x, \lambda) \\ y_0'(x, \lambda) & y_{2m}'(x, \lambda) \end{vmatrix} = 0. \tag{1.9}$$

Thus, λ is an eigenvalue if and only if λ satisfies (1.9). We note that the Wronskian in (1.9) is independent of x since (1.1) contains no first derivative of $y(x)$.

Our knowledge of the asymptotic behavior of $y_0(x, \lambda)$ and $y_{2m}(x, \lambda)$ is on the intervals $[X, +\infty)$ for $X > a_1$ and $(-\infty, X]$ for $X < a_{2m}$ respectively. Since these intervals are disjoint we cannot use this asymptotic behavior to study (1.9) directly. To overcome this difficulty we shall construct m pairs of regions and m pairs of linear independent solutions of (1.1) which will enable us to connect the two subdominant solutions of Lemma 1.1. We now state Lemma 1.2; the proof is given in § 4.

LEMMA 1.2. *There exist m pairs of regions $D_{+(2k-1)}$ and $D_{-(2k-1)}$, $k = 1, \cdots, m$, and m pairs of functions $u_{+(2k-1)}(x, \lambda)$ and $u_{-(2k-1)}(x, \lambda)$, $k = 1, \cdots, m$, and $\lambda_2 > 0$ such that*

(i) $$D_{+1} \cap D_{-1} \cap [X, +\infty) \neq \varnothing$$

for $X > a_1$,

$$D_{+(2k+1)} \cap D_{-(2k+1)} \cap D_{+(2k-1)} \cap D_{-(2k-1)} \neq \varnothing,$$

$k = 1, \cdots, m - 1$,

$$D_{+(2m-1)} \cap D_{-(2m-1)} \cap (-\infty, X] \neq \varnothing$$

for $X < a_{2m}$;

(ii) $$u_{\pm(2k-1)}(x, \lambda) - (-1)^{k+1}\lambda A(x) + \frac{1}{4}\frac{p'(x)}{p(x)} \sim P(x, (-1)^{k+1}\lambda)$$

as $|\lambda A(x)| \to \infty$ *for* $x \in D_{\pm(2k-1)}$ *and*

$$\lambda \in S_2 = \{\lambda | \operatorname{Re} \lambda > \lambda_2, |\operatorname{Im} \lambda| \leqq \lambda_1\};$$

(iii) $$y_{\pm(2k-1)}(x, \lambda) = \exp\left(\int^x u_{\pm(2k-1)}(s, \lambda)\, ds\right)$$

are linear independent solutions of the differential equation (1.1) *for* $k = 1, \cdots, m$ *and* $|\lambda|$ *sufficiently large with* $\lambda \in S_2$.

From Lemma 1.2 we know there exist constants, connection coefficients, depending only on λ such that

$$\begin{aligned}
y_0 &= C_0^{+1}(\lambda)y_{+1} &&+ C_0^{-1}(\lambda)y_{-1},\\
y_{+3} &= C_{+3}^{+1}(\lambda)y_{+1} &&+ C_{+3}^{-1}(\lambda)y_{-1},\\
y_{+(2j-1)} &= C_{+(2j-1)}^{+(2j-3)}(\lambda)y_{+(2j-3)} &&+ C_{+(2j-1)}^{-(2j-3)}(\lambda)y_{-(2j-3)},\\
y_{-(2j-1)} &= C_{-(2j-1)}^{+(2j-3)}(\lambda)y_{+(2j-3)} &&+ C_{-(2j-1)}^{-(2j-3)}(\lambda)y_{-(2j-3)},\\
y_{2m} &= C_{2m}^{+(2m-1)}(\lambda)y_{+(2m-1)} &&+ C_{2m}^{-(2m-1)}(\lambda)y_{-(2m-1)}
\end{aligned} \tag{1.10}$$

and

$$y_{2m} = C_{2m}^{+1}(\lambda)y_{+1} + C_{2m}^{-1}(\lambda)y_{-1},$$

where the (x, λ) dependence of the y's has been suppressed. We introduce the notation

$$W_{j,k}(\lambda) \equiv \begin{vmatrix} y_j(x, \lambda) & y_k(x, \lambda) \\ y_j'(x, \lambda) & y_k'(x, \lambda) \end{vmatrix}.$$

Hence (1.9) can be written as

$$W_{0,2m}(\lambda) = \begin{vmatrix} C_0^{+1} & C_0^{-1} \\ C_{2m}^{+1} & C_{2m}^{-1} \end{vmatrix} W_{+1,-1}(\lambda) = 0.$$

From Lemma 1.2, this is equivalent to the equation

$$(1.11)\qquad \begin{vmatrix} C_0^{+1} & C_0^{-1} \\ C_{2m}^{+1} & C_{2m}^{-1} \end{vmatrix} = 0.$$

By use of the system (1.10) we can evaluate equation (1.11) asymptotically using the asymptotic expansions given by Lemma 1.1 and Lemma 1.2. This enables us to prove the following.

THEOREM. *Let*

$$\lambda_k(l) = \frac{(l - 1/2)\pi}{\int_{a_{2k}}^{a_{2k-1}} \sqrt{|p(x)|}\, dx} + \sum_{n=1}^{\infty} \mathscr{E}_{k,n} l^{-n},$$

$k = 1, \cdots, m$, *be formal power series in* l^{-1} *which satisfy the formal equations*

$$\lambda_k \int_{a_{2k}}^{a_{2k-1}} \sqrt{|p(x)|}\, dx = \left(l - \frac{1}{2}\right)\pi - \frac{i}{2} \sum_{j=1}^{\infty} \left[\oint_{\Gamma_k} [(-1)^{k+1}\lambda_k A(x)]^{-j} P_j(x)\, dx \right],$$

where Γ_k *is a contour which encloses* a_{2k} *and* a_{2k-1} *only, and the integration is taken in the counterclockwise sense. Then there is a positive integer* l_0 *such that we can denote all large positive eigenvalues of the problem* (1.1), (1.2) *by* $\lambda_{k,l}$ *for* $l = l_0$, $l_0 + 1, \cdots$ *in such a way that*

$$\lambda_{k,l} = \frac{(l - 1/2)\pi}{\int_{a_{2k}}^{a_{2k-1}} \sqrt{|p(x)|}\, dx} + \sum_{n=1}^{N} \mathscr{E}_{k,n} l^{-n} + O(l^{-N-1})$$

as $l \to +\infty$, *where* N *is any positive integer.*

When $p(x)$ is a polynomial which possesses $2m$ simple real zeros and no other zeros, the problem (1.1), (1.2) has been considered by Evgrafov and Fedoryuk [2] and Sibuya [7]. By taking $p(x)$ to be a polynomial these authors were able to use the theory of irregular singular points at $x = \infty$. In the proof of Lemma 1.2 we need to study the properties of certain curves, the Stokes' curves, in the complex plane. These curves are, in general, very complicated and the assumption that $p(x)$ is a polynomial enables one to study the curves in the entire complex plane. By restricting our analysis to a local neighborhood of the real axis we are able to relax the requirement that $p(x)$ be a polynomial. The technique of utilizing the complex plane in order to bypass a real transition point appears to have its origin in a method suggested by Zwaan [9]. The mathematical rigor of Zwaan's treatment is unclear; see, for example, Birkhoff [1]. A similar approach of utilizing the complex plane has been employed by Fröman and Fröman [3].

Evgrafov and Fedoryuk computed connection coefficients about each transition point one at a time in order to connect the two subdominant solutions of Lemma 1.1. In our work the connection coefficients are computed about pairs of zeros. This technique was introduced by Sibuya and simplifies the computations.

Equations of the type considered here are of interest in the study of one-dimensional particle scattering and wave mechanics. In particular, we can study differential equations such as

$$y''(x) - \lambda^2\left(\frac{x^2 - 1}{x^2 + 1}\right) y(x) = 0.$$

2. A related integral equation. In this section we shall obtain an integral equation whose solution will yield solutions of the original differential equation (1.1). This technique has been used extensively in ordinary differential equations. The integral equation will be solved by the method of successive approximations.

The change of variables

$$y(x, \lambda) = \exp\left(\int^x u(s, \lambda)\, ds\right) \tag{2.1}$$

transforms the original differential equation into the Riccati differential equation

$$u'(x, \lambda) + u^2(x, \lambda) = \lambda^2 p(x). \tag{2.2}$$

The assumptions (H.1), (H.2) and (H.3) allow us to extend $p(x)$ as a complex analytic function in some region $\tilde{\mathscr{R}}$ of the complex plane containing the real axis. We further may assume that $p(x)$ possesses no zeros in the closure of $\tilde{\mathscr{R}}$ other than $x = a_1, x = a_2, \cdots$, and $x = a_{2m}$. Without loss of generality we may take $a_{2m} < 0 < a_1$ since this condition can always be achieved by a linear translation of the x variable. Consequently in $\tilde{\mathscr{R}} - \bigcup_{k=1}^{m} I_{2k-1}$, where

$$I_k = \{x | a_{k+1} \leqq x \leqq a_k\},$$

we can define a single-valued analytic branch $A(x)$ of $p^{1/2}(x)$ by taking $A(x) > 0$ for $x > a_1$. It is known [7, p. 238] that (2.2) possesses two formal solutions $u_+(x, \lambda)$ and $u_-(x, \lambda)$,

$$u_+(x, \lambda) = \lambda A(x) - \frac{1}{4}\frac{p'(x)}{p(x)} + P(x, \lambda), \tag{2.3}$$

$$u_-(x, \lambda) = -\lambda A(x) - \frac{1}{4}\frac{p'(x)}{p(x)} + P(x, -\lambda), \tag{2.4}$$

where

$$P(x, \lambda) = \sum_{n=1}^{\infty} [\lambda A(x)]^{-n} P_n(x) \tag{2.5}$$

and

$$P_1(x) = \frac{1}{2}\left\{\frac{1}{4}\left(\frac{p'(x)}{p(x)}\right)' - \frac{1}{16}\left(\frac{p'(x)}{p(x)}\right)^2\right\}, \tag{2.6}$$

$$P_{n+1}(x) = \frac{1}{2}\left\{\frac{1}{2}(n+1)\frac{p'(x)}{p(x)} P_n(x) - P_n'(x) - \sum_{j+h=n} P_j(x) P_h(x)\right\}, \qquad n \geqq 1. \tag{2.7}$$

We would like to show that (2.2) possesses actual solutions whose asymptotic behavior, as $|\lambda A(x)| \to \infty$ in some appropriate sector and x in some subregion of $\tilde{\mathscr{R}}$, is described by one of the forms (2.3) or (2.4). We note, however, that the zeros of $p(x)$ are singularities of the representations (2.3) and (2.4). Hence we would expect to have to restrict x to be bounded away from the transition points a_i. Let $B(a_i, \varepsilon)$ for $\varepsilon > 0$ be the set $\{x \in \tilde{\mathscr{R}} | \, |x - a_i| < \varepsilon\}$ and let $\mathscr{R} = \tilde{\mathscr{R}} - \bigcup_{k=1}^{m} I_{2k-1}$. Then in a region $\mathscr{R}_\varepsilon = \mathscr{R} - \bigcup_{i=1}^{2m} B(a_i, \varepsilon)$ we have that $A(x)$ is well-defined and x is bounded away from all transition points.

Let us suppose that λ lies in a strip $\tilde{S}$ given by

$$\tilde{S} = \{\lambda | \operatorname{Re} \lambda \geqq \Lambda_0 > 0, |\operatorname{Im} \lambda| \leqq \Lambda_1\}. \tag{2.8}$$

We observe that for $x \in \mathscr{R}_\varepsilon$, $\lambda \in \tilde{S}$, and $n = 1, 2, \cdots,$

$$|P_n(x)| \leqq M_n(\varepsilon) < \infty, \quad |P_n'(x)| \leqq M_n(\varepsilon) < \infty, \quad |P_n''(x)| \leqq M_n(\varepsilon) < \infty. \tag{2.9}$$

Our assumptions on $p(x)$ are such that for $x \in R_\varepsilon$ we have

$$|p(x)| \geqq \beta^2(\varepsilon) > 0$$

for some $\beta(\varepsilon) > 0$, and hence we have

$$|A(x)| \geqq \beta(\varepsilon) > 0 \tag{2.10}$$

for $x \in \mathscr{R}_\varepsilon$. Thus for $\lambda \in \tilde{S}$ and $x \in \mathscr{R}_\varepsilon$ we have

$$|\lambda A(x)| \geqq \Lambda_0 \beta(\varepsilon) > 0. \tag{2.11}$$

Furthermore, for $\lambda \in \tilde{S}$ we have

$$|\arg \lambda| \leqq \tan^{-1} (\Lambda_1/\Lambda_0) < \pi/2, \tag{2.12}$$

and for $x \in \mathscr{R}_\varepsilon$,

$$|\arg A(x)| \leqq \theta_1 < \infty \tag{2.13}$$

for some θ_1. From the inequalities (2.12) and (2.13) we see that $\lambda A(x)$ lies in a sector $\mathscr{S}_\varepsilon$ given by

$$\mathscr{S}_\varepsilon = \{\mu | |\mu| \geqq \Lambda_0 \beta(\varepsilon), |\arg \mu| < \theta_1 + \pi\}. \tag{2.14}$$

Because of the inequalities (2.9), we can use the Borel–Ritt theorem [7, § 9] to assert the existence of a function $g(x, \mu)$ and a $\mu_0 > 0$ such that:

(a) $g(x, \mu)$ is analytic in $\mathscr{R}_\varepsilon \times \mathscr{F}$, where $\mathscr{F}$ is the sector

$$\{\mu | |\mu| > \mu_0, |\arg \mu| < \theta_1 + 2\pi\};$$

(b)

$$\left| g(x, \mu) - \sum_{n=1}^{N} \mu^{-n} P_n(x) \right| \leqq E_N(\varepsilon) |\mu|^{-N-1}, \tag{2.15}$$

$$\left| \frac{\partial g}{\partial x}(x, \mu) - \sum_{n=1}^{N} \mu^{-n} P_n'(x) \right| \leqq E_N(\varepsilon) |\mu|^{-N-1}, \tag{2.16}$$

$$\left| \frac{\partial g}{\partial \mu}(x, \mu) - \sum_{n=1}^{N} (-n\mu^{-n-1} P_n(x)) \right| \leqq E_N(\varepsilon) |\mu|^{-N-2} \tag{2.17}$$

with $E_n(\varepsilon) < \infty$ for $n = 1, 2, \cdots$, $x \in \mathscr{R}_\varepsilon$ and $\mu \in \mathscr{F}$.

Using $g(x, \mu)$ we define another function $h_k(x, \lambda)$ by

$$h_k(x, \lambda) = g(x, (-1)^{k+1} \lambda A(x)). \tag{2.18}$$

We observe that by choosing Λ_0 sufficiently large we can assure that

$$|(-1)^{k+1} \lambda A(x)| \geqq \mu_0$$

and

$$|\arg((-1)^{k+1}\lambda A(x))| < \theta_1 + \pi + \pi/2 < \theta_1 + 2\pi.$$

Hence, $h_k(x, \lambda)$ is well-defined for $x \in \mathscr{R}_\varepsilon$ and λ in $\tilde{S}$ and satisfies:

(a) $h_k(x, \lambda)$ is analytic for $x \in \mathscr{R}_\varepsilon$ and $\lambda \in \tilde{S}$;

(b)

$$\left| h_k(x, \lambda) - \sum_{n=1}^{N} [(-1)^{k+1}\lambda A(x)]^{-n} P_n(x) \right| \leqq F_N(\varepsilon)|\lambda A(x)|^{-N-1}, \tag{2.19}$$

$$\begin{aligned}\left| h_k'(x, \lambda) - \sum_{n=1}^{N} [(-1)^{k+1}\lambda A(x)]^{-n} \left[P_n'(x) - \frac{n}{2}\frac{p'(x)}{p(x)} P_n(x) \right] \right| \\ \leqq F_N(\varepsilon)|\lambda A(x)|^{-N-1}\end{aligned} \tag{2.20}$$

for $x \in \mathscr{R}_\varepsilon$ and $\lambda \in \tilde{S}$.

Setting

$$u(x, \lambda) = (-1)^{k+1}\lambda A(x) - \frac{1}{4}\frac{p'(x)}{p(x)} + h_k(x, \lambda) + \tilde{u}_k(x, \lambda), \tag{2.21}$$

we have that $\tilde{u}_k(x, \lambda)$ satisfies the differential equation

$$\tilde{u}_k' + 2\left[(-1)^{k+1}\lambda A(x) + h_k(x, \lambda) - \frac{1}{4}\frac{p'(x)}{p(x)} \right] \tilde{u}_k + \tilde{u}_k^2 + H_k(x, \lambda) = 0, \tag{2.22}$$

where $H_k(x, \lambda)$ is calculated from (2.22), (2.21) and (2.18) and can be shown to satisfy

$$|H_k(x, \lambda)| \leqq C_N(\varepsilon)|\lambda A(x)|^{-N}, \qquad N = 1, 2, \cdots, \tag{2.23}$$

for $x \in \mathscr{R}_\varepsilon$ and $\lambda \in \tilde{S}$. If we now make the change of variables

$$\tilde{u}_k(x, \lambda) = \lambda A(x) v_k(x, \lambda) \tag{2.24}$$

in (2.22), we obtain

$$v_k' + 2[(-1)^{k+1}\lambda A(x) + h_k(x, \lambda)]v_k + \lambda A(x) v_k^2 + L_k(x, \lambda) = 0, \tag{2.25}$$

where

$$L_k(x, \lambda) = (\lambda A(x))^{-1} H_k(x, \lambda) \tag{2.26}$$

and $L_k(x, \lambda)$ satisfies

$$|L_k(x, \lambda)| \leqq D_N(\varepsilon)|\lambda A(x)|^{-N}, \qquad N = 1, 2, \cdots. \tag{2.27}$$
$$x \in \mathscr{R}_\varepsilon \quad \text{and} \quad \lambda \in \tilde{S}.$$

We can rewrite (2.25) as

$$v_k' + 2[(-1)^{k+1}\lambda A(x)]v_k = -2h_k(x, \lambda)v_k - \lambda A(x)v_k^2 - L_k(x, \lambda). \tag{2.28}$$

Treating the right-hand side of (2.28) as a nonhomogeneous term, we obtain the integral equation

$$
\begin{aligned}
v_k(x,\lambda) = \int^x & [-2h_k(s,\lambda)v_k(s,\lambda) - \lambda A(s)v_k^2(s,\lambda) - L_k(s,\lambda)] \\
& \cdot \exp\left(2\int_s^x (-1)^k \lambda A(\tau)\,d\tau\right) ds
\end{aligned}
\tag{2.29}
$$

by the method of variation of parameters. Note that we have not specified the initial point of integration or the path of integration in (2.1) and (2.29). One of the major aspects of our analysis will be to determine these quantities so that solutions of the integral equation (2.29) will yield solutions of the differential equation (1.1).

3. Existence of the subdominant solutions. Let $\varepsilon > 0$ and let $X_0 = a_1 + \varepsilon$. We shall consider the integral equation

$$
\begin{aligned}
v(x,\lambda) = \int_{+\infty}^x & [-2h_0(s,\lambda)v(s,\lambda) - \lambda A(s)v^2(s,\lambda) - L_0(s,\lambda)] \\
& \cdot \exp\left(2\int_s^x \lambda A(\tau)\,d\tau\right) ds
\end{aligned}
\tag{3.1}
$$

for $x \in [X_0, +\infty)$ and path of integration from $+\infty$ to x along the real axis.

Since $A(x) > 0$ for $x \in [X_0, +\infty)$ we have $A(x) \geqq \beta(\varepsilon) > 0$ from the inequality (2.10). Thus the exponential term appearing in (3.1) can be estimated by

$$
\left|\exp\left(2\lambda\int_s^x A(\tau)\,d\tau\right)\right| \leqq \exp[-2(\operatorname{Re}\lambda)\beta(\varepsilon)(s-x)].
\tag{3.2}
$$

From the hypotheses on $p(x)$ we know there exist finite, positive constants $\alpha_0(\varepsilon)$ and $\beta_0(\varepsilon)$ such that

$$
\alpha_0(\varepsilon) \leqq x^{-l/2}A(x) \leqq \beta_0(\varepsilon)
\tag{3.3}
$$

for $x \in [X_0, +\infty)$.

Let us define $F_f(x,\lambda)$ by

$$
\begin{aligned}
F_f(x,\lambda) = \int_{+\infty}^x & [-2h_0(s,\lambda)f(s,\lambda) - \lambda A(s)f^2(s,\lambda) - L_0(s,\lambda)] \\
& \cdot \exp\left(2\lambda\int_s^x A(\tau)\,d\tau\right) ds
\end{aligned}
\tag{3.4}
$$

for $x \in [X_0, +\infty)$, the path of integration from $+\infty$ to x along the real axis, $\lambda \in \tilde{S}$, and $f(x,\lambda)$ continuous in x and λ. Using the various estimates of the terms of the integrand we have the following lemma.

LEMMA 3.1. *If* $|f(x,\lambda)| \leqq |\lambda A(x)|^{-\nu}$ *for* $\nu = 1, 2, \cdots$, *then there exists a set of positive real numbers* λ_ν, $\nu = 1, 2, \cdots$, *such that* $\operatorname{Re}\lambda \geqq \lambda_\nu$ *implies*

$$
|F_f(x,\lambda)| \leqq |\lambda A(x)|^{-\nu}.
$$

Proof. By the triangle inequality applied to (3.4),

$$|F_f(x,\lambda)| \leqq \int_x^{+\infty} [|2h_0(s,\lambda)f(s,\lambda)| + |\lambda A(s)f^2(s,\lambda)| + |L_0(s,\lambda)|] \cdot \left|\exp\left(2\lambda \int_s^x A(\tau)\,d\tau\right)\right| ds.$$

Using the estimates (2.19), (2.27), (3.2) and (3.3) we have

$$|F_f(x,\lambda)| \leqq \{[2M_1\Lambda_0^{-1}\beta^{-1} + F_2\Lambda_0^{-2}\beta^{-2}]\alpha_0^{-\nu} + \Lambda_0^{1-\nu}\beta^{1-\nu}\alpha_0^{-\nu} + D_\nu\alpha_0^{-\nu}\} \cdot (|\lambda|^{-\nu}x^{-l\nu/2}) \int_x^{+\infty} \exp[-2(\operatorname{Re}\lambda)\beta(s-x)]\,ds,$$

where the dependence of the constants on ε has been suppressed. Hence, we have

$$|F_f(x,\lambda)| \leqq M(\varepsilon,\nu)(\operatorname{Re}\lambda)^{-1}|\lambda A(x)|^{-\nu},$$

where

$$M(\varepsilon,\nu) = \tfrac{1}{2}\beta_0^\nu\beta^{-1}\{[2M_1\Lambda_0^{-1}\beta^{-1} + F_2\Lambda_0^{-2}\beta^{-2}]\alpha_0^{-\nu} + \Lambda_0^{1-\nu}\beta^{1-\nu}\alpha_0^{-\nu} + D_\nu\alpha_0^{-\nu}\}.$$

Consequently by choosing $\operatorname{Re}\lambda \geqq \lambda_\nu$, where λ_ν is sufficiently large, we can achieve

$$|F_f(x,\lambda)| \leqq |\lambda A(x)^{-\nu}|.$$

Note that the λ_ν of Lemma 3.1 depend also upon ε, and as $\varepsilon \to 0$ the λ_ν will, in general, have to be taken larger. In a similar fashion we can obtain the following lemma.

LEMMA 3.2. *If* $|f_1(x,\lambda)| < |\lambda A(x)|^{-\nu}$ *and* $|f_2(x,\lambda)| \leqq |\lambda A(x)|^{-\nu}$, *then there exists a set of positive numbers* λ_ν, $\nu = 1, 2, \cdots$, *such that*

$$\|F_{f_2}(x,\lambda) - F_{f_1}(x,\lambda)\| \leqq r\|f_2(x,\lambda) - f_1(x,\lambda)\|$$

with $r < 1$, *provided* $\operatorname{Re}\lambda \geqq \lambda_\nu$ *and where* $\|\cdot\|$ *is the supremum norm over* $[X_0, +\infty) \times S_\nu$, *and*

$$S_\nu = \{\lambda | \operatorname{Re}\lambda \geqq \lambda_\nu, |\operatorname{Im}\lambda| \leqq \Lambda_1\}.$$

Lemma 3.1 and Lemma 3.2 both give sequences λ_ν, $\nu = 1, 2, \cdots$. We can form a third sequence $\lambda_\nu, 1, 2, \cdots$, by taking λ_ν to be the maximum of the corresponding λ_ν given by the lemmas. We can also assume $\lambda_1 \leqq \lambda_2 \leqq \lambda_3 \leqq \cdots$. For the remainder of this section, by λ_ν we shall mean a member of this third sequence. Again using our various estimates we have the following lemma.

LEMMA 3.3. *If* $f(x,\lambda)$ *is analytic and satisfies* $|f(x,\lambda)| \leqq |\lambda A(x)|^{-2}$ *for* $x \in [X_0, +\infty)$ *and*

$$\lambda \in \{\lambda | \operatorname{Re}\lambda \geqq \lambda_2, |\operatorname{Im}\lambda| < \Lambda_1\},$$

then $F_f(x,\lambda)$ *is analytic in the same region.*

Defining the successive approximations by

$$\begin{aligned} &v_0(x,\lambda) \equiv 0, \\ &v_{n+1}(x,\lambda) = F_{v_n}(x,\lambda), \qquad n = 1, 2, \cdots, \end{aligned} \tag{3.5}$$

we can use the lemmas of this section to apply the Banach fixed-point theorem and prove the convergence of the sequence $v_n(x, \lambda)$ to a function $v(x, \lambda)$. Setting $\tilde{u}_0(x, \lambda) = \lambda A(x)v(x, \lambda)$, defining $u_0(x, \lambda)$ by (2.21) with $k = 0$, and setting

$$y_0(x, \lambda) = \exp\left(\int_X^x u_0(s, \lambda)\, ds\right),$$

with a path of integration from X_0 to x along the real axis, we obtain the solution $y_0(x, \lambda)$ of Lemma 1.1. The square integrability of $y_0(x, \lambda)$ follows from the asymptotic behavior of $u_0(x, \lambda)$ as given by Lemma 1.1. It can be shown that for λ sufficiently large and x sufficiently large, the term $\lambda A(x)$ in the expression (i) of Lemma 1.1 implies the square integrability of $y_0(x, \lambda)$ since the terms $p'(x)/p(x)$ and $P(x, -\lambda)$ can be bounded. The treatment of $y_{2m}(x, \lambda)$ is analogous.

4. Construction of the connecting solutions. In § 3 we constructed two solutions of the differential equation (1.1) which were square integrable on semiaxes and which had known asymptotic behavior on disjoint semiaxes. We now shall construct m pairs of linearly independent solutions which will enable us to relate the two subdominant solutions $y_0(x, \lambda)$ and $y_{2m}(x, \lambda)$.

We again shall consider the integral equation (2.29). In the last section we employed the exponential decay for large $|x|$ of the term

$$\left|\exp\left[(-1)^k \int_s^x 2\lambda A(\tau)\, d\tau\right]\right| \tag{4.1}$$

in order to obtain the existence of the subdominant solutions. Now, however, x will lie in a bounded subset of $\mathscr{R}_\varepsilon$ and it will suffice to bound the exponential term (4.1).

We note that

$$\begin{aligned}&\left|\exp\left[(-1)^k \int_s^x 2\lambda A(\tau)\, d\tau\right]\right| \\ &= \exp\left[2(-1)^k\left(\operatorname{Re}\lambda \operatorname{Re}\left(\int_s^x A(\tau)\, d\tau\right) - \operatorname{Im}\lambda \operatorname{Im}\left(\int_s^x A(\tau)\, d\tau\right)\right)\right].\end{aligned}$$

For $\lambda \in S$ we have $|\operatorname{Im}\lambda| \leqq \Lambda_1$, and hence if x lies in a bounded region, then

$$\exp\left[2(-1)^{k+1} \operatorname{Im}\lambda \operatorname{Im}\left(\int_s^x A(\tau)\, d\tau\right)\right]$$

is bounded if the paths of integration are bounded. For $\lambda \in \tilde{S}$ we have $\operatorname{Re}\lambda$ unbounded and positive, and hence if $\exp[2(-1)^k \operatorname{Re}\lambda \operatorname{Re}(\int_s^x A(\tau)\, d\tau)]$ is to remain bounded we must be able to choose the paths of integration so that $(-1)^k \operatorname{Re}(\int_s^x A(\tau)\, d\tau)$ is nonpositive. In order to achieve this last condition we shall consider the mapping

$$z = (-1)^k \int^x A(\tau)\, d\tau, \tag{4.2}$$

and, in particular, we shall determine the curves in the x-plane along which $\operatorname{Re} z$ is constant. The basic tools for our analysis are the following two lemmas, whose proofs are included for completeness. In their present form they are due to Sibuya [7]. Similar results were obtained by Jenkins [5] and applied by Evgrafov and Fedoryuk [2].

LEMMA 4.1. *Let* $\operatorname{Re}(\int^x A(\tau)\,d\tau)$ *be constant along a smooth curve* $x = x(\rho)$ *not passing through any zeros of* (x); *then*

$$dx/d\rho = i\overline{A(x(\rho))}, \tag{4.3}$$

where the overbar denotes complex conjugation.

Proof. Suppose $\operatorname{Re}(\int^x A(\tau)\,d\tau)$ is constant along a path $x = \tau(p)$ and further suppose that the parameter p is chosen so that $\tau'(p) \neq 0$. We know

$$A(\tau(p)) \neq 0$$

and

$$\begin{aligned}\frac{d}{dp}\int^x A(y)\,dy &= \frac{d}{dp}\int^{\tau(p)} A(y)\,dy\\ &= A(\tau(p))\tau'(p) = if(p),\end{aligned}$$

where $f(p)$ is real since $\operatorname{Re}(\int^x A(y)\,dy)$ is constant. Moreover $f(p) \neq 0$. Hence we may introduce a new parameter ρ by

$$d\rho/dp = f(p)|A(\tau(p))|^{-2}.$$

Solving implicitly for $p = p(\rho)$ we obtain

$$\tau(\rho) = \tau(p(\rho)).$$

Moreover,

$$\begin{aligned}\frac{d}{d\rho}\tau(p(\rho)) &= \tau'(p(\rho))p'(\rho) = \tau'(p(\rho))\frac{|A(\tau(p(\rho)))|^2}{f(p(\rho))}\\ &= i\overline{A(\tau(p(\rho)))}\end{aligned}$$

since

$$\frac{\tau'(p(\rho))}{f(p(\rho))} = \frac{i}{A(\tau(p(\rho)))}.$$

This proves the lemma.

LEMMA 4.2. *Suppose* $x = x(\rho)$ *is a solution of the equation*

$$\frac{dx}{d\rho} = i\overline{A(x(\rho))}.$$

Then $\operatorname{Re}(\int^x A(\tau)\,d\tau)$ *is constant along* $x(\rho)$.

Proof. We observe that

$$\begin{aligned}\frac{d}{d\rho}\int^{x(\rho)} A(\tau)\,d\tau &= A(x(\rho))\frac{dx(\rho)}{d\rho}\\ &= iA(x(\rho))\overline{A(x(\rho))} = i|A(x(\rho))|^2.\end{aligned}$$

Hence by separating real and imaginary parts, we obtain

$$\frac{d}{d\rho}\operatorname{Re}\left(\int^{x(\rho)} A(\tau)\, d\tau\right) = 0$$

along $x(\rho)$. This implies $\operatorname{Re}(\int^x A(\tau)\, d\tau)$ is constant along $x = x(\rho)$.

Since $x = a_i$, $i = 1, \cdots, 2m$, is a simple zero of $p(x)$, it is shown [2, § 2] that there exist three trajectories meeting at angles of $2\pi/3$ at $x = a_i$ along which

$$(-1)^k \operatorname{Re}\left(\int_{a_i}^{x} A(\tau)\, d\tau\right) = 0.$$

These particular trajectories are referred to as Stokes' curves. On the cuts I_{2k-1}, $k = 1, \cdots, m$, $A(x)$ is purely imaginary, and hence,

$$(-1)^k \operatorname{Re}\left(\int_{a_{2i}}^{x} A(\tau)\, d\tau\right) = (-1)^k \operatorname{Re}\left(\int_{a_{2i-1}}^{x} A(\tau)\, d\tau\right) = 0$$

for $x \in I_{2i-1}$. Hence the cuts are Stokes' curves.

The trajectories of the autonomous differential equation (4.3), sufficiently close to the real axis for $m = 2$, are shown in Fig. 1. In particular the upper and lower cuts are trajectories and the arrows indicate increasing ρ.

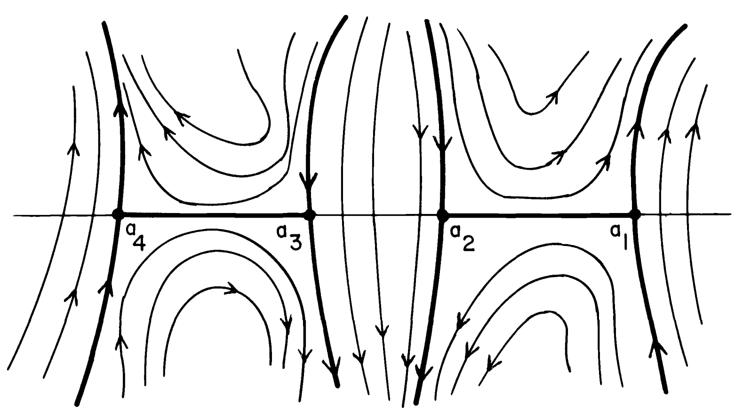

FIG. 1

In order to construct the region D_{+1} of Lemma 1.2, consider a region $\mathscr{D}$ as given in Fig. 2.

The transformation

$$z = -\int_{a_1}^{x} A(\tau)\, d\tau \tag{4.4}$$

maps the region $\mathscr{D}$ of Fig. 2 onto a multiply covered region $\mathscr{D}$ in the z-plane as indicated in Fig. 3 (where $\tilde{a}$ represents the image of a).

The use of the asymptotic representation requires that the paths of integration in the x-plane be bounded away from the turning points. This can be accomplished by requiring the corresponding paths in the z-plane to lie outside disks about the

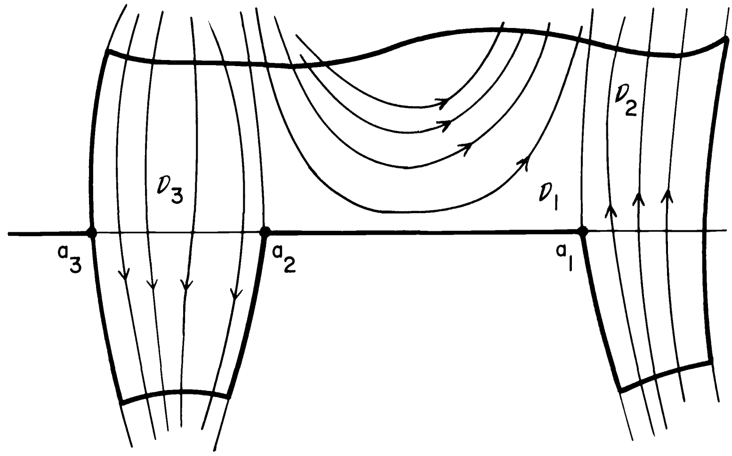

FIG. 2

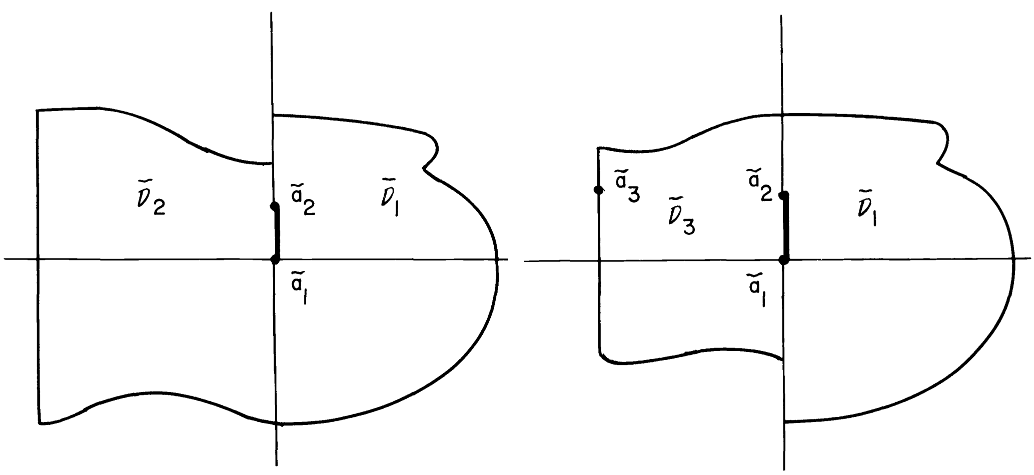

FIG. 3

images of the turning points. The additional restriction that Re z be nonincreasing along the paths then induces shadow regions, nonadmissible regions, as indicated by the cross-hatched areas in Fig. 4. Let $\tilde{X}_{+1}$ be the point indicated in Fig. 4, and let X_{+1} be the corresponding preimage in the x-plane. The desired region D_{+1} is then the preimage of the unshaded portion of $\mathscr{D}$ and is shown in Fig. 5.

We note that part of the upper cut is included in the region D_{+1}. By our construction any point $x \in D_{+1}$ can be connected to $X_{+1} \in D_{+1}$ by a smooth curve γ_x lying in D_{+1} such that $|\exp(-\int_s^x 2\lambda A(\tau)\,d\tau)|$ is bounded as s moves from X_{+1} to x along γ_x. In a similar fashion we can construct $2m - 1$ more regions D_{-1} and $D_{\pm(2k-1)}, k = 2, \cdots, m$, satisfying assertion (iii) of Lemma 1.2. We denote by $X_{\pm(2k-1)}$ the point in $D_{\pm(2k-1)}$ corresponding to X_{+1} in D_{+1}.

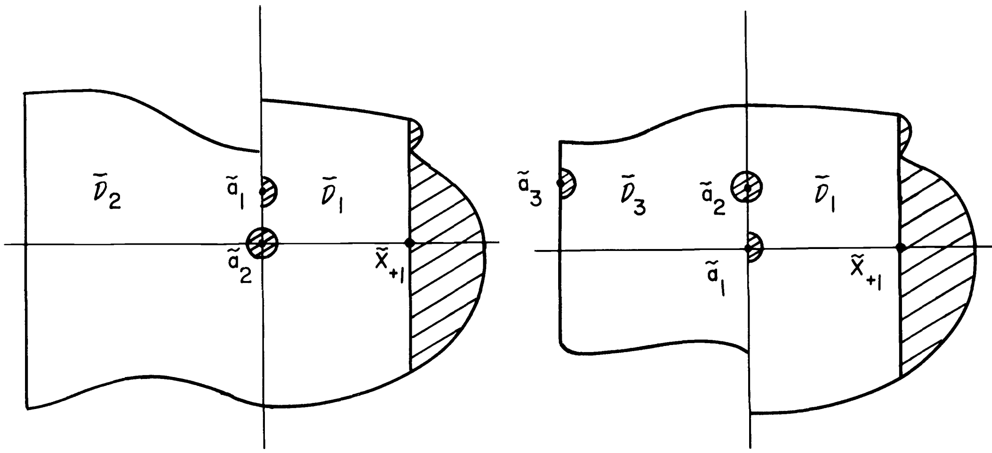

FIG. 4

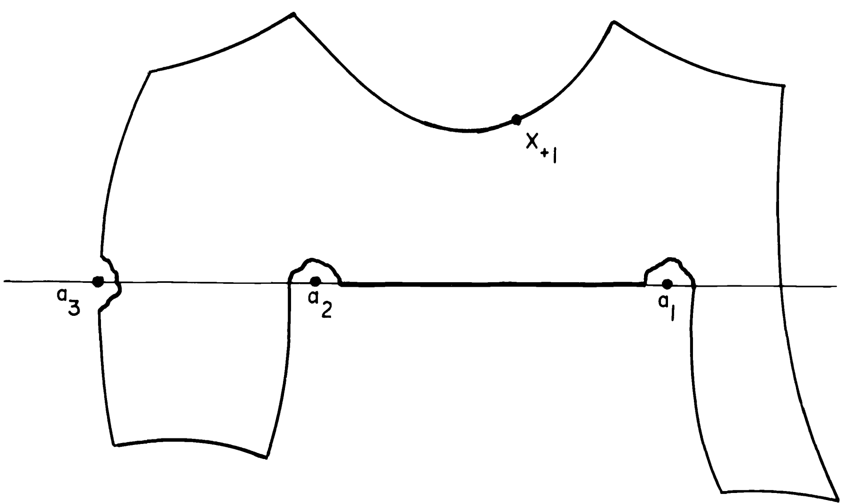

FIG. 5

We now define the successive approximations $v_n^k(x, \lambda)$ by

$$v_0^k(x, \lambda) \equiv 0,$$
$$v_{n+1}^k(x, \lambda) = F_{v_n^k}^k(x, \lambda), \qquad n = 1, 2, \cdots.$$

F_f^k is the analogue in the region $D_{+(2k-1)}$ of F_f in (3.4). By a procedure similar to that employed in the last section we can show that the successive approximations $v_n^k(x, \lambda)$ converge uniformly. This gives m functions, denoted $v_{+(2k-1)}(x, \lambda)$, analytic on $D_{+(2k-1)} \times S_2$. Setting

$$u_{+(2k-1)}(x, \lambda) = (-1)^{k+1}\lambda A(x) - \frac{1}{4}\frac{p'(x)}{p(x)} + h_k(x, \lambda) + \lambda A(x) v_{+(2k-1)}(x, \lambda)$$

and

$$y_{+(2k-1)}(x,\lambda) = \exp\left(\int_{b_{2k-1}}^{x} u_{+(2k-1)}(s,\lambda)\,ds\right)$$

for $x \in D_{+(2k-1)}$ with $b_{2k-1} = \frac{1}{2}[a_{2k} + a_{2k-1}]$ and the path of integration lying in $D_{+(2k-1)}$, we can satisfy conditions (i) and (ii) of Lemma 1.2. An analogous construction yields $y_{-(2k-1)}(x,\lambda)$.

The functions $y_{\pm(2k-1)}(x,\lambda)$ are solutions of the linear differential equation (1.1), which possesses no first derivative term. Consequently the Wronskian determinant W_k of $y_{+(2k-1)}$ and $y_{-(2k-1)}$ will be a function of λ only. Setting $x = b_{2k-1}$ we have that

$$W_k(\lambda) = u_{-(2k-1)}(b_{2k-1},\lambda) - u_{+(2k-1)}(b_{2k-1},\lambda).$$

Using the asymptotic behavior given by Lemma 1.2 we have

$$\begin{aligned} W_k(\lambda) = &\left[(-1)^{k+1}\lambda A(b_{2k-1}-) - \frac{1}{4}\frac{p'(b_{2k-1})}{p(b_{2k-1})} + O(|\lambda|)^{-1})\right] \\ &- \left[(-1)^{k}\lambda A(b_{2k-1}+) - \frac{1}{4}\frac{p'(b_{2k-1})}{p(b_{2k-1})} + O(|\lambda|^{-1})\right]. \end{aligned}$$

Since

$$\begin{aligned} A(b_{2k-1}+) &= -A(b_{2k-1}-) \\ &= (-1)^{k+1} i\sqrt{|p(b_{2k-1})|}, \end{aligned}$$

we have

$$W_k(\lambda) = -2\lambda i\sqrt{|p(b_{2k-1})|} + O(|\lambda|^{-1}).$$

Consequently if $\operatorname{Re}\lambda$ is sufficiently large, we have

$$W_k(\lambda) \neq 0,$$

and hence,

$$y_{+(2k-1)}(x,\lambda) \quad \text{and} \quad y_{-(2k-1)}(x,\lambda)$$

are linearly independent. This completes the proof of Lemma 1.2.

5. Proof of the theorem. Since the differential equation (1.1) is linear and $y_{+(2k-1)}(x,\lambda)$ and $y_{-(2k-1)}(x,\lambda)$ are linearly independent, we know there exist constants, the connection coefficients, depending on λ such that

$$\begin{aligned} y_0 &= C_0^{+1} y_{+1} + C_0^{-1} y_{-1}, \\ &\;\vdots \\ y_{+(2k-1)} &= C_{+(2k-1)}^{+(2k-3)} y_{+(2k-3)} + C_{+(2k-1)}^{-(2k-3)} y_{-(2k-3)}, \\ y_{-(2k-1)} &= C_{-(2k-1)}^{+(2k-3)} y_{+(2k-3)} + C_{-(2k-1)}^{-(2k-3)} y_{-(2k-3)}, \\ &\;\vdots \\ y_{2m} &= C_{2m}^{+(2m-1)} y_{+(2m-1)} + C_{2m}^{-(2m-1)} y_{-(2m-1)}, \\ y_{2m} &= C_{2m}^{+1} y_{+1} + C_{2m}^{-1} y_{-1}, \end{aligned} \tag{5.1}$$

where the λ dependence of the C's and the (x, λ) dependence of the y's has been suppressed.

When $y_0(x, \lambda)$ and $y_{2m}(x, y)$ are linearly dependent we shall have a solution to the problem (1.1), (1.2). We have linear dependence for those values of λ, the eigenvalues, for which

$$\begin{vmatrix} C_0^{+1}y_{+1} + C_0^{-1}y_{-1} & C_{2m}^{+1}y_{+1} + C_{2m}^{-1}y_{-1} \\ C_0^{+1}y'_{+1} + C_0^{-1}y'_{-1} & C_{2m}^{+1}y'_{+1} + C_{2m}^{-1}y'_{-1} \end{vmatrix} = 0. \tag{5.2}$$

By Lemma 1.2, equation (5.2) is equivalent to

$$\begin{vmatrix} C_0^{+1} & C_0^{-1} \\ C_{2m}^{+1} & C_{2m}^{-1} \end{vmatrix} = 0. \tag{5.3}$$

From the set of equations (5.1) we can obtain

$$\begin{aligned} y_{2m} &= (y_{+(2m-1)}, y_{-(2m-1)})\begin{pmatrix} C_{2m}^{+(2m-1)} \\ C_{2m}^{-(2m-1)} \end{pmatrix} \\ &= (y_{+(2m-3)}, y_{-(2m-3)})T_m\begin{pmatrix} C_{2m}^{+(2m-1)} \\ C_{2m}^{-(2m-1)} \end{pmatrix} \\ &= (y_{+(2m-5)}, y_{-(2m-5)})T_{m-1}T_m\begin{pmatrix} C_{2m}^{+(2m-1)} \\ C_{2m}^{-(2m-1)} \end{pmatrix} \\ &\;\;\vdots \\ &= (y_{+1}, y_{-1})T_2 \cdots T_m\begin{pmatrix} C_{2m}^{+(2m-1)} \\ C_{2m}^{-(2m-1)} \end{pmatrix}, \end{aligned}$$

where the T_k are 2×2 matrices of connection coefficients.

Consequently, (5.3) can be written as

$$(-C_0^{-1}, C_0^{+1})T_2 \cdots T_m\begin{pmatrix} C_2^{+(2m-1)} \\ C_{2m}^{-(2m-1)} \end{pmatrix} = 0. \tag{5.4}$$

By Cramer's rule we may solve the system (5.1) for the C's in terms of the appropriate Wronskians of the y's. Let

$$b_j = \tfrac{1}{2}(a_j + a_{j+1})$$

for $j = 1, \cdots, 2m - 1$,

$$b_0 = X_0, \qquad b_{2m} = X_{2m},$$

$$W_k = W_{+k,-k},$$

$$\eta(k) = [-2\lambda\sqrt{p(b_{2k-2})} + h_{k-1}(b_{2k-2}, \lambda) - h_k(b_{2k-2}, \lambda)],$$

$$I(j, k, \pm) = \int_{b_{2k-1}}^{b_{2j-2}} u_{\pm(2k-1)}(s, \lambda)\, ds,$$

and let $\mu(\lambda)$ be a generic term for a quantity which is asymptotically zero. Thus

$$T_k = W_{2k-3}^{-1}\begin{pmatrix} (\eta(k) + \mu(\lambda)) \exp[I(k,k,+) + I(k,k-1,-)] & (\eta(k) + \mu(\lambda)) \exp[I(k,k,-) + I(k,k-1,-)] \\ (-\eta(k) + \mu(\lambda)) \exp[I(k,k,+) + I(k,k-1,+)] & (-\eta(k) + \mu(\lambda)) \exp[I(k,k,-) + I(k,k-1,+)] \end{pmatrix}.$$

We can rewrite this last equation in the form

$$\begin{aligned} T_k = W_{2k-3}^{-1}&\begin{pmatrix} \exp(I(k,k-1,-)) & 0 \\ 0 & -\exp(I(k,k-1,+)) \end{pmatrix} \\ &\cdot\begin{pmatrix} (\eta(k) + \mu(\lambda)) \exp(I(k,k,+)) & (\eta(k) + \mu(\lambda)) \exp(I(k,k,-)) \\ (\eta(k) + \mu(\lambda)) \exp(I(k,k,+)) & (\eta(k) + \mu(\lambda)) \exp(I(k,k,-)) \end{pmatrix}. \end{aligned} \tag{5.5}$$

Using (5.5) in (5.4) we obtain

$$\begin{aligned} &W_1^{-1}W_3^{-1} \cdots W_{2m-3}^{-1}[\eta(2) + \mu(\lambda)] \cdots [\eta(m) + \mu(\lambda)](-C_0^{-1}, C_0^{+1}) \\ &\cdot\begin{pmatrix} \exp[I(2,1,-)] & 0 \\ 0 & -\exp[I(2,1,+)] \end{pmatrix} \\ &\cdots \begin{pmatrix} \exp[I(m,m,+)] & \psi(\lambda)\exp[I(m,m,-)] \\ \psi(\lambda)\exp[I(m,m,+)] & \psi(\lambda)\exp[I(m,m,-)] \end{pmatrix}\begin{pmatrix} C_{2m}^{+(2m-1)} \\ C_{2m}^{-(2m-1)} \end{pmatrix} = 0, \end{aligned} \tag{5.6}$$

where $\psi(\lambda) = 1 + \mu(\lambda)$.

We can compute $C_0^{-1}, C_0^{+1}, C_{2m}^{+(2m-1)}$ and $C_{2m}^{-(2m-1)}$ from the appropriate Wronskians in a similar manner to obtain the equation

$$\begin{aligned} &W_1^{-2}W_3^{-1} \cdots W_{2m-1}^{-1}[\eta(1) + \mu(\lambda)] \cdots [\eta(m+1) + \mu(\lambda)] \\ &\cdot(\exp[I(1,1,+)], [1 + \mu(\lambda)]\exp[I(1,1,-)]) \\ &\cdots \begin{pmatrix} \exp[I(m+1,m,-1)] \\ [-1 + \mu(\lambda)]\exp[I(m+1,m,-)] \end{pmatrix} = 0. \end{aligned} \tag{5.7}$$

We note that if $|\lambda|$ is sufficiently large, then

$$W_1^{-2}W_3^{-1} \cdots W_{2m-1}^{-1}[\eta(1) + \mu(\lambda)] \cdots [\eta(m+1) + \mu(\lambda)] \neq 0.$$

Thus, after rearrangement, (5.7) becomes

$$\begin{aligned} &(1, [-1 + \mu(\lambda)]\exp d_1)\begin{pmatrix} 1 & [-1 + \mu(\lambda)]\exp d_2 \\ 1 + \mu(\lambda) & [-1 + \mu(\lambda)]\exp d_2 \end{pmatrix} \\ &\cdots \begin{pmatrix} 1 + [-1 + \mu(\lambda)]\exp d_m \\ 1 + \mu(\lambda) + [-1 + \mu(\lambda)]\exp d_m \end{pmatrix} = 0, \end{aligned} \tag{5.8}$$

where

$$d_m + I(k,k,-) - I(k,k,+) - I(k+1,k,-) + I(k+1,k,+).$$

If we perform the indicated matrix multiplication, we obtain

$$(5.9)\quad [(1+\mu(\lambda)) + (-1+\mu(\lambda))\exp d_1] \cdots [(1+\mu(\lambda)) + (-1+\mu(\lambda))\exp d_m] = 0.$$

Equation (5.9) can be rewritten in the form

$$(5.10)\quad [1-\exp d_1]\cdots[1-\exp d_m] = \mu(\lambda).$$

Equation (5.10) will be satisfied if

$$(5.11)\quad 1-\exp d_k = \mu(\lambda)$$

for $k = 1, \cdots, m$. Equation (5.11) implies

$$(5.12)\quad d_k = -2\pi l i + \mu(\lambda), \quad l \text{ an integer},$$

for $k = 1, \cdots, m$. If we write d_k in terms of integrals, we obtain

$$\sum_{j=1}^{4} \int_{\sigma_{j,k}} \left[(-1)^{k+1}\lambda A(s) - \frac{1}{4}\frac{p'(s)}{p(s)} + h_k(s,\lambda) + \lambda A(s)\cdot v_{j,k}(s,\lambda)\right] ds = -2\pi l i + \mu(\lambda),$$

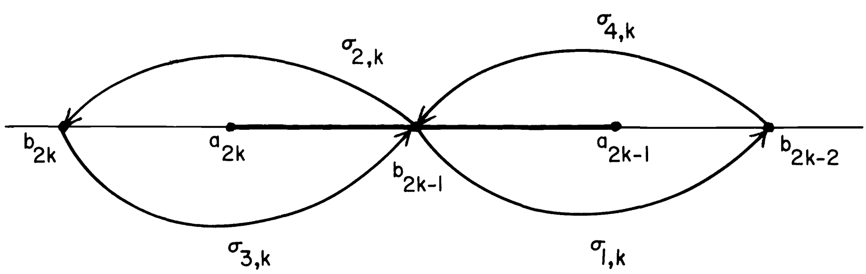

FIG. 6

where the paths of integration $\sigma_{j,k}$ are shown in Fig. 6, and where

$$v_{j,k}(s,\lambda) = v_{(-1)^j(2k-1)}(s,\lambda).$$

However,

$$\sum_{j=1}^{4} \int_{\sigma_{j,k}} |(-1)^{k+1}\lambda A(s)|\, ds = -2\lambda i \int_{a_{2k}}^{a_{2k-1}} \sqrt{|p(s)|}\, ds,$$

$$\sum_{j=1}^{4} \int_{\sigma_{j,k}} \frac{-1}{4}\frac{p'(s)}{p(s)}\, ds = -\pi i,$$

and formally,

$$\sum_{j=1}^{4} \int_{\sigma_{j,k}} [h_k(s,\lambda) + \lambda A(s) v_{j,k}(s,\lambda)]\, ds$$

$$= \sum_{n=1}^{\infty} \oint_{\Gamma_k} \{[(-1)^{k+1}\lambda A(s)]^{-n} P_n(s)\}\, ds,$$

$$\mu(\lambda) = 0,$$

where Γ_k is a contour contained in $D_{+(2k-1)} \cup D_{-(2k-1)}$ which encloses only the transition points $x = a_{2k-1}$ and $x = a_{2k}$ in a counterclockwise sense.

Hence,

$$-2\lambda i \int_{a_{2k}}^{a_{2k-1}} \sqrt{|p(s)|}\, ds - \pi i + \sum_{n=1}^{\infty} \oint_{\Gamma_k} [(-1)^{k+1}\lambda A(s)]^{-n} P_n(s)\, ds = -2l\pi i$$

provided $|\lambda|$ is sufficiently large and $\lambda \in S_2$. Thus, as $l \to +\infty$ we have that the large positive eigenvalues $\lambda_{k,l}$ of the problem (1.1), (1.2) satisfy the formal equations

$$2\lambda_{k,l} i\left(\int_{a_{2k}}^{a_{2k-1}} \sqrt{|p(s)|}ds\right) + \pi i - \sum_{n=1}^{\infty} \oint_{\Gamma_k} [(-1)^{k+1}\lambda_{k,l} A(s)]^{-n} P_n(s)\, ds = 2l\pi i. \tag{5.13}$$

We note that the existence of eigenvalues for $|\lambda|$ sufficiently large is guaranteed by Rouché's theorem. Equation (5.13) can be rewritten as

$$\lambda_{k,l} \int_{a_{2k}}^{a_{2k-1}} \sqrt{|p(x)|}\, dx = \left(l - \frac{1}{2}\right)\pi - \frac{i}{2} \sum_{n=1}^{\infty} \oint_{\Gamma_k} [(-1)^{k+1}\lambda_{k,l} A(x)]^{-n} P_n(x)\, dx. \tag{5.14}$$

If the $\lambda_k(l)$ as given by the theorem satisfy (formally) (5.14), then

$$\lambda_{k,l} - \left[\frac{(l - 1/2)\pi}{\int_{a_{2k}}^{a_{2k-1}} \sqrt{|p(x)|}\, dx} + \sum_{n=1}^{N} \mathscr{E}_{k,l} l^{-n}\right] = O(l^{-N-1})$$

as $l \to +\infty$ for $N = 1, 2, \cdots$.

Let λ be such an eigenvalue as found above and let y be the corresponding solution of (1.1). If we multiply (1.1) by y and integrate from $-\infty$ to $+\infty$, we obtain

$$\int_{-\infty}^{+\infty} yy'' - \lambda^2 \int_{-\infty}^{+\infty} y^2 p = 0.$$

Integrating the first term by parts we obtain

$$\int_{-\infty}^{+\infty} y'^2 - \lambda^2 \int_{-\infty}^{+\infty} y^2 p = 0.$$

Since $y' \not\equiv 0, \int_{-\infty}^{+\infty} y^2 p \neq 0$ and λ^2 is real. From the distribution of eigenvalues given by formula (5.13) it follows that the eigenvalues of large magnitude obtained above are real.

This proves the theorem.

Acknowledgments. The author wishes to thank Professors W. A. Harris, Jr. and Professor Y. Sibuya for their invaluable help and encouragement in the preparation of this paper, and also the referee for his helpful suggestions.

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SIAM J. MATH. ANAL.
Vol. 2, No. 4, November 1971

THE BEHAVIOR AS $\varepsilon \to +0$ OF SOLUTIONS TO $\varepsilon\nabla^2 w = \partial w/\partial y$ IN $|y| \leqq 1$ FOR DISCONTINUOUS BOUNDARY DATA*

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Abstract. The title problem is considered for boundary data $w(x, -1) = f(x)$, $w(x, 1) = g(x)$. Here f, g are infinitely differentiable except at $x = 0, a$ respectively, where they have right- and left-hand derivatives of all orders. With $g = 0$ five regions are distinguished: the core $0 < x_0 \leqq |x|$ and the free layer $\varepsilon^{-1/2}|x| \leqq X_\infty$, excluding $\varepsilon^{-1/2}|x| \leqq X_0$, $|y + 1| \leqq y_{-1}$, in $-1 \leqq y \leqq y_1 < 1$; their boundary layers in $\varepsilon^{-1}(1 - y) \leqq Y_\infty$; and the excluded region $\varepsilon^{-1}|x| \leqq X_{*\infty}$, $\varepsilon^{-1}(1 + y) \leqq y_{*\infty}$. The solution for $f = 0$ is asymptotically zero everywhere except in the boundary layer, where $0 < x_a \leqq |x - a|$ is distinguished from the transition zone $\varepsilon^{-1}|x - a| \leqq x_{*\infty}$. By means of Fourier transforms it is shown that the method of matched asymptotic expansions gives approximations to all orders in each of the regions, and that the latter can be extended to overlap. For the excluded region, which gives birth to the "parabolic" free layer, this contradicts what has previously been supposed. Of particular interest is the transition zone, which resolves a breakdown in the "hyperbolic" boundary layer. The expansion in the core is determined independently of the others, but not that in the free layer. As a consequence, the odd powers of $\varepsilon^{1/2}$ which appear in the free layer are absent in the core. Other assumptions concerning f and g are also considered.

1. Introduction. We propose to study the asymptotic properties, as $\varepsilon \to +0$, of the solution of the elliptic equation

$$\varepsilon\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) - \frac{\partial w}{\partial y} = 0 \quad \text{on the strip } |y| \leqq 1, \tag{1a}$$

which satisfies the boundary conditions

$$w(x, -1) = f(x), \qquad w(x, 1) = g(x). \tag{1b}$$

The equation arises in magnetohydrodynamics, where ε measures the importance of viscous force relative to the electromagnetic force, and in the theory of plate-membranes under tension in the y-direction, where ε measures the bending stiffness [6]. In either case the region is bounded, and treatment of the strip is intended to be a first step in understanding that more complicated situation. Certainly boundedness in the y-direction is the more important feature. For this reason we have not mentioned ordinary hydrodynamics, where the equation arises in Oseen's approximation: the region there is unbounded in the y-direction and the questions are of quite different character.

The present type of problem has been considered with varying degrees of generality by several authors. The classic paper on the subject is by Eckhaus and de Jager [3]. Certain aspects have been followed up by Mauss [7]–[10] and Grasman [5] as well as by Eckhaus himself [2]. But nobody has faced the question we shall treat: proving that the formal method of matched asymptotic expansions is correct to all orders, where we are especially interested in discontinuities in f, g, or their derivatives.

* Received by the editors October 29, 1970.

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The proof consists in deriving these asymptotic expansions directly from the exact solution, which is expressed in terms of the Green's function. Difficulty arises when this Green's function is written as the infinite sum of Bessel functions, corresponding to the fundamental solution and images in the two boundaries (see end of § 3). Although it is easily seen that all but the first few Bessel functions are asymptotically zero, it is difficult to see how the remaining ones should be manipulated to yield the various asymptotic expansions. However, when the Bessel functions are replaced by their Fourier integrals, the terms in the expansions are obtained simply from the Taylor series (in ε) of the corresponding transforms, and the validity of the expansions is established by estimating remainders. In short, we take Fourier transforms from the start and find, as is often the case, that it is relatively easy to deal with the transform of the Green's function.

Some care is still required to ensure that inverses exist and remainders are estimated correctly. Since there are enough of these questions to deal with, we shall ignore the more trivial ones such as whether an integral can be differentiated. In other words, a formal step will only receive attention if it is in fact not valid.

2. The method of matched asymptotic expansions.[1] The solution is assumed to have an asymptotic expansion[2]

$$w \sim \sum_{k=0}^{\infty} w_k^I(x, y)\varepsilon^k. \tag{2}$$

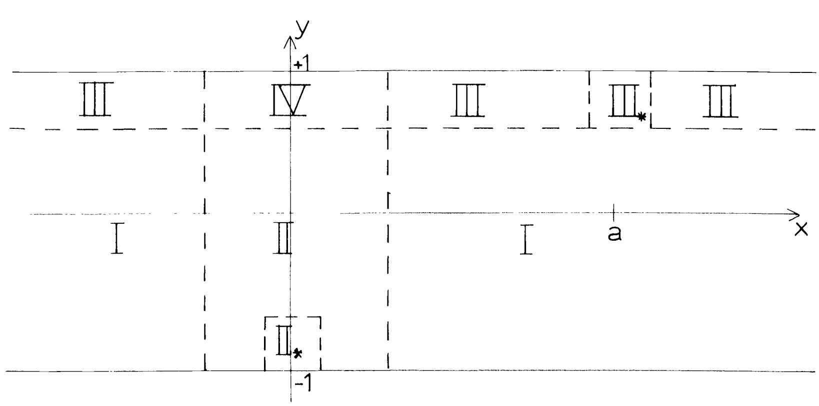

FIG. 1

Substitution in the boundary value problem (1) then yields the recurrence relation

$$\frac{\partial w_k^I}{\partial y} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) w_{k-1}^I \tag{3a}$$

for the coefficient functions (assuming that the derivatives are $O(1)$). At each stage a first order equation has to be solved, so that only one boundary condition can be

[1] See, for example, [1], [11].

[2] The superscripts I, II, etc. correspond to the regions in Fig. 1.

satisfied; for reasons that will be clear later, this must be

$$w_k^I(x, -1) = \begin{cases} f(x), & k = 0, \\ 0, & k \neq 0, \end{cases} \tag{3b}$$

and not the one at $y = 1$. There is no difficulty in calculating as many terms as desired, but the very first term

$$w_0^I(x, y) = f(x)$$

shows that the asymptotic expansion cannot be uniformly valid. The boundary condition at $y = 1$ is in general violated, and a discontinuity (implying that the x-derivative is not $O(1)$) occurs across any vertical line through a point of discontinuity of f. Note that these two types of breakdown are different: the first occurs however smooth f and g are at the value of x considered; the second is a direct consequence of a discontinuity in f.

For simplicity we shall assume that f has a single discontinuity at $x = 0$. Then, recognizing that the solution must have rapid changes across $x = 0$, we introduce the new coordinate

$$X = \varepsilon^{-1/2} x \tag{4}$$

so as to make $\varepsilon(\partial^2/\partial x^2) = \partial^2/\partial X^2$ explicitly comparable to $\partial/\partial y$ in the original equation (1a). The solution is now assumed to have the asymptotic expansion

$$w \sim \sum_{k=0}^{\infty} w_k^{II}(X, y)\varepsilon^{k/2}, \tag{5}$$

which leads to the recurrence relation

$$\left(\frac{\partial^2}{\partial X^2} - \frac{\partial}{\partial y}\right) w_k^{II} = -\frac{\partial^2}{\partial y^2} w_{k-2}^{II} \tag{6a}$$

and the boundary condition

$$w_k^{II}(X, -1) = f^{(k)}(\pm 0)\frac{X^k}{k!} \quad \text{when } X \gtrless 0 \tag{6b}$$

for the coefficient functions. Here $f^{(k)}(\pm 0)$ are right and left kth derivatives of f at $x = 0$. Once more only one boundary condition can be satisfied, and it must be the one at the lower boundary. However, because of the singularity at $X = 0$, $y = -1$, the w_k^{II} are not determined to within certain singular solutions of the homogeneous diffusion equation. Leaving aside this question for the moment, we see that in practice only a few terms can be calculated since at each stage an inhomogeneous diffusion equation must be solved; though, in principle, all terms can be determined. Note how half-integer powers are induced by the boundary values $f(x)$. No such terms arise for $g(x)$, which is associated with there being no equivalent to region II.

The expansion (5) cannot be valid near $X = 0$, $y = -1$, as is easily seen for the special case

$$f(\pm 0) = \pm 1, \quad f'(\pm 0) = 0, \quad f''(\pm 0) = 0.$$

The functions

$$w_0^{II} = \operatorname{erf}\left(\frac{X}{2(y+1)^{1/2}}\right), \quad w_1^{II} = 0, \quad w_2^{II} = \frac{X^3}{8\pi^{1/2}(y+1)^{5/2}} \exp\left(-\frac{X^2}{4(y+1)}\right)$$

satisfy all conditions, and w_2^{II} becomes unbounded as the discontinuity in f is approached along any path $X/(y+1)^{1/2} = \text{const.} \neq 0$. This is hardly surprising since we are attempting to approximate the solution of the elliptic equation (1a) near a singularity in its boundary data by means of the corresponding singular solutions of parabolic equations.

To take proper account of the rapid changes near the discontinuity we introduce the coordinates

$$X_* = \varepsilon^{-1/2} X, \qquad y_* = \varepsilon^{-1}(1+y) \tag{7}$$

so as to make the neglected term $\varepsilon(\partial^2/\partial y^2) = \varepsilon^{-1}(\partial^2/\partial y_*^2)$ explicitly comparable to $\partial^2/\partial X^2 - \partial/\partial y = \varepsilon^{-1}(\partial^2/\partial X^2 - \partial/\partial y_*)$. The asymptotic expansion

$$w \sim \sum_{k=0}^{\infty} w_k^{II*}(X_*, y_*)\varepsilon^k \tag{8}$$

then leads to the full equation

$$\left(\frac{\partial^2}{\partial X_*^2} + \frac{\partial^2}{\partial y_*^2} - \frac{\partial}{\partial y_*}\right) w_k^{II*} = 0 \tag{9a}$$

for each of the coefficient functions, and the boundary conditions

$$w_k^{II*}(X_*, 0) = f^{(k)}(\pm 0)\frac{X_*^k}{k!} \quad \text{when } X_* \gtrless 0. \tag{9b}$$

The solution at each stage is not completely determinate. However, there is only one solution which does not grow exponentially as $y_* \to \infty$, and this must be selected to ensure matching.

It is then through this matching that the indeterminacy in region II (noted above) is resolved. In particular, one finds that the homogeneous diffusion solution

$$\frac{3}{4\sqrt{\pi}} \frac{X}{(y+1)^{3/2}} \exp\left(-\frac{X^2}{4(y+1)}\right)$$

must be added to w_2^{II}. It is interesting to note that this choice satisfies the principle of minimum singularity: as $y \to -1$ the original w_2^{II} becomes a multiple of $\delta'(X)$ and the added term just cancels this behavior.

We now consider the violated boundary condition. It is convenient to treat the part of the solution due to g separately and to consider $g = 0$ first. The anticipated rapid change in the solution as $y \to 1$ suggests the new coordinate

$$Y = \varepsilon^{-1}(1-y), \tag{10}$$

so that $\varepsilon(\partial^2/\partial y^2) = \varepsilon^{-1}(\partial^2/\partial Y^2)$ is explicitly comparable to $\partial/\partial y = -\varepsilon^{-1}(\partial/\partial Y)$ in (1a). The asymptotic expansion

$$w \sim \sum_{k=0}^{\infty} w_k^{III}(x, Y)\varepsilon^k \tag{11}$$

then leads to the recurrence relation

$$\left(\frac{\partial^2}{\partial Y^2} + \frac{\partial}{\partial Y}\right) w_k^{III} = -\frac{\partial^2}{\partial x^2} w_{k-2}^{III} \tag{12a}$$

and the boundary condition

$$w_k^{III}(x, 0) = 0 \tag{12b}$$

for the coefficient functions. This time the functions are not uniquely determined since at each stage an integration constant is introduced. The N constants in the truncation of the series (11) after N terms are obtained by the matching principle. Y is replaced by $\varepsilon^{-1}(1 - y)$ in the truncated series, which is then expanded to $O(\varepsilon^M)$. Similarly y is replaced by $1 - \varepsilon Y$ in the M-term truncation of the series (2), which is then expanded to $O(\varepsilon^N)$. The results must be identical under the transformation (10).

It is now clear why the boundary condition (3b) had to be satisfied: the matching procedure would fail at $y = -1$. With $Y = \varepsilon^{-1}(1 + y)$, the operator in the recurrence relation (12a) is replaced by $\partial^2/\partial Y^2 - \partial/\partial Y$, so that the function e^Y appears giving terms which cannot be matched since they are of exponential order in ε for fixed y.

Near $x = 0$ we must also introduce the coordinate X and write

$$w \sim \sum_{k=0}^{\infty} w_k^{IV}(X, Y)\varepsilon^{k/2}. \tag{13}$$

The recurrence relation is now

$$\left(\frac{\partial^2}{\partial Y^2} + \frac{\partial}{\partial Y}\right) w_k^{IV} = -\frac{\partial^2}{\partial X^2} w_{k-2}^{IV} \tag{14a}$$

with the boundary condition

$$w_k^{IV}(X, 0) = 0. \tag{14b}$$

Once again there is an integration constant at each stage which is determined by matching with the expansion (5).

Finally we consider the part of the solution due to g. Except near $y = 1$ it will be asymptotically zero, as is seen by setting $f \equiv 0$ above. Near $y = 1$ we again use the coordinate (10), the expansion (11), and the recurrence relation (12a); but instead of the boundary condition (12b), we take

$$w_k^{III}(x, 0) = \begin{cases} g(x) & \text{for } k = 0, \\ 0 & \text{for } k \neq 0. \end{cases} \tag{12c}$$

The integration constant at each stage is determined by matching with zero. Note that $w_k^{III}(x, Y) = 0$ for k odd; in other words, the odd powers of ε are induced by the solution away from $y = 1$. The first coefficient function is clearly

$$w_0^{III}(x, Y) = g(x)\, e^{-Y}$$

so that a discontinuity occurs across any vertical line extending down a distance $O(\varepsilon)$ from a point of discontinuity of g.

For simplicity we shall assume that g has a single discontinuity at $x = a$. Then, introducing

$$x_* = \varepsilon^{-1}(x - a), \tag{15}$$

so as to make $\varepsilon(\partial^2/\partial x^2) = \varepsilon^{-1}(\partial^2/\partial x_*^2)$ explicitly comparable to $\varepsilon^{-1}(\partial^2/\partial Y^2 - \partial/\partial Y)$, we set

$$w \sim \sum_{k=0}^{\infty} w_k^{III*}(x_*, Y)\varepsilon^k. \tag{16}$$

The corresponding recurrence relation is

$$\left(\frac{\partial^2}{\partial x_*^2} + \frac{\partial^2}{\partial Y^2} + \frac{\partial}{\partial Y}\right) w_k^{III*} = 0 \tag{17a}$$

with the boundary condition

$$w_k^{III*}(x_*, 0) = g^{(k)}(a \pm 0)(x_*^k/k!) \quad \text{for } x_* \gtrless 0, \tag{17b}$$

where $g^{(k)}(a \pm 0)$ are right and left kth derivatives of g at $x = a$. At each state we must select the (unique) solution which vanishes as $Y \to \infty$, in order to match with zero.

These then are the results obtained by the method of matched asymptotic expansions, and the object of the present paper is to show that they are uniformly valid representations of the exact solution to all orders in ε. To this end it is necessary to place certain conditions on f, g and to make more precise the regions of validity.

The conditions:

(18a) f is infinitely differentiable for $x \neq 0$ and $f^{(k)}(\pm 0)$ exist for all k;

(18b) g is infinitely differentiable for $x \neq a$ and $g^{(k)}(a \pm 0)$ exist for all k;

are implicitly assumed in using the method. For example, $w_k^I(w_k^{III})$ involves any given derivative of $f(g)$ for k sufficiently large. The further conditions

$$\int_{-\infty}^{\infty} |f^{(k)}(x)|\, dx, \int_{-\infty}^{\infty} |g^{(k)}(x)|\, dx < \infty \quad \text{for all } k \tag{18c}$$

are then a technicality: the derivatives must now die out sufficiently rapidly at $\pm\infty$, but the data there has no asymptotic influence at any finite point.

The regions of validity for the f-expansions are:

(a) I: $x_0 \leqq |x|, \quad -1 \leqq y \leqq y_1 < 1$;

(b) II: $|X| \leqq X_\infty, \quad -1 \leqq y \leqq y_1 < 1$ excluding $|X| \leqq X_0$, $y + 1 \leqq y_{-1}$;

(c) II_*: $|X_*| \leqq X_{*\infty}, \quad 0 \leqq y_* \leqq y_{*\infty}$;

(d) III: $x_0 \leqq |x|, \quad 0 \leqq Y \leqq Y_\infty$;

(e) IV: $|X| \leqq X_\infty, \quad 0 \leqq Y \leqq Y_\infty$.

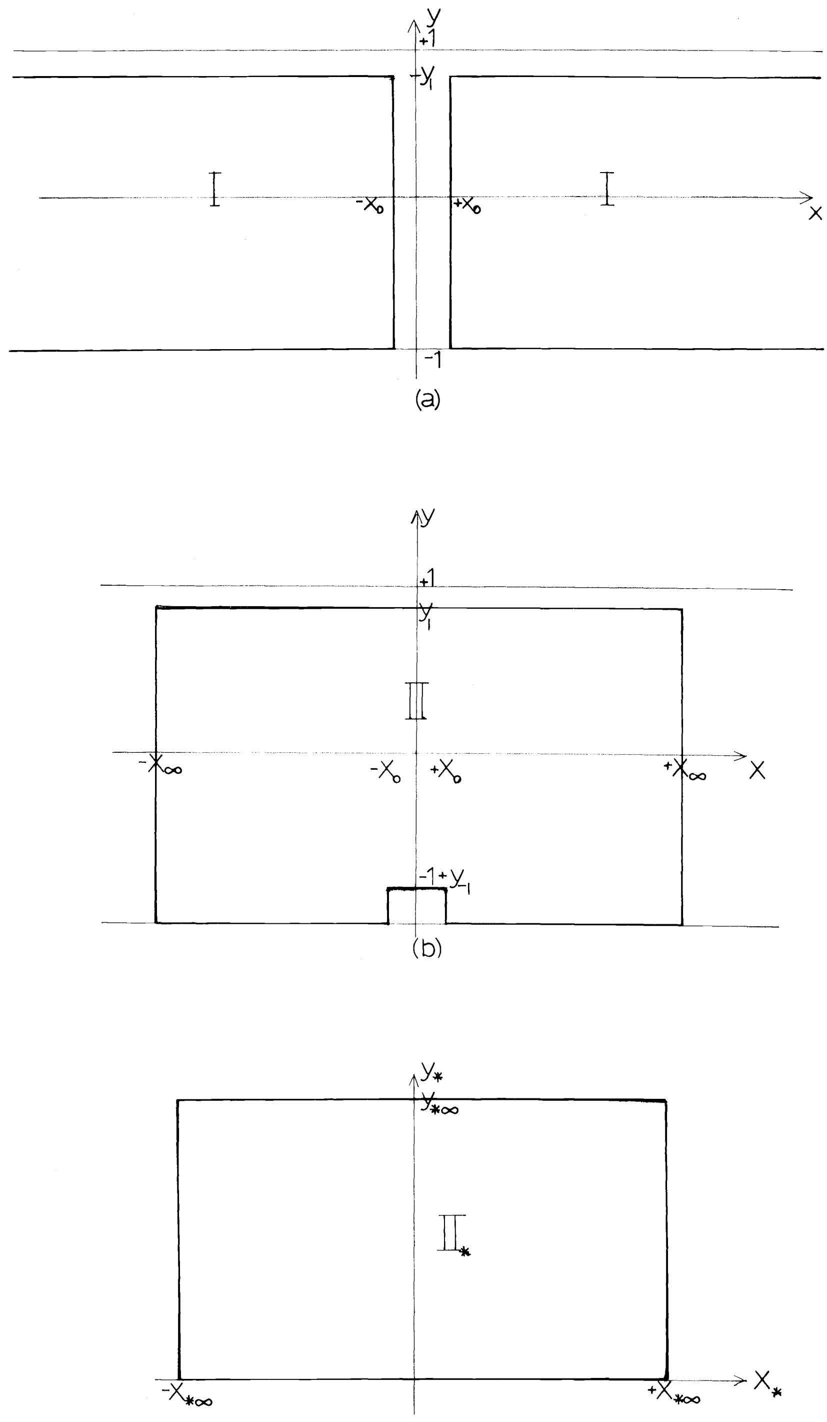
y
+1
y_1
I
$-x_0$
$+x_0$
I
x
-1
(a)
y
+1
y_1
II
$-X_\infty$
$-X_0$
$+X_0$
$+X_\infty$
X
$-1+y_{-1}$
(b)
y_*
$y_{*\infty}$
II_*
$-X_{*\infty}$
$+X_{*\infty}$
X_*
(c)

FIG. 2

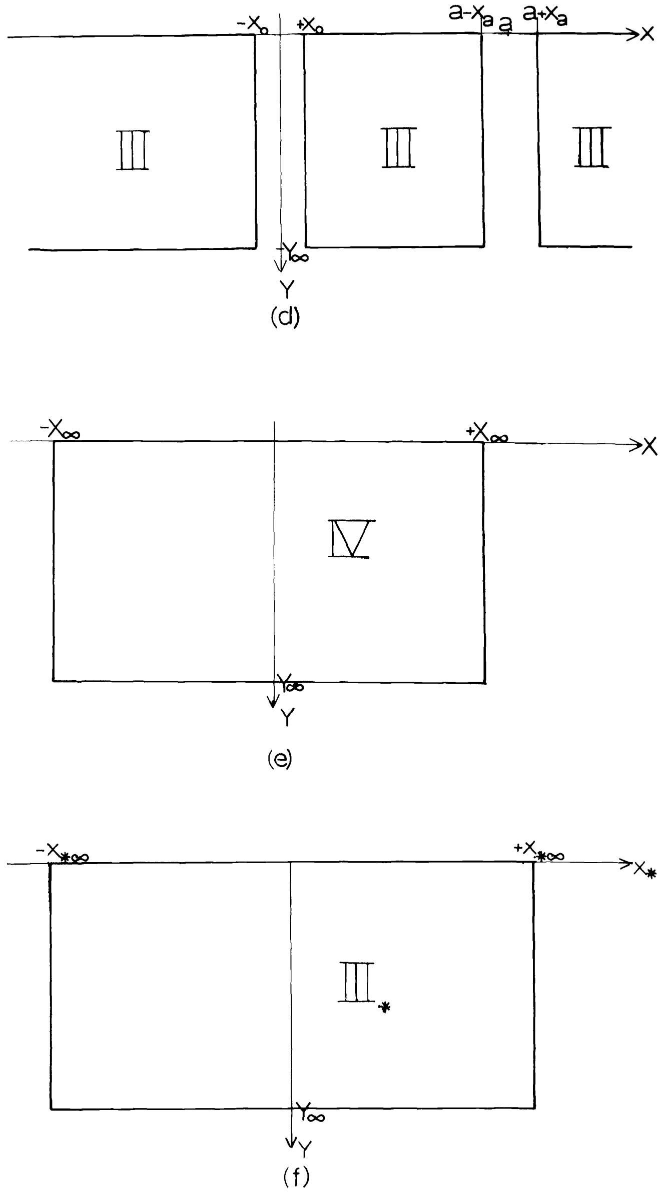

FIG. 2 (*continued*)

For the g-expansions we have

(d) $\quad III: \quad x_a \leqq |x - a|, \qquad 0 \leqq Y \leqq Y_\infty;$

(f) $\quad III_*: \quad |x_*| \leqq x_{*\infty}, \qquad 0 \leqq Y \leqq Y_\infty.$

The lettering corresponds to the parts of Fig. 2.

Here $x_0, y_1, X_\infty, X_0, y_{-1}, X_{*\infty}, y_{*\infty}, Y_\infty, x_a, x_{*\infty}$ are in the first instance fixed positive numbers; but we shall show that the regions can be extended to

$$x_0 = \varepsilon^{1/2-\delta}, \quad 1 - y_1 = \varepsilon^{1-\delta}, \quad X_\infty = \varepsilon^{-1/2+\delta}, \quad X_0 = \varepsilon^{1/4-\delta}, \quad y_{-1} = \varepsilon^{1/2-\delta},$$
$$X_{*\infty} = \varepsilon^{-1+\delta}, \quad y_{*\infty} = (2-\delta)\varepsilon^{-1}, \quad Y_\infty = \varepsilon^{-1+\delta}, \quad x_a = \varepsilon^{1-\delta}, \quad x_{*\infty} = \varepsilon^{-1+\delta},$$

if weaker asymptotic approximations are allowed. Here $\delta > 0$ is arbitrarily small; and, in the case of g, the Y_∞ can in fact be arbitrarily large. Adjacent regions now overlap.

Note that X_0, y_{-1} do not reach the scale of the inner region II_* while $X_{*\infty}$, $y_{*\infty}$ go beyond the scale of region II. Extension is therefore not necessarily a guide to the new scale.

Because the solution is governed by the inhomogeneous diffusion equation (6a) in the region II, Eckhaus and de Jager [3] have called the latter a parabolic layer. Note that there is no difficulty in applying the method of matched asymptotic expansions to the origin of this layer, namely the region II_* (cf. Grasman [5]). For similar reasons we may call the region III a hyperbolic layer and IV a hyperbolic intersection region. The transition zone III_*, and in particular its essential difference from the singular region II_*, has been overlooked in the literature.

3. The exact solution. Taking the Fourier transform

$$\overline{(\cdot)} = \int_{-\infty}^{\infty} e^{-i\xi x}(\cdot)\,dx$$

of the differential equation (1a) and using the boundary condition (1b) we find

$$\bar{w}(\xi, y) = e^{(y+1)/(2\varepsilon)}\left[\frac{e^{-r(y+1)} - e^{r(y-3)}}{1 - e^{-4r}}\right]\bar{f}(\xi) + e^{(y-1)/(2\varepsilon)}\left[\frac{e^{r(y-1)} - e^{-r(y+3)}}{1 - e^{-4r}}\right]\bar{g}(\xi), \tag{19}$$

where

$$r = \sqrt{1 + 4\varepsilon^2\xi^2}/(2\varepsilon).$$

The exact solution of our boundary value problem is then the inverse

$$w(x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi x}\bar{w}(\xi, y)\,d\xi.$$

We are concerned with the asymptotic properties of this solution and will use the abbreviation a.e.s. for "asymptotically exponentially small." Any function which is uniformly a.e.s. in a region can be omitted. Thus e^{-4r} is a.e.s. uniformly in ξ and so will lead to functions which can be ignored throughout the strip. We may therefore write

$$\bar{w} = e^{(y+1)/(2\varepsilon)}[e^{-r(y+1)} - e^{r(y-3)}]\bar{f} + e^{(y-1)/(2\varepsilon)}[e^{r(y-1)} - e^{-r(y+3)}]\bar{g}(\xi).$$

The solution may be further simplified by separating regions near $y = 1$ from the others. Thus in regions I, II, II_*,

$$\bar{w} \sim \exp\left[\left(\frac{1}{2\varepsilon} - r\right)(y + 1)\right]\bar{f} \tag{19'a}$$

since the remaining terms are uniformly a.e.s., provided y_1 approaches 1 more slowly than ε tends to zero. In regions III, III_*, IV,

$$\begin{aligned}\bar{w} \sim \left\{\exp\left[\left(\frac{1}{2\varepsilon} - r\right)(2 - \varepsilon Y)\right] - \exp\left[\frac{2 - \varepsilon Y}{2\varepsilon} - r(2 + \varepsilon Y)\right]\right\}\bar{f}& \\ + \exp\left[-\left(\frac{1}{2\varepsilon} + r\right)\varepsilon Y\right]\bar{g},&\end{aligned} \tag{19'b}$$

the remaining terms being uniformly a.e.s., provided Y_∞ tends to infinity more slowly than ε^{-1}.

The brackets multiplying $\bar{f}$ and $\bar{g}$ in the Fourier transform (19) of the exact solution come from the Green's function, and in order to see the separate effects of the Bessel functions mentioned in the Introduction, their common denominator should be expanded as a series of exponentials. The first term in the $\bar{f}$-bracket then gives rise to

$$\exp\left(\frac{y + 1}{2\varepsilon}\right)\exp\left[-r(y + 4k + 1)\right], \qquad k = 0, 1, 2, 3, \cdots,$$

where the second factor is the transform of the normal derivative at the lower boundary of a Bessel function representing the (equal) effects of the fundamental solution and the images at $y + 4k$ and $-(y + 4k + 2)$. Similarly the second term yields

$$-\exp\left(\frac{y + 1}{2\varepsilon}\right)\exp\left[-r(2 - y + 4k + 1)\right], \qquad k = 0, 1, 2, 3, \cdots,$$

representing the (equal) effects of images at $2 - y + 4k$ and $y - 4k - 4$. Together these account for the fundamental solution and all images in $y = \pm 1$, at $2l + (-1)^l y$, $l = 0, \pm 1, \pm 2, \cdots$. Similarly the bracket multiplying $\bar{g}$ leads to

$$\exp\left(\frac{y - 1}{2\varepsilon}\right)\exp\left[-r(2 - y + 4k - 1)\right] \quad \text{and}$$

$$-\exp\left(\frac{y - 1}{2\varepsilon}\right)\exp\left[-r(y + 4k + 4 - 1)\right],$$

where the second factors are derived from the same Bessel functions, this time the normal derivative at the upper boundary being taken. The pairings $2 - y + 4k$, $y - 4k$ and $y + 4k + 4$, $-(y + 4k + 2)$ are different because the boundary is, but the same points are involved.

Only the fundamental solution and its image in the lower boundary contribute to the simplified form (19′a). In addition to these, only the image of the fundamental solutions in the upper boundary plus its further image in the lower boundary contribute to the $\bar{f}$-term in (19′b). (Note that y and $2 - y$ approach each other as

$y \to 1$.) The $\bar{g}$-term has only the fundamental solution and its image in the upper boundary.

4. The core region I. Here we have the representation (19′a), which may be written

$$\bar{w} \sim \bar{K}\bar{f}, \quad \text{where} \quad \bar{K}(\xi, y; \varepsilon) = \exp[(1/(2\varepsilon) - r)(y + 1)]. \tag{20}$$

$\bar{K}$ is in fact the transform of the Green's function of the original equation (1a) for the half-plane $y \geqq -1$, so that it satisfies

$$\frac{\partial}{\partial y}\bar{K} = \varepsilon\left(\frac{\partial^2}{\partial y^2} - \xi^2\right)\bar{K}, \qquad \bar{K}(\xi, -1; \varepsilon) = 1 \tag{21}$$

as can easily be seen directly.

Consider now the Taylor expansion,

$$\bar{K}(\xi, y; \varepsilon) = \sum_{k=0}^{m-1} \bar{K}^{(k)}(\xi, y; 0)\frac{\varepsilon^k}{k!} + \bar{R}_m(\xi, y; \varepsilon), \tag{22}$$

where

$$\bar{R}_m(\xi, y; \varepsilon) = \bar{K}^{(m)}(\xi, y; t\varepsilon)\frac{\varepsilon^m}{m!}, \qquad 0 < t < 1,$$

to any order m. The series apparently leads to the first m terms of the expansion (2) with coefficient functions which are the inverses of $\bar{f}\bar{K}^{(k)}/k!$. However these inverses do not exist (in the ordinary sense) since $\bar{K}^{(k)}$ is $O(\xi^{2k})$ for large ξ and, in order to obtain ones which do, we replace f with a function F such that

$$\begin{gathered} F \equiv f \quad \text{for } x_0/2 \leqq |x|, \\ F \in C^\infty \quad \text{and} \quad \int_{-\infty}^{\infty} |F^{(k)}(x)|\, dx < \infty \quad \text{for all } k. \end{gathered} \tag{23}$$

The construction and properties of F are given in the Appendix. Since it has integrable derivatives of all orders, its transform is $o(\xi^{-N})$ for every N and the inverses mentioned above exist for every k.

We may therefore write

$$w \sim \sum_{k=0}^{m-1} w_k^I \varepsilon^k + R_m * F + K * (f - F),$$

where

$$w_k^I = \frac{1}{2\pi k!}\int_{-\infty}^{\infty} \bar{K}^{(k)}(\xi, y; 0)\bar{F}(\xi)\, e^{i\xi x}\, d\xi, \tag{24a}$$

$$R_m * F = \frac{\varepsilon^m}{2\pi m!}\int_{-\infty}^{\infty} \bar{K}^{(m)}(\xi, y; t\varepsilon)\bar{F}(\xi)\, e^{i\xi x}\, d\xi, \tag{24b}$$

$$K * (f - F) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{K}(\xi, y; \varepsilon)[\bar{f}(\xi) - \bar{F}(\xi)]\, e^{i\xi x}\, d\xi. \tag{24c}$$

We shall now show that: (i) the w_k^I satisfy the recurrence relation (3a) and the boundary conditions (3b); and that, uniformly in the region I (with x_0, y_1 fixed),

(ii) $R_m * F = O(\varepsilon^m)$ for every m and (iii) $K * (f - F)$ is a.e.s. The validity of the expansion (2) will thereby be established.

(i) By substituting the expansion (22) into (21) we find

$$\frac{\partial}{\partial y}\bar{K}^{(k)}(\xi, y; 0) = k\left(\frac{\partial^2}{\partial y^2} - \xi^2\right)\bar{K}^{(k-1)}(\xi, y; 0), \qquad \bar{K}^{(k)}(\xi, -1, 0) = \begin{cases} 1, & k = 0, \\ 0, & k \neq 0. \end{cases}$$

From these it is easily seen that the integrals (24a) satisfy the recurrence relation and boundary conditions.

(ii) Since $\bar{K}^{(m)}(\xi, y; \varepsilon)$ is the sum of terms of the form

$$\varepsilon^{\alpha_1}(1 + 4\varepsilon^2\xi^2)^{-\alpha_2/2}(1 + (1 + 4\varepsilon^2\xi^2)^{1/2})^{-\alpha_3}(y + 1)^{\beta}\xi^{2\gamma}\exp\left[\frac{-2\varepsilon\xi^2(y + 1)}{1 + \sqrt{1 + 4\varepsilon^2\xi^2}}\right],$$

where $\alpha_1, \alpha_2, \alpha_3, \beta, \gamma$ are nonnegative integers with $\gamma \leqq m$, there exists a constant C_m such that

$$|\bar{K}^{(m)}(\xi, y; t\varepsilon)| \leqq C_m(1 + \xi^2)^m$$

when ε is bounded. It follows that

$$|R_m * F| < \frac{C_m\varepsilon^m}{2\pi m!}\int_{-\infty}^{\infty}(1 + \xi^2)^m|\bar{F}(\xi)|\, d\xi = O(\varepsilon^m) \tag{25}$$

since $\bar{F}$ is $o(\xi^{-N})$ for every N.

(iii) From the convolution theorem and the definition (24c) we may write

$$K * (f - F) = \frac{y + 1}{2\pi\varepsilon}e^{(y+1)/(2\varepsilon)}\int_{-x_0/2}^{x_0/2}\frac{K_1(\sqrt{(x - x')^2 + (y + 1)^2}/(2\varepsilon))}{\sqrt{(x - x')^2 + (y + 1)^2}} \cdot [f(x') - F(x')]\, dx', \tag{26}$$

where K_1 is the modified Bessel function. Since the integration variable x' is always bounded away from x when the latter lies in region I, we deduce from the exponential behavior of K_1 for large values of its argument that $K * (f - F)$ is a.e.s. uniformly in the region I.

There remains the question of extending the region of validity by accepting a weaker asymptotic approximation. The limitation

$$(1 - y_1) = \varepsilon^{1-\delta} \tag{27}$$

is accepted in using the simplified form (19′a) and no further condition is imposed by the analysis of the present section. Hence we may concentrate on x_0, and the problem is to determine the asymptotic behavior of $R_m * F$ and $K * (f - F)$ when

$$x_0 = \varepsilon^{\kappa} \quad \text{with} \quad \kappa > 0.$$

$R_m * F$ requires a more careful estimate of $\bar{F}$, which can be obtained from

$$\bar{F}(\xi) = (i\xi)^{-(2m+1)}\int_{-\infty}^{\infty}F^{(2m+1)}(x)\, e^{-i\xi x}\, dx.$$

Since

$$F^{(2m+1)}(x_0 x) = x_0^{-(2m+1)} F_m(x; x_0),$$

where F_m is bounded as $x_0 \to 0$ (see Appendix), it follows that

$$\bar{F}\left(\frac{\xi}{x_0}\right) = x_0 (i\xi)^{-(2m+1)} \int_{-\infty}^{\infty} F_m(x; x_0)\, e^{-i\xi x}\, dx$$

is bounded by $x_0 A_m \xi^{-(2m+2)}$ as $|\xi| \to \infty$. Here A_m depends on x_0 but, being bounded as $x_0 \to 0$, it may be replaced by a constant. By changing ξ^2 into $(1 + \xi^2)$ and the constant A_m appropriately, we then have a bound for all ξ, which can be used in the estimate (25) to give

$$|R_m * F| < \frac{AC_m}{2\pi m!} \varepsilon^m x_0^{-2m} \int_{-\infty}^{\infty} \frac{(x_0^2 + \xi^2)^m}{(1 + \xi^2)^{m+1}}\, d\xi = O(\varepsilon^{m(1-2\kappa)}).$$

Hence asymptotic approximation on a weaker scale is obtained provided

$$\kappa < \tfrac{1}{2}.$$

$K * (f - F)$ remains a.e.s. for such an x_0.

5. The free layer *II*. The last limitation suggests introducing the coordinate (4) to describe the solution near $x = 0$. Correspondingly the transform variable is changed to

$$\eta = \varepsilon^{1/2} \xi$$

so that (with tildes denoting the new transforms)

$$\tilde{w} \sim \varepsilon^{-1/2} \tilde{L} \bar{f}(\varepsilon^{-1/2}\eta),$$

where

$$\tilde{L}(\eta, y; \varepsilon) = \exp\left[\left(\frac{1}{2\varepsilon} - s\right)(y + 1)\right] \quad \text{and} \quad s = \frac{\sqrt{1 + 4\varepsilon\eta^2}}{2\varepsilon}. \tag{28}$$

The kernel $\tilde{L}$ satisfies

$$\left(\frac{\partial}{\partial y} + \eta^2\right)\tilde{L} = \varepsilon \frac{\partial^2}{\partial y^2}\tilde{L}, \qquad \tilde{L}(\eta, -1; \varepsilon) = 1. \tag{29}$$

As before, the Taylor expansion

$$\tilde{L}(\eta, y; \varepsilon) = \sum_{k=0}^{m-1} \tilde{L}^{(k)}(\eta, y; 0)\frac{\varepsilon^k}{k!} + \tilde{S}_m(\eta, y; \varepsilon), \tag{30}$$

where

$$\tilde{S}_m(\eta, y; \varepsilon) = \tilde{L}^{(m)}(\eta, y; t\varepsilon)\frac{\varepsilon^m}{m!},$$

will be needed. Simultaneous expansion of $\bar{f}$ and inversion of the coefficients of successive powers of ε then apparently lead to the first m terms of the series (5).

However this involves divergent integrals, which may be avoided by using the convolution theorem to invert before expanding f. With

$$f(\varepsilon^{1/2}x) = \sum_{k=0}^{m-1} f^{(k)}(\pm 0)X^k\frac{\varepsilon^{k/2}}{k!} + f^{(m)}(t\varepsilon^{1/2}X)X^m\frac{\varepsilon^{m/2}}{m!} \quad \text{for } X \gtrless 0,$$

we find

$$w \sim \sum_{k=0}^{m-1} w_k^{II}\varepsilon^{k/2} + S_m * f + T_m,$$

where

(31a) $$\omega_k^{II} = \sum_{j=0}^{[k/2]} \frac{1}{j!(k-2j)!}\int_{-\infty}^{\infty} L^{(j)}(X - X', y; 0) f_{k-2j}(X')X'^{(k-2j)}\,dX',$$

(31b) $$S_m * f = \frac{\varepsilon^n}{n!}\int_{-\infty}^{\infty} L^{(n)}(X - X', y; t\varepsilon) f(\varepsilon^{1/2}X')\,dX',$$

(31c) $$T_m = \varepsilon^{m/2}\sum_{j=0}^{n-1} \frac{1}{j!(m-2j)!}\int_{-\infty}^{\infty} L^{(j)}(X - X', y; 0) f^{(m-2j)}(t\varepsilon^{1/2}X')X'^{(m-2j)}\,dX',$$

with

(31′) $$n = \left[\frac{m+1}{2}\right] = \begin{cases}(m+1)/2 & \text{for } m \text{ odd},\\ m/2 & \text{for } m \text{ even},\end{cases}$$

$$f_{k-2j}(X) = f^{(k-2j)}(\pm 0) \quad \text{for } X \gtrless 0,$$

and t may vary from function to function. We shall now show that: (i) the w_k^{II} satisfy the recurrence relation (6a) and the boundary conditions (6b); and that (ii) $S_m * f$ and T_m are $O(\varepsilon^{m/2})$, for every m, uniformly in the region II (with X_0, X_∞, y_{-1} and y_1 fixed). Proof of matching with the II_*-expansion will however be postponed to § 6. Expansion (5) will then have been validated.

(i) Substitution of the expansion (30) into the equations (29) and inversion show that

$$\left(\frac{\partial^2}{\partial X^2} - \frac{\partial}{\partial y}\right)L^{(k)}(X, y; 0) = -k\frac{\partial^2}{\partial y^2}L^{(k-1)}(X, y; 0),$$

$$L^{(k)}(X, -1; 0) = \begin{cases}\partial(X), & k = 0,\\ 0, & k \neq 0.\end{cases}$$

The sum of integrals (31a) is now seen to satisfy the recurrence relation and boundary conditions.

(ii) We need only prove that each of the integrals (31b), (31c) is bounded in the region II whenever ε is bounded. But some care is needed, as is easily seen for $S_m * f$ from the terms

(32) $$(1 + 4\varepsilon\eta^2)^{-\alpha/2}[1 + (1 + 4\varepsilon\eta^2)^{1/2}]^{-\beta}(y+1)^\gamma\eta^{2\delta}\exp[(\tfrac{1}{2\varepsilon} - s)(y+1)]$$

which form $\tilde{L}^{(j)}(\eta, y; \varepsilon)$ for $j \neq 0$, where α, β, γ, δ are positive integers with $\alpha + \beta \geqq 3$ and $4 \leqq \delta \leqq 2j$. At $y = -1$ no help is obtained from the exponential, and the corresponding inversion integral is divergent for all x.

Nevertheless, the integral has a limit as $y \to -1$ if $X \neq 0$, so we shall manipulate it until this limit is exhibited when y is set equal to -1. The first step is to write the corresponding term in $L^{(j)}(X, y; \varepsilon)$ in the form

$$\frac{(-1)^\delta}{2\pi}(y+1)^\gamma \frac{\partial^{2\delta}}{\partial X^{2\delta}} \int_{-\infty}^{\infty} (1 + 4\varepsilon\eta^2)^{-\alpha/2} \cdot [1 + (1 + 4\varepsilon\eta^2)^{1/2}]^{-\beta} \exp[(\tfrac{1}{2\varepsilon} - s)(y+1)]\, e^{i\eta X}\, d\eta$$

and integrate on X' by parts 2δ times. Because of the discontinuities in f and its derivatives at $X = 0$, the integrated terms will be multiples of

$$\varepsilon^{\delta - (\mu+1)/2}(y+1)^\gamma \int_{-\infty}^{\infty} (1 + 4\varepsilon\eta^2)^{-\alpha/2}[1 + (1 + 4\varepsilon\eta^2)^{1/2}]^{-\beta} \cdot \exp[(\tfrac{1}{2\varepsilon} - s')(y+1)](i\eta)^\mu e^{i\eta X}\, d\eta, \qquad \text{where} \quad 0 \leqq \mu \leqq 2\delta - 1; \tag{33a}$$

the remaining integral is a multiple of

$$\varepsilon^\delta (y+1)^\gamma \int_{-\infty}^{\infty} f^{(2\delta)}(\varepsilon^{1/2}X')\, dX' \int_{-\infty}^{\infty} (1 + 4\varepsilon\eta^2)^{-\alpha/2} \cdot [1 + (1 + 4\varepsilon\eta^2)^{1/2}]^{-\beta} \exp[(\tfrac{1}{2\varepsilon} - s')(y+1)]\, e^{i\eta(X - X')}\, d\eta. \tag{33b}$$

Here M_2' denotes M_2 with ε replaced by $t\varepsilon$. The integral in (33a) is bounded for all X, ε when y is bounded away from -1; but it is still divergent for $y = -1$. Convergence at $y = -1$ for X positive can be ensured by bending the ends of the integration path upwards in the complex η-plane, so that they asymptote at an angle to the real axis. The integral is then seen to be convergent for $y = -1$ and bounded for all y, when X is bounded away from zero. Deform downwards for X negative. In short, the terms (33a) are uniformly bounded in the region *II*. In bounding the integral in (33b) we note that the η-integral is bounded by a multiple of $\varepsilon^{-1/2}$, as can be seen by using $t^{1/2}\varepsilon^{1/2}\eta$ for integration variable and remembering $\alpha + \beta > 1$. The absolute integrability of $f^{(2\delta)}$ then ensures the contribution (33b) to be bounded in region *II* (in fact $O(\varepsilon^{\delta - 1})$) since $\delta \geqq 1$.

The treatment of (31c) is similar. Each integral (including $j = 0$ now) involves the sum of terms of the form (32) with $\varepsilon = 0$, on each of which integration by parts is performed 2δ times. In place of the expression (33a) we now have

$$\varepsilon^{\nu/2}(y+1)^\gamma \int_{-\infty}^{\infty} \exp[-\eta^2(y+1)](i\eta)^\mu e^{i\eta X}\, d\eta = \sqrt{\pi}\,\varepsilon^{\nu/2}(y+1)^{\gamma - 1/2} \frac{\partial^\mu}{\partial X^\mu} \exp\left[\frac{-X^2}{y+1}\right], \tag{34a}$$

where ν is a nonnegative integer, which is clearly uniformly bounded in the region

II. The expression (33b) is likewise replaced by

$$(34b)\quad \begin{aligned}&(y+1)^{\gamma}\int_{-\infty}^{\infty}\mathscr{F}(X')\,dX'\int_{-\infty}^{\infty}\exp\left[-\eta^2(y+1)\right]e^{i\eta(X-X')}\,d\eta\\ &\qquad=(y+1)^{\gamma}\int_{-\infty}^{\infty}\mathscr{F}(X')\left(\frac{\pi}{y+1}\right)^{1/2}\exp\left[\frac{-(X-X')^2}{4(y+1)}\right]dX',\end{aligned}$$

where

$$\mathscr{F}(X')=\frac{\partial^{2\delta}}{\partial X'^{2\delta}}\left[f^{(m-2j)}(t\varepsilon^{1/2}X')X'^{(m-2j)}\right]$$

is bounded by a power of X'. It follows that the integral is uniformly bounded in the region *II*.

Extension of the region upwards to

$$1-y_1=\varepsilon^{1-\delta}$$

is valid, as in § 4. Sideways, the limitation

$$X_{\infty}=\varepsilon^{-1/2+\delta}$$

arises from the integrals (34b). For $X=\varepsilon^{-\kappa}$ they behave like $\varepsilon^{\delta-\kappa(m-2j)}$, of which the worst is $\varepsilon^{-\kappa m}$. From (31c) we then see that κ must be less than 1/2.

Extension towards the point $X=0$, $y=-1$ is limited by the exponent in the integrals (33a), which for points near $X_0=\varepsilon^{\kappa}$, $y_{-1}=\varepsilon^{\lambda}$ with $\kappa,\lambda>0$ becomes

$$\frac{-2\eta^2}{1+(1+4\varepsilon\eta^2)^{1/2}}\varepsilon^{\lambda}\pm i\eta\varepsilon^{\kappa}.$$

Both terms have negative real parts (after deformation), one of which may be prevented from vanishing in the limit $\varepsilon\to 0$ by the transformation $\eta=\varepsilon^{-\kappa}\tau$ when $\lambda\geqq 2\kappa$ or $\eta=\varepsilon^{-\lambda/2}\tau$ when $\lambda\leqq 2\kappa$. The terms (33a) are then of order $\varepsilon^{\delta-(\mu+1)/2}$ times $\varepsilon^{-(\mu+1)\kappa}$ or $\varepsilon^{-(\mu+1)\lambda/2}$ so that the worst is of order $\varepsilon^{-4\kappa n}$ or $\varepsilon^{-2\lambda n}$. From (31b) we then see that λ can be arbitrarily large provided

$$\kappa<\tfrac{1}{4},$$

while κ can be arbitrarily large provided

$$\lambda<\tfrac{1}{2}.$$

It is noteworthy that the region *II* can only be extended down to half the scale of II_*: extension is not a reliable guide to the new scale.

6. The singular region II_*. The limitations on the extension of region *II* suggest that the coordinates (7) are needed to describe the solution near $X=0$, $y=-1$. The transform variable is likewise changed to

$$\eta_*=\varepsilon^{1/2}\eta$$

so that (with hats denoting the new transforms)

$$\hat{w}\sim\varepsilon^{-1}\hat{L}_*\bar{f}(\varepsilon^{-1}\eta_*),$$

where

$$\hat{L}_*(\eta_*, y_*) = \exp[(\tfrac{1}{2} - s_*)y_*] \quad \text{and} \quad s_* = \sqrt{1 + 4\eta_*^2}/2.$$

The kernel $\hat{L}_*$, which is now independent of ε, satisfies

$$\left(\frac{\partial^2}{\partial y_*^2} - \frac{\partial}{\partial y_*} - \eta_*^2\right)\hat{L}_* = 0, \qquad \hat{L}_*(\eta_*, 0) = 1; \tag{35}$$

and, in place of a second boundary condition, it does not grow exponentially as $y_* \to \infty$.

No expansion of the kernel is involved this time, but it is again necessary to invert by convolution before expanding f to avoid divergent integrals. We find

$$w \sim \sum_{k=0}^{m-1} w_k^{II*}(X_*, y_*)\varepsilon^k + L_* * f^{(m)},$$

where

$$w_k^{II*} = \frac{1}{k!}\int_{-\infty}^{\infty} L_*(X_* - X'_*, y_*) f_k(X'_*) X'^k_* \, dX'_*, \tag{36a}$$

$$L_* * f^{(m)} = \frac{\varepsilon^m}{m!}\int_{-\infty}^{\infty} L_*(X_* - X'_*, y_*) f^{(m)}(t\varepsilon X'_*) X'^m_* \, dX'_*. \tag{36b}$$

There is no difficulty in showing that: (i) the w_k^{II*} satisfy the equation (9a), the boundary conditions (9b), and the matching conditions noted after them; and that (ii) $L_* * f^{(m)} = O(\varepsilon^m)$ uniformly in the region II_* (with $X_{*\infty}$, $y_{*\infty}$ fixed). The validity of the expansion (8) is thereby established.

(i) Substitute the integrals (36a) directly into the equation and boundary conditions to show that they satisfy them by virtue of the equations (35). The series formed from them matches that formed from the integrals (31a) by virtue of the matching of $\hat{L}_*$ and $\tilde{L}$.

(ii) The integral in $L_* * f^{(m)}$ is actually

$$\frac{y_* e^{y_*/2}}{2\pi}\int_{-\infty}^{\infty} \frac{K_1(\frac{1}{2}\sqrt{(X_* - X'_*)^2 + y_*^2})}{\sqrt{(X_* - X'_*)^2 + y_*^2}} X'^m_* f^{(m)}(t\varepsilon X'_*) \, dX'_*$$

which is seen to be bounded in II_* when ε is bounded.

Extending the region of validity to

$$X_{*\infty} = \varepsilon^{-\kappa}, \qquad y_{*\infty} = \varepsilon^{-\lambda} \quad \text{with } \kappa, \lambda > 0$$

requires an estimation of the last integral for such values of X_*, y_*. On interchanging $X_* - X'_*$ and X'_* and noting that $f^{(m)}$ is bounded, we see that there remains

$$\begin{aligned} &y_* e^{y_*/2}\int_{-\infty}^{\infty} \frac{K_1(\frac{1}{2}\sqrt{X'^2_* + y_*^2})}{\sqrt{X'^2_* + y_*^2}} \sum_{s=0}^{m} \binom{m}{s} |X_*|^{m-s}|X'_*|^s \, dX'_* \\ &\qquad = \sum_{s=0}^{m} c_s |X_*|^{m-s} y_*^{(s+1)/2} e^{y_*/2} K_{(1-s)/2}\left(\frac{y_*}{2}\right), \end{aligned} \tag{37}$$

where

$$c_s = 2^{s+1}\Gamma\left(\frac{s+1}{2}\right)\binom{m}{s}.$$

The terms in the series are clearly of order $\varepsilon^{-(m-s)\kappa}\varepsilon^{-s\lambda/2} = \varepsilon^{-m\kappa+s(\kappa-\lambda/2)}$, there being no singularities in the y_*-functions at $y_* = 0$. The worst term is of order

$$\varepsilon^{-m\kappa} \quad \text{for } \kappa \geqq \lambda/2 \quad \text{and} \quad \varepsilon^{-m\lambda/2} \quad \text{for } \kappa \leqq \lambda/2;$$

in either case we must have

$$\kappa < 1, \qquad \lambda < 2$$

if the remainder (36b) is to be asymptotically small (on a weaker scale).

Thus the expansion gives an approximation even in the extended free layer itself. In fact, since its accuracy in the region *II* is the same as that of expansion (5), it must then be identical and this is easily checked. Note that values of λ greater than 1 are not of interest here since we have already accepted the limitation (27).

7. The boundary layer *III*: $g \equiv 0$. We must now use the representation (19′b) with $\bar{g} \equiv 0$, which will be written

$$\bar{w} \sim \bar{\mathscr{K}}\bar{f},$$

where

$$\bar{\mathscr{K}}(\xi, Y; \varepsilon) = \exp\left[\left(\frac{1}{2\varepsilon} - r\right)(2 - \varepsilon Y)\right] - \exp\left[\frac{(2-\varepsilon Y)}{2\varepsilon} - r(2 + \varepsilon Y)\right].$$

Clearly,

$$\left(\frac{\partial^2}{\partial Y^2} + \frac{\partial}{\partial Y}\right)\bar{\mathscr{K}} = \varepsilon^2\xi^2\bar{\mathscr{K}}, \qquad \bar{\mathscr{K}}(\xi, 0; \varepsilon) = 0, \tag{38}$$

but there is no second boundary condition. Instead we shall show that $\bar{\mathscr{K}}$ matches the $\bar{K}$ of definition (20) to all orders.

First note that there is no contribution from $\exp[(2 - \varepsilon Y)/(2\varepsilon) - r(2 + \varepsilon Y)]$: with $Y = (1 - y)/\varepsilon$ and $-1 \leqq y \leqq y_1$, it is a.e.s. uniformly for ξ real. Accordingly we must show that $\bar{K} = \exp[(1/(2\varepsilon) - r)(y + 1)] = \exp[(1/(2\varepsilon) - r)(2 - \varepsilon Y)]$ satisfies the matching principle. While it would be difficult to believe otherwise, a formal proof is as follows. $\bar{K}$ has the expansion (22) for $|y| \leqq 1$ for every m. Moreover, since $\bar{K}^{(k)}(\xi, y; 0)$ is a polynomial in $(y + 1)$ of degree k (with coefficients depending on ξ^2—see § 4(ii)), it certainly has an inner expansion (in Y). Hence, according to Fraenkel's Theorem 1 [4], the matching principle holds to all orders.

We now introduce the Taylor expansion

$$\bar{\mathscr{K}}(\xi, Y; \varepsilon) = \sum_{k=0}^{m-1} \bar{\mathscr{K}}^{(k)}(\xi, Y; 0)\frac{\varepsilon^k}{k!} + \bar{\mathscr{R}}_m(\xi, Y; \varepsilon), \tag{39}$$

where

$$\bar{\mathscr{R}}_m(\xi, Y; \varepsilon) = \bar{\mathscr{K}}^{(m)}(\xi, Y; t\varepsilon)\frac{\varepsilon^m}{m!}.$$

The inverse of $\bar{\mathscr{K}}^{(k)}$ does not exist, as may be expected from the similar difficulty in the core region since there is matching. As in § 4 term-by-term inversion must be done after F, the smoothed version (23) of f, has been introduced; so that

$$w \sim \sum_{k=0}^{m-1} w_k^{III}\varepsilon^k + \mathscr{R}_m * F + \mathscr{K} * (f - F),$$

where

(40a)
$$w_k^{III} = \frac{1}{2\pi k!}\int_{-\infty}^{\infty} \bar{\mathscr{K}}^{(k)}(\xi, Y; 0)\bar{F}(\xi)\, e^{i\xi x}\, d\xi,$$

(40b)
$$\mathscr{R}_m * F = \frac{\varepsilon^m}{2\pi m!}\int_{-\infty}^{\infty} \bar{\mathscr{K}}^{(m)}(\xi, Y; t\varepsilon)\bar{F}(\xi)\, e^{i\xi x}\, d\xi,$$

(40c)
$$\mathscr{K} * (f - F) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{\mathscr{K}}(\xi, Y; \varepsilon)[\bar{f}(\xi) - \bar{F}(\xi)]\, e^{i\xi x}\, d\xi.$$

It remains to be shown that (i) the w_k^{III} satisfy the recurrence relation (12a), the boundary conditions (12b), and the matching conditions mentioned after them; and that (ii) $\mathscr{R}_m * F = O(\varepsilon^m)$ for every m and (iii) $\mathscr{K} * (f - F)$ is a.e.s., both uniformly in the region III (with x_0, Y_∞ fixed). In other words, the validity of the expansion (11) will be established. The proof is similar to that in region I (§ 4).

(i) By substituting the expansion (39) into the equations (38), we find

$$\left(\frac{\partial^2}{\partial Y^2} + \frac{\partial}{\partial Y}\right)\bar{\mathscr{K}}^{(k)}(\xi, Y; 0) = k(k-1)\xi^2\bar{\mathscr{K}}^{(k-2)}(\xi, Y; 0), \qquad \bar{\mathscr{K}}^{(k)}(\xi, 0; 0) = 0$$

for all k. Hence the integrals (40a) satisfy the recurrence relation and boundary conditions. The fact that the series formed from them matches the series formed from the integrals (24a) follows from the matching of $\bar{\mathscr{K}}$ and $\bar{K}$ as proved above.

(ii) $\bar{\mathscr{K}}^{(m)}(\xi, Y; \varepsilon)$ is the sum of terms of the form

(41)
$$(\varepsilon Y)^{\alpha_1}(1 + 4\varepsilon^2\xi^2)^{-\alpha_2/2}(1 + \sqrt{1 + 4\varepsilon^2\xi^2})^{-\alpha_3} Y^\beta \xi^{2\gamma}$$

times

$$\exp\left[\frac{-2\varepsilon\xi^2(2 - \varepsilon Y)}{1 + \sqrt{1 + 4\varepsilon^2\xi^2}}\right] \quad \text{or} \quad \exp\left[\frac{-4\varepsilon\xi^2}{1 + \sqrt{1 + 4\varepsilon^2\xi^2}} - \frac{(1 + \sqrt{1 + 4\varepsilon^2\xi^2})Y}{2}\right],$$

where $\alpha_1, \alpha_2, \alpha_3, \beta, \gamma$ are nonnegative integers with $\beta \leqq m - \gamma$ and $\gamma \leqq m$. Hence $\bar{\mathscr{K}}^{(m)}$ can be bounded as in § 4(ii), so that the smoothness of F ensures $\mathscr{R}_m * F$ is $O(\varepsilon^m)$.

(iii) The inverse of $\bar{\mathscr{K}}$ is

$$\frac{\exp[(2 - \varepsilon Y)/(2\varepsilon)]}{2\pi\varepsilon}\left\{\frac{2 - \varepsilon Y}{\sqrt{x^2 + (2 - \varepsilon Y)^2}} K_1\left(\frac{\sqrt{x^2 + (2 - \varepsilon Y)^2}}{2\varepsilon}\right) - \frac{2 + \varepsilon Y}{\sqrt{x^2 + (2 + \varepsilon Y)^2}} K_1\left(\frac{\sqrt{x^2 + (2 + \varepsilon Y)^2}}{2\varepsilon}\right)\right\}$$

so that the convolution argument in § 3(iii) shows $\mathscr{K} * (f - F)$ to be a.e.s. uniformly in region III.

Extending the region of validity by setting

$$x_0 = \varepsilon^k, \qquad Y_\infty = \varepsilon^{-\lambda} \qquad \text{with } \kappa, \lambda > 0$$

follows the same lines as for region I (see end of § 4). Because of the occurrence of Y in the terms (41) making up the derivatives of $\bar{\mathscr{K}}$, each of the terms in $\mathscr{R}_m * F$ must be bounded separately using different estimates of $\bar{F}$. Anticipating that λ is not greater than 1 (so that εY is bounded), we see that the worst terms are those containing $Y^{m-\gamma}\xi^{2\gamma}$; and if the estimate of $\bar{F}$ obtained from $F^{(2\gamma+1)}$ is used (cf. end of § 3), their contribution to $\mathscr{R}_m * F$ is seen to be at most $O(\varepsilon^m x_0^{-2\gamma} Y_\infty^{m-\gamma})$. Letting γ range from 0 to m now shows that we must have

$$\kappa < \tfrac{1}{2} \quad \text{and} \quad \lambda < 1.$$

It is easily checked that $\mathscr{K} * (f - F)$ remains a.e.s. for such x_0, Y_∞.

8. The intersection IV of the layers: $g = 0$. To describe the solution near $x = 0$ we once more introduce the coordinate (4) and the transform variable of § 5. Then

$$\tilde{w} \sim \varepsilon^{-1/2}\tilde{\mathscr{L}}\bar{f}(\varepsilon^{-1/2}\eta),$$

where

$$\tilde{\mathscr{L}}(\eta, Y; \varepsilon) = \exp\left[\left(\frac{1}{2\varepsilon} - s\right)(2 - \varepsilon Y)\right] - \exp\left[\frac{2 - \varepsilon Y}{2\varepsilon} - s(2 + \varepsilon Y)\right].$$

(s is given by the formula (28).) We have

$$\left(\frac{\partial^2}{\partial Y^2} + \frac{\partial}{\partial Y}\right)\tilde{\mathscr{L}} = \varepsilon\eta^2\tilde{\mathscr{L}}, \qquad \tilde{\mathscr{L}}(\eta, 0; \varepsilon) = 0; \tag{42}$$

and in place of a second boundary condition, the matching of $\tilde{\mathscr{L}}$ with the $\tilde{L}$ of definition (28) to all orders (as will now be shown).

The argument is similar to that in § 6. The term $\exp[(2 - \varepsilon Y)/(2\varepsilon) - s(2 + \varepsilon Y)]$ can be neglected: with $Y = (1 - y)/\varepsilon$ and $-1 \leqq y \leqq y_1$, it is a.e.s. uniformly for η real. We need only show that $\tilde{L} = \exp[(1/(2\varepsilon) - s)(y + 1)] = \exp[(1/(2\varepsilon) - s) \cdot (2 - \varepsilon Y)]$ satisfies the matching principle. But $\tilde{L}$ has the expansion (30) for $|y| \leqq 1$ and every m, while $\tilde{L}^{(k)}(\eta, y; 0)$ has an inner (Y) expansion to all orders. Fraenkel's Theorem 1 therefore ensures that the matching principle holds to all orders.

Once again a Taylor expansion

$$\tilde{\mathscr{L}}(\eta, Y; \varepsilon) = \sum_{k=0}^{m-1} \tilde{\mathscr{L}}^{(k)}(\eta, y; 0)\frac{\varepsilon^k}{k!} + \tilde{\mathscr{S}}_m(\eta, y; \varepsilon), \tag{43}$$

where

$$\tilde{\mathscr{S}}_m(\eta, y; \varepsilon) = \tilde{\mathscr{L}}^{(m)}(\eta, Y; t\varepsilon)\varepsilon^m/m!,$$

will be needed. Then, if as before (§ 5) divergent integrals are avoided by inverting through the convolution theorem before expanding f, we obtain

$$w \sim \sum_{k=0}^{m-1} w_k^{IV}(X, Y)\varepsilon^{k/2} + \mathscr{S}_m * f + \mathscr{T}_m,$$

where

$$\text{(44a)} \qquad w_k^{IV} = \sum_{j=0}^{[k/2]} \frac{1}{j!(k-2j)!} \int_{-\infty}^{\infty} \mathscr{L}^{(j)}(X - X', Y; 0) f_{k-2j}(X') X'^{(k-2j)}\, dX',$$

$$\text{(44b)} \qquad \mathscr{S}_m * f = \frac{\varepsilon^n}{n!} \int_{-\infty}^{\infty} \mathscr{L}^{(n)}(X - X', Y; t\varepsilon) f(\varepsilon^{1/2}X')\, dX',$$

$$\text{(44c)} \qquad \mathscr{T}_m = \varepsilon^{m/2} \sum_{j=0}^{n-1} \frac{1}{j!(m-2j)!} \int_{-\infty}^{\infty} \mathscr{L}^{(j)}(X - X', Y; 0) f^{(m-2j)}(t\varepsilon^{1/2}X') X'^{(m-2j)}\, dX',$$

and the definitions (31′) still hold. It remains to be shown that (i) the w_k^{IV} satisfy the recurrence relation (14a), the boundary conditions (14b), and the matching conditions mentioned after them; and that (ii) $\mathscr{S}_m * f$ and $\mathscr{T}_m$ are $O(\varepsilon^{m/2})$, for every m, uniformly in the region IV (with X_∞, Y_∞ fixed). In this case the validity of the expansion (13) is established. The proof is similar to that in region II (§ 5).

(i) On substituting the expansion (43) into (42) and inverting, we find

$$\left(\frac{\partial^2}{\partial Y^2} + \frac{\partial}{\partial Y}\right) \mathscr{L}^{(k)}(X, Y; 0) = -k \frac{\partial^2 \mathscr{L}^{(k-1)}}{\partial X^2}(X, Y; 0), \qquad \mathscr{L}^{(k)}(X, 0; 0) = 0$$

for all k. Hence the sums of integrals (44a) satisfy the recurrence relation and boundary conditions. Moreover, the series formed from them matches the series formed from the integrals (31a) by virtue of the matching of $\mathscr{L}$ and L, which is ensured by the matching of $\tilde{\mathscr{L}}$ and $\tilde{L}$ proved above.

(ii) We have to show that each of the integrals (44b), (44c) is bounded in the region IV for ε sufficiently small; this turns out to be more straightforward than in § 5. As there, we must look at the individual terms in $\tilde{\mathscr{L}}^{(j)}(\eta, Y; \varepsilon)$. They are

$$\text{(45)} \qquad (\varepsilon Y)^{\alpha_1}(1 + 4\varepsilon\eta^2)^{-\alpha_2/2}(1 + \sqrt{1 + 4\varepsilon\eta^2})^{-\alpha_3} Y^\beta \eta^{2\gamma}$$

times

$$\exp\left[\frac{-2\eta^2(2 - \varepsilon Y)}{1 + \sqrt{1 + 4\varepsilon\eta^2}}\right] \quad \text{or} \quad \exp\left[\frac{-4\eta^2}{1 + \sqrt{1 + 4\varepsilon\eta^2}} - (1 + \sqrt{1 + 4\varepsilon\eta^2})\frac{Y}{2}\right],$$

where $\alpha_1, \alpha_2, \alpha_3, \beta, \gamma$ are nonnegative integers with

$$\text{(45′)} \qquad \beta \leqq \gamma \quad \text{and} \quad \beta + \gamma \leqq 2j.$$

The exponential factors ensure that the inverse of any such term is bounded for all X provided Y is bounded and, for the first factor, ε is sufficiently small to give a negative exponent. An immediate estimate of the integral in $\mathscr{S}_m * f$ is therefore $O(\varepsilon^{-1/2})$, but this can be improved by expanding $\mathscr{L}^{(n)}$ once more to give

$$\int_{-\infty}^{\infty} \mathscr{L}^{(n)}(X - X', Y; 0) f(\varepsilon^{1/2}X')\, dX'$$

plus a remainder which is now $O(\varepsilon^{1/2})$. So we are left with this last integral, which can be treated along with the integrals (44c).

For $\varepsilon = 0$ the nonzero terms (45) have the inverses

$$(-1)^\gamma Y^\beta \, \partial^{2\gamma}[\sqrt{\pi/2}\exp(-X^2/8)]/\partial X^{2\gamma}$$

times 1 or e^{-Y}, so that we are concerned with integrals of the form

$$\int_{-\infty}^{\infty} Y^\beta \exp[-X'^2/8]\mathscr{F}(X - X')\,dX', \tag{46}$$

where

$$\mathscr{F}(X) = \frac{\partial^{2\gamma}}{\partial X^{2\gamma}}[f^{(m-2j)}(t\varepsilon^{1/2}X')X'^{(m-2j)}], \qquad j = 0, 1, \cdots, n-1 \text{ or } m/2.$$

Note that X' and $X - X'$ have been interchanged, and that the limitations for $j = m/2$ are $\beta \leqq \gamma$, $\beta + \gamma \leqq 2n$ (and not m). Clearly these integrals are bounded if X, Y and ε are.

Extension of the region to

$$X_\infty = \varepsilon^{-\kappa}, \quad Y_\infty = \varepsilon^{-\lambda} \quad \text{with } \kappa, \lambda > 0$$

affects these results in two ways. Anticipating that λ is not greater than 1, so that εY is bounded, we see that the critical factor in the terms (45) is Y^β, which at worst changes the bound on their inverses to $O(\varepsilon^{-\lambda j})$. The contribution to $\mathscr{S}_m * f$ (corresponding to the remainder above) is then $O[\varepsilon^{n+1/2-\lambda(n+1)}]$ so that for

$$\lambda < 1$$

a weaker asymptotic approximation is attained as soon as n is larger than $(\lambda - 1/2)/(1 + \lambda)$.

The same change occurs in each of the integrals (46), but in addition the powers of X which arise from expanding the powers of $(X - X')$ will provide at worst $O(\varepsilon^{-\kappa'(m-2j-2\gamma)})$, where $\kappa' = 1/2$ or κ accordingly as $m - 2j \gtrless 2\gamma$, if we anticipate $\kappa < 1/2$. Thus the integrals (46) for a given $j \neq m/2$ are at worst $O(\varepsilon^{-q})$, where $q = \max[\lambda\beta + \kappa'(m - 2j - 2\gamma)]$ on the triangle (45′). But for $\varepsilon = 0$ the nonzero terms (45) have $\gamma \geqq j$, so that for fixed j the maximum value of the bracket is $\kappa' m + (\lambda - 4\kappa')j$, attained for $\beta = \gamma = j$. Hence $q = \lambda m/4$ or κm according as $\kappa \lessgtr \lambda/4$, and we must have

$$\kappa < \tfrac{1}{2}.$$

(No further restriction arises for $j = m/2$.)

9. The boundary layer III: $f \equiv 0$. Outside the boundary layer the part of the solution due to g is uniformly a.e.s., as has already been noted in writing down the representation (19′a). Inside the boundary layer the representation (19′b) gives

$$\bar{w} \sim \bar{\mathscr{K}}_0\bar{g}, \quad \text{where} \quad \bar{\mathscr{K}}_0(\xi, Y; \varepsilon^2) = \exp\left[-\left(\frac{1}{2\varepsilon} + r\right)(\varepsilon Y)\right].$$

Clearly,

$$\left(\frac{\partial^2}{\partial Y^2} + \frac{\partial}{\partial Y}\right)\bar{\mathscr{K}}_0 = \varepsilon^2\xi^2\bar{\mathscr{K}}_0, \qquad \bar{\mathscr{K}}_0(\xi, 0; \varepsilon^2) = 1. \tag{47}$$

Once again there is no second boundary condition, but instead $\bar{\mathscr{K}}_0$ is seen to match the zero function in y.

The Taylor expansion of the kernel in ε^2 is

$$\bar{\mathscr{K}}_0(\xi, Y; \varepsilon^2) = \sum_{k=0}^{m-1} \bar{\mathscr{K}}_0^{(k)}(\xi, Y; 0)\frac{\varepsilon^{2k}}{k!} + \bar{\mathscr{R}}_{om}(\xi, Y; \varepsilon^2), \tag{48}$$

where

$$\bar{\mathscr{R}}_{0m}(\xi, Y; \varepsilon^2) = \bar{\mathscr{K}}_0^{(m)}(\xi, Y; t\varepsilon^2)\varepsilon^{2m}/m!$$

and derivatives are taken with respect to ε^2. Inversion term-by-term can only be carried out after a smoothed version of g has been introduced (cf. §§ 4, 7). With

$$G \equiv g \quad \text{for} \quad x_0/2 \leqq |x - a|,$$

$$G \in C^\infty \quad \text{and} \quad \int_{-\infty}^{\infty} |G^{(k)}(X)|\, dX < \infty \quad \text{for all } k$$

(constructed in the Appendix), we may write

$$w \sim \sum_{k=0}^{m-1} w_{2k}^{III}\varepsilon^{2k} + \mathscr{R}_{0m} * G + \mathscr{K}_0 * (g - G),$$

where

$$w_{2k}^{III} = \frac{1}{2\pi k!}\int_{-\infty}^{\infty} \bar{\mathscr{K}}_0^{(k)}(\xi, Y; 0)\bar{G}(\xi)\, e^{i\xi x}\, d\xi, \tag{49a}$$

$$\mathscr{R}_{0m} * G = \frac{\varepsilon^{2m}}{2\pi m!}\int_{-\infty}^{\infty} \bar{\mathscr{K}}_0^{(m)}(\xi, Y; t\varepsilon^2)\bar{G}(\xi)\, e^{i\xi x}\, d\xi, \tag{49b}$$

$$\mathscr{K} * (g - G) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{\mathscr{K}}_0(\xi, Y; \varepsilon^2)[\bar{g}(\xi) - \bar{G}(\xi)]\, e^{i\xi x}\, d\xi. \tag{49c}$$

We shall show that (i) the w_{2k}^{III} satisfy the recurrence relation (12a), the boundary conditions (12c), and the matching conditions mentioned after them; and that (ii) $\mathscr{R}_{0m} * G = O(\varepsilon^{2m})$ for every m and (iii) $\mathscr{K} * (g - G)$ is a.e.s., both uniformly in the region III (with x_0, Y_∞ fixed). The expansion (11), containing only even powers of ε, will then have been proved valid. The steps are similar to but simpler than those for f in § 7.

(i) According to equations (47), the coefficient functions in the expansion (48) satisfy

$$\left(\frac{\partial^2}{\partial Y^2} + \frac{\partial}{\partial Y}\right)\bar{\mathscr{K}}^{(k)}(\xi, Y; 0) = k\xi^2\bar{\mathscr{K}}_0(\xi, Y; 0),$$

$$\bar{\mathscr{K}}_0^{(k)}(\xi, 0; 0) = \begin{cases} 1 & \text{for } k = 0, \\ 0 & \text{for } k \neq 0. \end{cases}$$

It follows that the integrals (49a) satisfy the recurrence relation and boundary conditions. Matching with the zero function outside the boundary layer is ensured by the matching of $\bar{\mathscr{K}}_0$ with it.

(ii) The derivative $\bar{\mathscr{K}}^{(m)}(\xi, Y; \varepsilon^2)$ is the sum of terms

$$(1 + 4\varepsilon^2\xi^2)^{-\alpha/2} Y^{\beta}\xi^{2m} \exp\left[-(1 + \sqrt{1 + 4\varepsilon^2\xi^2})Y/2\right], \tag{50}$$

where α and $\beta \leqq m$ are nonnegative integers. Hence it can be bounded as in § 4 (ii), so that the smoothness of G ensures $\mathscr{R}_{0m} * G$ is $O(\varepsilon^m)$.

(iii) In convolution form

$$\mathscr{K} * (g - G) = \frac{Y}{2\pi} e^{-Y/2} \int_{a-(x_0/2)}^{a+(x_0/2)} \frac{K_1(\sqrt{(x - x')^2 + \varepsilon^2 Y^2}/(2\varepsilon))}{\sqrt{(x - x')^2 + \varepsilon^2 Y^2}} [g(x') - G(x')]\, dx'$$

which is seen to be a.e.s. uniformly in the region III.

Extension of the region to

$$x_a = \varepsilon^{\kappa}, \quad Y_{\infty} = \varepsilon^{-\lambda} \quad \text{with} \quad \kappa, \lambda > 0$$

uses a simple version of the argument at the end of § 7. Because of the exponential, the terms (50) are worst when Y is finite, and then they contribute at most $O(\varepsilon^{2m}x_a^{-2m})$ to $\mathscr{R}_{0m} * G$. Hence

$$\kappa < 1$$

but λ is arbitrary. In fact the two parts of the extended region may be joined across $x = x_a$ for any value of Y which tends to infinity algebraically in $1/\varepsilon$. No further restriction comes from $\mathscr{K} * (g - G)$, which remains a.e.s.

Note that the excluded region does not shrink down to a point, but only to the line $x = x_a$ in the boundary layer. Even though the regions II_* and III_* have the same asymptotic dimensions, the character of the solution in them is quite different.

10. The transition zone III_*. The solution near the vertical line $x = a$ in the boundary layer through the discontinuity in g is described by means of the coordinate (15) and the corresponding transform variable

$$\xi_* = \varepsilon\xi.$$

The different notation X_*, y_* and x_*, Y is designed to emphasize the different nature of the regions II_* and III_*: the former resolves a breakdown in a parabolic layer where two coordinates are involved; the latter resolves a breakdown in a hyperbolic layer where only one coordinate is involved.

We must now consider (using hats again for the transform)

$$\hat{w} \sim \varepsilon^{-1}\hat{\mathscr{K}}_*\bar{g}(\varepsilon^{-1}\xi_*)\, e^{ia\xi_*/\varepsilon},$$

where

$$\hat{\mathscr{K}}_*(\xi_*, Y) = \exp\left[(-\tfrac{1}{2} + r_*)Y\right] \quad \text{and} \quad r_* = \sqrt{(1 + 4\xi_*^2)}/2.$$

Clearly,

$$\left(\frac{\partial^2}{\partial Y^2} + \frac{\partial}{\partial Y} - \xi_*^2\right)\hat{\mathscr{K}}_* = 0, \qquad \hat{\mathscr{K}}_*(\xi_*, 0) = 1, \tag{51}$$

and, in place of a second boundary condition, $\hat{\mathscr{K}}_*$ is seen to match with the zero function in y.

As in region II_* no expansion is involved, but inversion by convolution is necessary before expanding g. We find

$$w \sim \sum_{k=0}^{m-1} w_k^{III*}(x_*, Y)\varepsilon^k + \mathscr{K}_* * g^{(m)},$$

where

$$w_k^{III*} = \frac{1}{k!}\int_{-\infty}^{\infty} \mathscr{K}_*(x_* - x'_*, Y) g_k(x'_*) x'^k_* \, dx'_*, \tag{52a}$$

$$\mathscr{K}_* * g^{(m)} = \frac{\varepsilon^m}{m!}\int_{-\infty}^{\infty} \mathscr{K}_*(x_* - x'_*, Y) g^{(m)}(a + t\varepsilon x'_*) x'^m_* \, dx'_*, \tag{52b}$$

with

$$g_{(k)}(x_*) = g^{(k)}(a \pm 0) \quad \text{for } x_* \gtrless 0. \tag{52$'$}$$

The expansion (16) will therefore be established if we show that (i) the w_k^{III*} satisfy (17a), the boundary conditions (17b) and the matching conditions noted after them; and that (ii) $\mathscr{K}_* * g^{(m)} = O(\varepsilon^m)$ uniformly in the region III_* (with $x_{*\infty}$, Y_∞ fixed).

(i) Substitute the integrals (52a) directly into the equation and boundary conditions to show that they satisfy them by virtue of (51). The series formed from them matches the zero function in y because $\hat{\mathscr{K}}_*$ does.

(ii) The integral in $\mathscr{K}_* * g^{(m)}$ is actually

$$\frac{Ye^{-Y/2}}{2\pi}\int_{-\infty}^{\infty} \frac{K_1(\sqrt{(x_* - x'_*)^2 + Y^2}/2)}{\sqrt{(x_* - x'_*)^2 + Y^2}} g^{(m)}(a + t\varepsilon x'_*) x'^m_* \, dx'_*$$

which is bounded in III_* so long as ε is bounded.

That the region can be extended to

$$x_{*\infty} = \varepsilon^{-\kappa}, \quad Y_\infty = \varepsilon^{-\lambda}, \quad \text{with} \quad \kappa, \lambda > 0,$$

is seen from the corresponding treatment of II_* (§ 6). The expression (37) is replaced by

$$e^{-Y}\left[\sum_{s=0}^{m} c_s |x_*|^{m-s} Y^{(s+1)/2} e^{Y/2} K_{(1-s)/2}(Y/2)\right].$$

The extra exponential factor results from the change $e^{y_*/2}$ to $e^{-Y/2}$, which in turn is traceable to the kernel having M_{1*} in place of M_{2*}. It suppresses the powers of Y so that Y_∞ plays no role. Thus

$$\kappa < 1 \quad \text{and any } \lambda$$

will do.

In fact, for any Y which tends to infinity algebraically in $1/\varepsilon$ every remainder is a.e.s. and the expansion asymptotes zero. In particular, this holds in the core, where εY is constant.

11. Concluding remarks. There are two variations of the basic conditions (18) on f and g which are of considerable importance. First the condition (18a) may be strengthened to:

$$f^{(k)} \text{ is continuous at } x = 0 \text{ for } k \leqq k_0. \tag{53}$$

The question then is at what stage the free layer must be introduced. Similarly the condition (18b) may be strengthened. On the other hand, the condition (18a) may be weakened to:

$$f^{(k)}(\pm 0) \text{ exist for } k \leqq k_0,$$

while still insisting that

$$\int_{-\infty}^{\infty} |f^{(k)}(x)|\,dx < \infty \quad \text{for } k \leqq k_0 + 1$$

remains from condition (18c), though nothing is said about later derivatives. The question then is the order to which the various asymptotic expansions are valid. Clearly a similar question arises for g. We shall consider these points in turn.

There is no need to introduce the free shear layer until the expansion in region I fails to be valid near $x = 0$, i.e., so long as F can be avoided. We must therefore determine the largest integer m for which the integral in the remainder (24b) can still be bounded when F is replaced by f. But by analogy with the bound (25), $\bar{f}$ must be small compared to $\xi^{-(2m+1)}$ as $\xi \to \infty$, while the condition (53) ensures that $\bar{f}$ is at worst of order $\xi^{-(k_0+1)}$. Hence

$$\max m = [(k_0 - 1)/2].$$

Obviously the expansion in region III is also valid to this order near $x = 0$. Similar arguments apply to g.

When only a finite number of left- and right-derivatives of f exist at $x = 0$, the expansions fail first in the free layer and its intersection with the boundary layer. In fact, they never fail outside if we still require

$$\left(\int_{-\infty}^{-x_0} + \int_{x_0}^{\infty} \right) |f^{(k)}(x)|\,dx < \infty \quad \text{for all } k;$$

see, for example, the bounding (25).

In region II all derivatives up to $f^{(2\delta-1)}(\pm 0)$ are used in estimating the remainder as well as $\int_{-\infty}^{\infty} |f^{(2\delta)}(x')|\,dx' < \infty$; see § 5(ii). Since $\delta \leqq 2j$ and $j \leqq n$ we must have $4n \leqq k_0 + 1$ so that

$$\max m = 2[(k_0 + 1)/4].$$

This applies also in region IV, and no further restriction arises in region II_*. Similar arguments apply to g.

Note that there is no difficulty in calculating the coefficient functions (31a) much further, namely up to $k = k_0$ corresponding to

$$\max m = k_0 + 1.$$

However we can only prove the approximation is $O(\varepsilon^{[(k_0+1)/4]})$. For example, if $k_0 = 3$ the four coefficient functions w_0^{II}, w_1^{II}, w_2^{II}, w_3^{II} can be calculated, but the resulting approximation is only known to be $O(\varepsilon)$; w_2^{II} and w_3^{II} are useless.

Appendix. The construction of $F(x)$, the smoothed version of f introduced in § 4, will be based on the C^∞-function

$$\sigma(x) = \begin{cases} e^{-1/x} & \text{for } x > 0, \\ 0 & \text{for } x < 0. \end{cases}$$

Consider

$$\tau(x) = \int_{-\infty}^{x} \sigma(x' - \tfrac{1}{2})\sigma(1 - x')\,dx' \Big/ \int_{-\infty}^{\infty} \sigma(x' - \tfrac{1}{2})\sigma(1 - x')\,dx',$$

where the common integrand vanishes for $x' < \frac{1}{2}$ and $x' > 1$. Clearly τ is C^∞ and takes the values

$$\tau(x) = \begin{cases} 0 & \text{for } x < \frac{1}{2}, \\ 1 & \text{for } x > 1. \end{cases}$$

Now set

$$F(x) = [\tau(2x/x_0) + \tau(-2x/x_0)]f(x);$$

then F has the properties (23), the last by virtue of $\int_{-\infty}^{\infty} |f^{(k)}(x)|\,dx < \infty$ for all k. (In fact, it is zero for $|x| \leqq x_0/4$.)

As $x_0 \to 0$ the most singular contributions to $F^{(2m+1)}$ come from letting all $2m + 1$ derivatives fall on the functions $\tau(\pm 2x/x_0)$. Thus the worst terms in $F^{(2m+1)}(x_0 x)$ are $(\pm 2/x_0)^{2m+1}\tau^{(2m+1)}(\pm 2x)f(x)$, so that it can be written $x_0^{-(2m+1)}F_m(x; x_0)$, where F_m is bounded as $x_0 \to 0$. This property is used at the end of § 4.

Similarly the smoothed version of g used in § 9 is

$$G(x) = [\tau(2(x - a)/x_a) + \tau(2(a - x)/x_a)]g(x).$$

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SIAM J. MATH. ANAL.
Vol. 2, No. 4, November 1971

THE SUMMATION OF SERIES*

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Abstract. A number of formulas are presented for representing a variety of infinite series in terms of rapidly convergent definite integrals. Several new explicit summations are obtained by these methods.

Physics and chemistry abound with slowly convergent infinite series. The purpose of this paper is to supplement an earlier paper of a similar title by A. D. Wheelon [2] with a variety of methods for treating a large number of types of infinite series. The methods themselves are undoubtedly not new but no attempt has been made to search the mathematical literature for their sources. Likewise, our intention has been to be heuristic and the detailed conditions under which the various procedures are valid are not given in detail. However, in applications these are usually self-evident or can be found by testing the convergence of the integrals and sums in question. As an example of how our considerations may be useful consider the simple one-dimensional phase modulated coulomb sum which might occur in the study of a one-dimensional lattice or a long polymer chain:

$$S = \sum_{l=-\infty}^{\infty} \frac{e^{ikl}}{|l-x|}.$$

We have taken the lattice constant or monomer spacing as unity and by periodicity there is no loss in generality by taking $0 < x < 1$. By direct summation it would take (depending on k) as many as 100,000 terms to give S accurately to several decimal places. By the use of classical methods, such as the Ewald procedure for calculating Madelung energies, this can be reduced to as few as 50 terms. However, by (3) and (4) of this paper we find quite simply

$$S = \frac{1}{x} + \int_0^\infty \frac{(\cos k - e^{-t})}{\cosh t - \cos k} \cosh (xt)\, dt + i \sin k \int_0^\infty \frac{\sinh (xt)}{\cosh t - \cos k}\, dt.$$

One of these integrals can be evaluated exactly and the remaining one has a smooth exponentially decaying integrand and a value which is given to ten places by seven-point Gaussian quadrature for $k \not\equiv 0 \pmod{2\pi}$ and all $0 < x < 1$.

We begin by considering Fourier series. A number of methods for summing these have been summarized by McFadden [1] but all of these are suitable for a restricted class of summands and are much more complicated than the very simple procedure presented here. We start from the expansion

$$(e^x - e^{it})^{-1} = e^{-it} \sum_{k=1}^{\infty} e^{ikt} e^{-kx}, \qquad 0 \leqq t < 2\pi, \quad x > 0. \tag{1}$$

* Received by the editors February 16, 1971, and in revised form April 12, 1971.

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Next let the Laplace transform of the real function $f(x)$ be $F(k)$; then by multiplying (1) through by $f(x)$ and integrating from 0 to ∞, we find

$$\sum_{k=1}^{\infty} e^{ikt}F(k) = e^{it}\int_0^\infty (e^x - e^{it})^{-1} f(x)\,dx. \tag{2}$$

Taking the real and imaginary parts of (2), we obtain

$$\sum_{k=1}^{\infty} F(k)\cos kt = \frac{1}{2}\int_0^\infty \frac{(\cos t - e^{-x})}{\cosh x - \cos t} f(x)\,dx, \tag{3}$$

$$\sum_{k=1}^{\infty} F(k)\sin kt = \frac{1}{2}\sin t\int_0^\infty \frac{f(x)\,dx}{\cosh x - \cos t}. \tag{4}$$

Equations (3) and (4) give the interesting relation between corresponding sine and cosine series:

$$\sum_{k=1}^{\infty} F(k)\cos kt = \cot t\sum_{k=1}^{\infty} F(k)\sin kt - \frac{1}{2}\int_0^\infty \frac{e^{-x}f(x)\,dx}{\cosh x - \cos t}. \tag{5}$$

As an example, let $F(k) = k^{-1}\ln k$; then $f(x) = -\ln(\gamma x)$, where γ is Euler's constant. The integral which occurs on the right-hand side of (4) is tabulated:[1]

$$\int_0^\infty \frac{\ln(\gamma x)}{\cosh x - \cos t}\,dx = (\pi - t)\csc t\ln(2\pi\gamma) + \pi\csc t\ln\left[\frac{\pi\csc(t/2)}{\Gamma^2(t/2\pi)}\right], \tag{6}$$

and we obtain Kummer's series

$$\sum_{k=1}^{\infty} k^{-1}\ln k\sin kt = \frac{1}{2}(t - \pi)\ln(2\pi\gamma) + \frac{\pi}{2}\ln\left[\pi^{-1}\Gamma^2\left(\frac{t}{2\pi}\right)\sin\left(\frac{t}{2}\right)\right], \tag{7}$$
$$0 < t < \pi.$$

As a second example, take $f(x) = \tanh x$. Then, since

$$\int_0^\infty \frac{\tanh x}{\cosh x - \cos t}\,dx = -\sec t\ln(2\sin^2 t), \tag{8}$$

$$\int_0^\infty e^{-kx}\tanh x\,dx = \tfrac{1}{2}\{\psi[(1/4)(k+2)] - \psi(k/4)\} - k^{-1}, \tag{9}$$

we have

$$\sum_{k=1}^{\infty} \{\psi[(1/4)(k+2)] - \psi(k/4)\}\sin kt = \pi - t - \tan t\ln(2\sin^2 t). \tag{10}$$

This is one of a large number of new closed form summations that can be obtained by inspecting a table of Laplace transforms.

By using Parseval's theorem for the Fourier transform in $L_1(0, \infty)$, equation (4) can be written in the form

$$\sum_{k=1}^{\infty} F(k)\sin kt = \int_0^\infty \frac{\sinh(\pi - t)x}{\sinh \pi x}\varphi_c(x)\,dx, \tag{11}$$

---

[1] All integrals referred to as tabulated have been checked independently by the author.

where $\varphi_c(x)$ is the Fourier cosine transform of $f(x)$:

$$\varphi_c(x) = \frac{1}{2}\operatorname{Re} F(ix) + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{F(iu)}{x^2 - u^2} u\, du. \tag{12}$$

For example, if $F(k) = k^{-1}$, then $\varphi_c(x) = (\pi/2)\delta(x)$ (Dirac delta function) and we obtain the well-known series $\sum k^{-1} \sin kt = \frac{1}{2}(\pi - t)$, $0 < t < 2\pi$. Equations (3) and (4) easily reproduce all the Fourier series listed in standard compilations.

Next we note that (3) and (4) can be adapted to the summation of other function series. For example, if both sides of (4) are multiplied by $\pi^{-1}\sin(z \sin t)$ and integrated with respect to t from 0 to π, we find

$$\begin{aligned} \sum_{k\,\mathrm{odd}} F(k)J_k(z) &= \frac{1}{2\pi}\int_0^{\infty} dx \int_0^{\pi} dt f(x)\frac{\sin t \sin(z \sin t)}{\cosh x - \cos t} \\ &= \frac{1}{\pi}\int_0^{\infty} dx \cosh x f(x) \int_0^1 \frac{\sin(zt)}{t^2 + \sinh^2 x} \frac{t\, dt}{\sqrt{1 - t^2}}. \end{aligned} \tag{13}$$

As an example, for $F(k) = k^{-1}, f(x) = 1$ and we find

$$\sum_{k\,\mathrm{odd}} k^{-1}J_k(z) = \frac{\pi}{4}\mathbf{H}_0(z). \tag{14}$$

If we put $t = 0, \pi$ in (3), we obtain

$$\sum_{k=1}^{\infty} F(k) = \int_0^{\infty} \frac{f(x)}{e^x - 1} dx, \qquad \sum_{k=1}^{\infty} (-1)^k F(k) = -\int_0^{\infty} \frac{f(x)}{e^x + 1} dx. \tag{15}$$

The first of these transformations was noted by Wheelon [2]. The second can also be used to sum a class of transformed series as follows. Let

$$g(y) = \left(\frac{2}{\pi}\right)\int_0^{\infty} \cos(xy)F(x)\, dx.$$

Then, since

$$\begin{aligned} 1 + 2\sum_{n=1}^{\infty} (-1)^n \cos ny &= \lim_{\varepsilon\to 0} \frac{\sinh \varepsilon}{\cosh \varepsilon + \cos y} = 2\lim_{\varepsilon\to 0} \frac{\varepsilon}{\varepsilon^2 + 2(1 + \cos y)} \\ &= 2\pi\delta\left[2\cos\frac{y}{2}\right] = \sum_{k=-\infty}^{\infty} \delta[y - (2k + 1)\pi], \end{aligned}$$

we have

$$\sum_{k=1}^{\infty} (-1)^k F(k) = \tfrac{1}{2}F(0) - \pi \sum_{k\,\mathrm{odd}} g(k\pi), \tag{16}$$

or

$$\sum_{k\,\mathrm{odd}} g(k\pi) = \frac{1}{2\pi}\int_0^{\infty} g(y)\, dy - \frac{1}{\pi}\int_0^{\infty} \frac{f(x)}{e^x + 1} dx, \tag{17}$$

where $f(x)$ is the inverse Laplace transform of the cosine transform of $g(y)$. For example, let $g(y) = (y^2 + a^2)^{-1}$. Then $f(x) = (\pi/2a)\,\delta(x - a)$, and we find

$$\sum_{k\,\mathrm{odd}} \frac{1}{(k\pi)^2 + a^2} = \frac{1}{4a} \tanh \frac{a}{2}. \tag{18}$$

As an example of the first transformation in (15), we note the Laplace transform pair

$$\sin [a(1 - e^{-x})] \cdot \sim \cdot a^{-k}\Gamma(k)U_{k+1}(2a, 0), \tag{19}$$

where $U_{k+1}(x, y)$ is Lommel's function. Thus, we immediately obtain

$$\sum_{k=1}^{\infty} a^{-k}\Gamma(k)U_{k+1}(2a, 0) = \mathrm{Si}\,(a); \tag{20}$$

that is, we have summed the double series

$$\sum_{k=1}^{\infty} \sum_{m=0}^{\infty} (-1)^m a^{2m+1} \frac{\Gamma(k)}{\Gamma(2m + k + 2)} = \mathrm{Si}\,(a). \tag{21}$$

One more interesting series comes from the Laplace transform pair

$$(1 - e^{-x})^{\mu/2} J_\mu[a(1 - e^{-x})^{1/2}] \cdot \sim \cdot \Gamma(k)(2/a)^k J_{k+\mu}(a). \tag{22}$$

This gives, by using the first formula in (15),

$$\sum_{k=1}^{\infty} \Gamma(k) z^k J_{k+\mu}(2/z) = 2\int_0^1 J_\mu(2x/z)x^{\mu-1}\,dx. \tag{23}$$

Thus, for $\mu = 1$ we have the interesting sum

$$\sum_{k=1}^{\infty} \Gamma(k) z^k J_{k+1}(2/z) = z[1 - J_0(2/z)]. \tag{24}$$

These procedures may be viewed in reverse to evaluate certain definite integrals in terms of infinite series. As an example, by using (15), we find

$$\int_0^\infty \frac{\sin^2 ax}{e^x - 1}\,dx = 2a^2 \sum_{k=1}^{\infty} \frac{1}{k[k^2 + (2a^2)]}. \tag{25}$$

Series such as that in (25) are easily summed in terms of the digamma function $\psi(z)$, and we have

$$\int_0^\infty \frac{\sin^2 ax}{e^x - 1}\,dx = \tfrac{1}{2}[\gamma + \mathrm{Re}\,\psi(2ia)], \tag{26}$$

a simple formula which does not appear to be listed anywhere. The integrals $\int_0^\infty \sin^{2n} ax(e^x - 1)^{-1}\,dx$ can be obtained similarly.

By using general properties of the Laplace transform, various other transformation formulas can be derived. For example, if $F(k) \cdot \sim \cdot f(x)$, then $k^{-1}\int_k^\infty F(u)\,du \cdot \sim \cdot \int_0^x y^{-1} f(y)\,dy$. Therefore

$$\sum_{k=1}^{\infty} \frac{1}{k}\int_k^\infty F(u)\,du = -\int_0^\infty f(x) \ln (1 - e^{-x})x^{-1}\,dx. \tag{27}$$

Similarly,

$$\sum_{k=1}^{\infty} F[g(k)] = \int_0^{\infty}\int_0^{\infty} dx\,dy\,\frac{h(x,y)f(y)}{e^x - 1},$$

where $h(x, y)$ is the inverse Laplace transform of $\exp\{-yg(k)\}$. For example,

$$\begin{aligned}\sum_{k=1}^{\infty} F(k^{1/3}) &= \frac{1}{3\pi}\int_0^{\infty} y^{3/2}\,dy\int_0^{\infty}\frac{dx}{x^{3/2}}K_{1/3}[2(y/3)^{3/2}x^{-1/2}]f(y)(e^x - 1)^{-1} \\ &= \frac{4}{9\pi}\int_0^{\infty}(e^{1/u^2} - 1)^{-1}\,du\int_0^{\infty} dz\,z^{2/3}f(z^{2/3})K_{1/3}\left(\frac{2}{3\sqrt{3}}uz\right),\end{aligned} \tag{28}$$

$$\sum_{k=1}^{\infty} F(k^{1/2}) = \frac{1}{2\sqrt{\pi}}\int_0^{\infty} dx\int_0^{\infty} y\,dy\frac{e^{-y^2/4x}f(y)}{e^x - 1} \tag{29}$$

and

$$\sum_{k=1}^{\infty} F(\log k) = \int_0^{\infty} dx\int_0^{1} dy\frac{x^{y-1}f(y)}{\Gamma(y)(e^x - 1)} + \int_1^{\infty} dy\,f(y)\zeta(y). \tag{30}$$

In many cases, at least one of the integrals can be performed explicitly and the second converges rapidly.

Finally, convenient transformation formulas can also be obtained by using other integral transforms. For example, from the expansion

$$\sum_{k=0}^{\infty}(-1)^k\frac{a^kx^k}{k!} = e^{-ax}, \tag{31}$$

by multiplying both sides by $f(x)$ and integrating from 0 to ∞, we have

$$\sum_{k=0}^{\infty}(-1)^k\frac{a^k}{k!}M(k + 1) = F(k), \tag{32}$$

where $M(k)$ and $F(k)$ are the Mellin and Laplace transforms of $f(x)$, respectively. Alternatively,

$$\sum_{k=0}^{\infty}(-1)^k\frac{a^k}{k!}F(k) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)a^s\Gamma(-s)\,ds = \varphi(\ln a), \tag{33}$$

where $\varphi(p)$ is the inverse Laplace transform of $F(s)\Gamma(-s)$.

Finally we have

$$f(k + a) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} M(s)(k + a)^{-s}\,ds, \tag{34}$$

so if the allowed range of c in (34) includes 1, we have

$$\sum_{k=0}^{\infty} f(k + a) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} M(s)\zeta(s, a)\,ds, \qquad c > 1, \tag{35}$$

where $\zeta(s, a)$ is the generalized Riemann zeta function. $\zeta(s, a)$ has only one simple pole $s = 1$ with unit residue, so the integral on the right-hand side of (35) can generally be evaluated by residues. As an example we consider a series which occurs in the theory of the diamagnetism of an electron gas:

$$S = \sum_{k=1}^{n} (1 - kx)^{3/2}, \tag{36}$$

where n is the largest integer less than or equal to $1/x$. Here we have $f(k) = \theta(1 - kx)(1 - kx)^{3/2}$ and $a = 1$, where $\theta(x)$ denotes the unit step function. By (35) we have ($\zeta(s, 1) = \zeta(s)$)

$$S = \frac{\Gamma(5/2)}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s)\zeta(s)}{\Gamma(5/2 + s)} x^{-s}\, ds, \qquad c > 1. \tag{37}$$

The integrand has simple poles at $s = 1, 0, -1, -3, \cdots$. Where $x > 1$ we can close the contour to the right and $S = 0$ as expected. For $x < 1$ by closing to the left we find the asymptotic representation

$$S = \frac{2}{5x}\left[1 - \frac{5x}{4} - \frac{15\sqrt{\pi}}{8} \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n} B_n}{\Gamma(2n + 1)\Gamma(7/2 - 2n)}\right], \tag{38}$$

where we have used the values $\zeta(0) = -\frac{1}{2}$, $\zeta(1 - 2n) = (-1)^n B_n/2n$ and the B_n are Bernoulli numbers. Equation (38) is valuable for studying (36) at small x.

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SIAM J. MATH. ANAL.
Vol. 2, No. 4, November 1971

UNCERTAINTY INEQUALITIES FOR HANKEL TRANSFORMS*

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Abstract. In this paper an uncertainty inequality for Hankel transforms is obtained.
Let $\nu > 0$ be fixed. We set

$$d\mu_\nu(x) = c_\nu^{-1} x^{2\nu}\, dx, \qquad c_\nu = 2^{\nu - 1/2}\Gamma(\nu + \tfrac{1}{2}),$$

and

$$\mathbf{J}_\nu(x) = c_\nu x^{-\nu + 1/2} J_{\nu - 1/2}(x),$$

where $J_{\nu-1/2}(x)$ is a Bessel function of the first kind of order $\nu - \frac{1}{2}$. We define

$$f\hat{\ }(t;\nu) = \int_0^\infty f(x)\mathbf{J}_\nu(xt)\, d\mu_\nu(x).$$

A probability frequency function with respect to $d\mu_\nu$ is defined as a nonnegative function in $L_\nu^1(0, \infty)$ with norm one, and the generalized variance of a probability frequency function $F(x)$ is defined by

$$V_\nu[F] = \int_0^\infty x^2 F(x)\, d\mu_\nu(x).$$

Let $f(x)$ belong to $L_\nu^2(0, \infty)$ with norm one. By Parseval's equality $|f(x)|^2$ and $|f\hat{\ }(x;\nu)|^2$ can be considered as probability frequency functions. The uncertainty inequality

$$V_\nu[|f(x)|^2]V_\nu[|f\hat{\ }(x;\nu)|^2] \geqq (\nu + \tfrac{1}{2})^2$$

is proved, and the constant $(\nu + \frac{1}{2})^2$ is shown to be the best possible.

1. Introduction. For $f \in L^2(-\infty, \infty)$ with $\|f\| = 1$ and $g(y) \sim \int_{-\infty}^{\infty} f(x)e^{-2\pi i x y}\, dx$, Weyl [12] has shown that

$$V[|f|^2]V[|g|^2] \geqq 1/(16\pi^2),$$

where

$$V[F] = \int_{-\infty}^{\infty} (x - m)^2 F(x)\, dx$$

and

$$m = \int_{-\infty}^{\infty} x F(x)\, dx.$$

In this paper an analogous inequality will be established for Hankel transforms. Let $\nu > 0$ be fixed but arbitrary. We set

$$d\mu_\nu(x) = c_\nu^{-1} x^{2\nu}\, dx,$$

where $c_\nu = 2^{\nu - 1/2}\Gamma(\nu + \frac{1}{2})$. We define $L_\nu^p(0, \infty)$, $1 \leqq p \leqq \infty$, as the Banach space of those real measurable functions on $(0, \infty)$ for which

$$\|f\|_{p,\nu} = \left[\int_0^\infty |f(x)|^p\, d\mu_\nu(x)\right]^{1/p}$$

* Received by the editors March 31, 1970, and in final revised form April 2, 1971.
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is finite. We write L_ν^p. Let $\mathbf{J}_\nu(x)$ be the function defined by

$$\mathbf{J}_\nu(x) = c_\nu x^{-\nu+1/2} J_{\nu-1/2}(x),$$

where $J_{\nu-1/2}(x)$ is a Bessel function of the first kind of order $(\nu - \frac{1}{2})$. For $f(x) \in L_\nu^1$ we define the Hankel transform $f\hat{\ }(t; \nu)$ of $f(x)$ by

$$f\hat{\ }(t; \nu) = \int_0^\infty f(x)\mathbf{J}_\nu(xt)\, d\mu_\nu(x), \qquad 0 \leqq t < \infty.$$

Unless there is confusion about the order ν, we write $f\hat{\ }(t)$.

DEFINITION 1.1. We say a function $F(x)$ on $(0, \infty)$ is a *probability frequency function* with respect to $d\mu_\nu$ if:

(a) $$F(x) \geqq 0 \quad \text{for} \quad 0 < x < \infty$$

and

(b) $$F(x) \in L_\nu^1 \quad \text{with} \quad \|F\|_{1,\nu} = 1.$$

For the remainder of this paper the following assumptions are made for all functions:

(1) $$f(x) \in L_\nu^2$$

and

(2) $$\|f\|_{2,\nu} = 1.$$

By Parseval's equality (see Guy [5]) these functions will satisfy

(3) $$\|f\hat{\ }\|_{2,\nu} = 1.$$

Hence, $|f(x)|^2$ and $|f\hat{\ }(x)|^2$ can be considered as probability frequency functions with respect to $d\mu_\nu$.

Next we define the function $D_\nu(x, y, z)$ by

$$D_\nu(x, y, z) = \frac{2^{(3\nu-5/2)}\Gamma(\nu + \frac{1}{2})^2}{\Gamma(\nu)\pi^{1/2}}(xyz)^{1-2\nu}\Delta(x, y, z)^{2\nu-2},$$

where $\Delta(x, y, z)$ is the area of a triangle whose sides are x, y, z if there is such a triangle and otherwise $D_\nu(x, y, z)$ is zero.

For any locally integrable function $f(x)$ we define the associated function $f(x, y)$ by

$$f(x, y) = \int_0^\infty f(s)D_\nu(x, y, s)\, d\mu_\nu(s), \qquad 0 < x, y < \infty.$$

For a fixed y, $0 < y < \infty$, the operator T_y defined by $T_y[f(x)] = f(x, y)$ is a "translation operator." This motivates the following definition.

DEFINITION 1.2. The *generalized second moment* about the point c, $V_\nu[F, c]$, of a frequency function F with respect to $d\mu_\nu$ is defined by

$$V_\nu[F, c] = \int_0^\infty g(x, c)F(x)\, d\mu_\nu(x),$$

where $g(t) = t^2$ and $0 < c < \infty$. Referring to Cholewinski and Haimo [3, p.8] we have that

$$V_\nu[F, c] = \int_0^\infty (x^2 + c^2)F(x)\, d\mu_\nu(x)$$
$$= \int_0^\infty x^2 F(x)\, d\mu_\nu(x) + c^2.$$

DEFINITION 1.3. We define the *generalized variance* of a frequency function F as

$$V_\nu[F] = \int_0^\infty x^2 F(x)\, d\mu_\nu(x).$$

Note that $V_\nu[F, 0] = V_\nu[F]$.

It is now agreed to drop the "with respect to $d\mu_\nu$" when referring to frequency functions and generalized variances.

Functions in these spaces appear in a natural way in n-dimensional Euclidean space. Let $F(x_1, \cdots, x_n)$ be a radial function in n-space; that is, there is a function F_0 on $[0, \infty)$ such that $F(x_1, \cdots, x_n) = F_0(r)$, where $r = (x_1^2 + x_2^2 + \cdots + x_n^2)$. Suppose $F_0(r) \in L^1(0, \infty)$ with norm one. It is clear that the mean of F in n-space is zero. Furthermore if one looks at the "absolute variance" of F in n-space, that is,

$$\int_{-\infty}^\infty \cdots \int_{-\infty}^\infty \sum_{k=1}^n (x_k - y_k)^2 F(x_1, \cdots, x_n)\, dx_1 \cdots dx_n,$$

then by changing to spherical coordinates, we have

$$\int_{-\infty}^\infty \cdots \int_{-\infty}^\infty \sum_{k=1}^n (x_k - y_k)^2 F(x_1, \cdots, x_n)\, dx_1 \cdots dx_n$$
$$= (2\pi)^{\nu+1/2}\left\{\int_0^\infty r^2 F_0(r)\, d\mu_\nu(r) + r_y^2\right\},$$

where $2\nu = n - 1$ and r_y is the Euclidean distance of y from the origin. Observe the correspondence of this to the generalized variance.

2. An uncertainty inequality. This section turns to the task of proving the uncertainty inequality for Hankel transforms,

$$V_\nu[|f|^2]V_\nu[|f\hat{\ }|^2] \geqq (\nu + 1/2)^2,$$

that is, that the generalized variances of $|f|^2$ and $|f\hat{\ }|^2$ are not both small.

We first prove an intermediate result concerning $df\hat{\ }(x)/dx$.

THEOREM 2.1. *If $f \in L_\nu^2 \cap L_{\nu+1}^2$, then $f\hat{\ }(x)$ is absolutely continuous on $(0, \infty)$ and*

$$\int_0^\infty \left[\frac{d}{dx} f\hat{\ }(x)\right]^2 d\mu_\nu(x) = \int_0^\infty [tf(t)]^2\, d\mu_\nu(t).$$

Proof. Suppose first that in addition $f \in L_\nu^1 \cap L_{\nu+1}^1$. Then

$$f\hat{\ }(x) = \int_0^\infty f(t)\mathbf{J}_\nu(xt)\, d\mu_\nu(t)$$

converges absolutely and defines $f^{\wedge}(x)$ as a continuous function on $(0, \infty)$. We have

$$\frac{d}{dz}\mathbf{J}_\nu(z) = -z\mathbf{J}_{\nu+1}(z)c_\nu c_{\nu+1}^{-1},$$

and since $t\mathbf{J}_{\nu+1}(t) \in L^\infty(0, \infty)$, we have

$$\frac{d}{dx}f^{\wedge}(x) = -\int_0^\infty tf(t)[xt\mathbf{J}_{\nu+1}(xt)c_\nu c_{\nu+1}^{-1}]\,d\mu_\nu(t)$$

which can be rewritten as

$$-x^{-1}\frac{d}{dx}f^{\wedge}(x) = \int_0^\infty f(t)\mathbf{J}_{\nu+1}(xt)\,d\mu_{\nu+1}(t). \tag{4}$$

By Parseval's formula for $\nu + 1$,

$$\int_0^\infty x^{-2}\left[\frac{d}{dx}f^{\wedge}(x)\right]^2 d\mu_{\nu+1}(x) = \int_0^\infty [f(t)]^2\,d\mu_{\nu+1}(t)$$

which is the same as

$$\int_0^\infty \left|\frac{d}{dx}f^{\wedge}(x)\right|^2 d\mu_\nu(x) = \int_0^\infty [tf(t)]^2\,d\mu_\nu(t).$$

Suppose now that $f \in L_\nu^2 \cap L_{\nu+1}^2$ but not that $f \in L_\nu^1 \cap L_{\nu+1}^1$. Choose $f_n(t)$ such that $f_n(t) \in L_\nu^2 \cap L_{\nu+1}^2 \cap L_\nu^1 \cap L_{\nu+1}^1$ and such that

$$\|f - f_n\|_{2,\nu} \to 0, \qquad \|f - f_n\|_{2,\nu+1} \to 0$$

as $n \to \infty$. It follows easily from (4) that

$$\begin{aligned} -\int_{x_1}^{x_2} xf_n^{\wedge}(x; \nu+1)\,dx &= -\int_{x_1}^{x_2} x\int_0^\infty f_n(t)\mathbf{J}_{\nu+1}(xt)\,d\mu_{\nu+1}(t)\,dx \\ &= -\int_{x_1}^{x_2} x\left[-x^{-1}\frac{d}{dx}f_n^{\wedge}(x; \nu)\right]dx \\ &= f_n^{\wedge}(x_2; \nu) - f_n^{\wedge}(x_1; \nu). \end{aligned} \tag{5}$$

By Parseval's equality we have that

$$\|f^{\wedge}(x; \nu) = f_n^{\wedge}(x; \nu)\|_{2,\nu} = \|f(x) - f_n(x)\|_{2,\nu} \to 0,$$
$$\|f^{\wedge}(x; \nu+1) - f_n^{\wedge}(x; \nu+1)\|_{2,\nu+1} = \|f(x) - f_n(x)\|_{2,\nu+1} \to 0$$

as $n \to \infty$.

Taking the limit as $n \to \infty$ in (5), we have that

$$f^{\wedge}(x_2; \nu) - f^{\wedge}(x_1; \nu) = -\int_{x_1}^{x_2} xf^{\wedge}(x; \nu+1)\,dx, \quad \text{a.e.}$$

So $f^{\wedge}(x; \nu)$ (suitably redefined on a set of measure zero) is absolutely continuous (and hence continuous) on $[a, b]$ for all $0 < a < b < \infty$, and thus

$$-x^{-1}\frac{d}{dx}f\hat{\ }(x) = f\hat{\ }(x; v+1) \quad \text{a.e.}$$

As before, applying Parseval's equality for $v + 1$, we have our result.

THEOREM 2.2. *Let* $\|f\|_{2,v} = 1$. *Then*

$$V_v[|f|^2]V_v[|f\hat{\ }|^2] \geqq (v + 1/2)^2.$$

Proof. We may assume that $V_v[|f|^2]$ and $V_v[|f\hat{\ }|^2]$ are both finite, since otherwise there is nothing to prove. Hence, $f \in L^2_{v+1}$ and $f\hat{\ } \in L^2_{v+1}$. It follows from Theorem 2.1 that $df\hat{\ }(x)/dx \in L^2_v$. We have

$$\frac{d}{dx}[x^{2v+1}f\hat{\ }(x)^2] = (2v+1)x^{2v}f\hat{\ }(x)^2 + 2\{x^v[xf\hat{\ }(x)]\}\left\{x^v\frac{d}{dx}f\hat{\ }(x)\right\}$$

which implies that $d[x^{2v+1}f\hat{\ }(x)^2]/dx \in L^1(0, \infty)$. Consequently the limits

$$\lim_{x\to 0^+} x^{2v+1}f\hat{\ }(x)^2 \quad \text{and} \quad \lim_{x\to\infty} x^{2v+1}f\hat{\ }(x)^2$$

exist. Since $f\hat{\ } \in L^2_v$, both limits must be zero.

Let

$$I(a, b) = \int_a^b xf\hat{\ }(x)\frac{d}{dx}f\hat{\ }(x)\, d\mu_v(x).$$

Integrating by parts, we obtain

$$I(a, b) = -[xf\hat{\ }(x)^2c_v^{-1}x^{2v}]_a^b + (v + \tfrac{1}{2})\int_a^b f\hat{\ }(t)^2\, d\mu_v(t),$$

and using the limit relations above, we have

$$I(0, \infty) = (v + \tfrac{1}{2})\int_0^\infty f\hat{\ }(t)^2\, d\mu_v(t) = v + \tfrac{1}{2}.$$

By Schwarz's inequality and Theorem 2.1,

$$\begin{aligned} I(0, \infty) &\leqq \int_0^\infty [xf\hat{\ }(x)]^2\, d\mu_v(x)\int_0^\infty \left[\frac{d}{dx}f\hat{\ }(x)\right]^2 d\mu_v(x) \\ &= \int_0^\infty [xf\hat{\ }(x)]^2\, d\mu_v(x)\int_0^\infty [tf(t)]^2\, d\mu_v(t), \end{aligned}$$

and the proof is complete.

It is clear that equality occurs in Theorem 2.2 if and only if equality occurs in our application of Schwarz's inequality. Since necessary and sufficient conditions for equality in Schwarz's inequality are well known, one can easily find *all* the extremal functions. An example of such a function is

$$F_v(x; t) = \left(\frac{1}{t}\right)^{v/2+1/4} e^{-x^2/(4t)}, \qquad t > 0.$$

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SIAM J. MATH. ANAL.
Vol. 2, No. 4, November 1971

ALMOST EVERYWHERE CONVERGENCE OF FOURIER SERIES ON THE RING OF INTEGERS OF A LOCAL FIELD*

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Abstract. It is shown that the partial sums of the Fourier series of $L^p(\mathfrak{D})$-functions ($p > 1$) converge almost everywhere (a.e.), where $\mathfrak{D}$ is the ring of integers in a local field K. This includes the case where K is a p-adic number field as well as the case where $\mathfrak{D}$ is the Walsh–Paley or dyadic group 2^ω. The techniques are essentially those used by Carleson [2] in establishing the a.e. convergence of trigonometric Fourier series for $L^2(-\pi, \pi)$-functions as modified by Hunt [4] to obtain this same result for $L^p(-\pi, \pi)$-functions, $p > 1$. The necessary modifications for the local field setting are made in the context of the Sally–Taibleson [7] development of harmonic analysis on local fields and by use of Taibleson's multiplier theorem [11]. These same results for 2^ω have already been obtained by Billiard ($L^2(2^\omega)$) [1] and by Sjölin ($L^p(2^\omega)$, $p > 1$) [8]. Many advantages (in particular the non-Archimedean nature of the valuation) of the local field case over the trigonometric case have been utilized. Consequently many purely technical elements of the trigonometric case have disappeared and one is left only with elements of the proof which bear on the central idea. For this reason the proof given can be used to obtain a clearer understanding of the proof for trigonometric Fourier series.

CONTENTS

1. Introduction and statement of results. Let K be a totally disconnected, locally compact, nondiscrete, complete field. The locally compact, nondiscrete, complete fields have been completely characterized and are either connected (the real and complex number fields) or are totally disconnected. The totally disconnected fields can be of characteristic zero or have finite characteristic. Those of finite characteristic are a field of formal Laurent series over a finite field. Those of characteristic zero are a $\mathfrak{p}$-adic number field or a finite algebraic extension of such a field.

Various aspects of harmonic analysis on such "local fields" K have been studied in [7], [3], [5], [6], [9], [10], [11] and [12]. In this paper we shall study Fourier series of functions defined on the ring of integers of a local field K.

We briefly present some notation which will allow us to state our results:

Let $\mathfrak{D}$ denote the ring of integers in K and consider $f \in L^1(\mathfrak{D}, dx)$, where dx denotes normalized Haar measure on $\mathfrak{D}$ ($\int_{\mathfrak{D}} dx = 1$). $S_n f(x)$ will denote the nth

* Received by the editors December 12, 1970.

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partial sum of the Fourier series of f with respect to a suitably ordered complete set of characters on $\mathfrak{O}$. Let $Mf(x) = \sup_n |S_n f(x)|$. $\|f\|_p$ will denote the usual L^p-norm on $L^p(\mathfrak{O}, dx)$, $1 \leqq p \leqq \infty$, and we write $|E| = \int_E dx$ for E a measurable subset of $\mathfrak{O}$. Recall that

$$\log^+ t = \begin{cases} \log t, & t \geqq 1, \\ 0, & \text{otherwise}. \end{cases}$$

In terms of this notation our results are the following theorems.

THEOREM 1. *If $f \in L^p$, $1 < p < \infty$, then there is a constant $A_p > 0$, independent of f, such that $\|Mf\|_p \leqq A_p \|f\|_p$.*

THEOREM 2. *If $f \in L^\infty$, $y > 0$, then there are constants $B_1, B_2 > 0$, independent of f and y, such that $|\{x \in \mathfrak{O}: Mf(x) > y\}| \leqq B_1 \exp\{-B_2 y/\|f\|_\infty\}$.*

THEOREM 3. *If $\int_{\mathfrak{O}} |f(x)| \{\log^+ |f(x)|\}^2 \, dx < \infty$, then there is a constant $C > 0$ independent of f such that $\|Mf\|_1 \leqq C \int_{\mathfrak{O}} |f(x)| \{\log^+ |f(x)|\}^2 \, dx + C$.*

THEOREM 4. *If $\int_{\mathfrak{O}} |f(x)| \log^+ |f(x)| \log^+ \log^+ |f(x)| \, dx < \infty$, then $S_n f(x) \to f(x)$ $(n \to \infty)$ for a.e. $x \in \mathfrak{O}$.*

DEFINITION. f is said to be a *special function* if $f = gI_F$, where I_F is the characteristic function of a measurable subset F of $\mathfrak{O}$, and g is a measurable function with values in $(1/2, 1]$.

(1.1) BASIC RESULT. *Suppose $1 < p < \infty$, $y > 0$, and f is a special function on $\mathfrak{O}$. (For $p = 2$, f can be an arbitrary function in L^2.) Then there is a constant $C_p > 0$, independent of f and y, $C_p \leqq Cp^2/(p-1)$ (C an absolute depending only on K), such that*

$$|\{x \in \mathfrak{O}: Mf(x) > y\}|^{1/p} \leqq C_p y^{-1} \|f\|_p.$$

In § 2 we obtain Theorems 1–4 as a consequence of the Basic Result, (1.1). We note that it is sufficient to establish the Basic Result for characteristic functions (see Hunt [4] and Sjölin [8]). E. M. Stein suggested the use of special functions as technically and intuitively advantageous. These advantages are seen, in particular, in the derivation of Theorems 2–4. For Theorem 1 we use only the fact that the Basic Result holds for characteristic functions; which are, of course, special functions. It should be noted that the proofs (given the Basic Result) of Theorems 2–3 in § 2 are purely measure theoretic and hold on any finite measure space.

In § 3 we collect notation and preliminary results needed for the proof of the Basic Result. This includes a brief review of properties of K, $\mathfrak{O}$, characters on $\mathfrak{O}$, the ordering of the characters, and properties of the Dirichlet kernel.

Section 4 contains the proof of our Basic Result. The proof is essentially Hunt's L^p variant of Carleson's original proof of the a.e. convergence of the partial sums of the Fourier series of functions in $L^2(-\pi, \pi)$. (See Carleson [2] and Hunt [4].) Modifications necessary to obtain local field results depend on Sally and Taibleson's development of harmonic analysis on local fields [7] and Taibleson's multiplier theorem [11].

In this paper we have utilized many advantages of the local field case over the trigonometric case. Consequently many purely technical elements of the trigonometric case have disappeared and we are left with only elements of the proof which bear on the central idea. Because of this, the present proof provides a clearer understanding of the trigonometric proof.

It is worthwhile to note that local field results include the case where $\mathfrak{O}$ is the Walsh–Paley group 2^ω. 2^ω can be identified with the additive group of the ring of integers in the 2-series field; i.e., the field of formal Laurent series over $GF(2)$. In this case $\mathfrak{O}$ can be represented as the interval $[0, 1]$ and the characters on $\mathfrak{O}$ are just the familiar Walsh functions, and the properties listed in § 3 can be verified without any special knowledge of local fields.

Billard [1] has obtained the a.e. convergence of Walsh–Fourier series of L^2-functions by modifying Carleson's original proof. Theorems 1–4 for Walsh–Fourier series were proved by Sjölin [8] by modifying the basic result of Hunt's L^p variant of Carleson's proof.

2. Reduction to the Basic Result. Given only that the Basic Result, (1.1), holds for characteristic functions, Theorems 1–3 are proved in Hunt [4] and Theorem 4 in Sjölin [8]. In this section we restrict ourselves to the proofs of Theorems 2–4, and we exploit the technical advantages of using special functions. The intuitive advantage follows from the fact that any function can be written as a countable sum of multiples of such special functions, and that the Basic Result implies that, restricted to any multiple of a special function f, the map $f \to Mf$ is of weak-type (p, p) for $1 < p < \infty$.

Proof of Theorem 2. We may assume that f is nonnegative and bounded by 1. (If necessary, replace f with $f/\|f\|_\infty$. If $\|f\|_\infty = 0$, there is nothing to prove.) We need to show that there exist $B_1, B_2 > 0$, independent of $f(\|f\|_\infty = 1)$ and $y > 0$ such that

$$|\{x \in \mathfrak{O}: Mf(x) > y\}| \leqq B_1 \exp(-B_2 y).$$

Write $f = \sum_{k=1}^\infty f_k$, where

$$f_k(x) = \begin{cases} f(x), & x \in F_k = \{x \in \mathfrak{O}: 2^{-k} < f(x) \leqq 2^{-k+1}\}, \\ 0, & x \notin F_k, \end{cases}$$

$$\begin{aligned} |\{x \in \mathfrak{O}: Mf(x) > y\}| &\leqq \sum_{k=1}^\infty |\{x \in \mathfrak{O}: Mf_k(x) > 2^{-k}y\}| \\ &\leqq \{C_p y^{-1}\}^p \sum_{k=1}^\infty 2^{kp} \int_{\mathfrak{O}} |f_k(x)|^p \, dx \end{aligned}$$

for each p, $1 < p < \infty$. But,

$$\begin{aligned} \sum_{k=1}^\infty 2^{kp} \int_{\mathfrak{O}} |f_k(x)|^p \, dx &\leqq \sum_{k=1}^\infty [2^k 2^{-k+1}]^p |F_k| \\ &\leqq 2^p \sum_{k=1}^\infty |F_k| \leqq 2^p |\mathfrak{O}| = 2^p. \end{aligned}$$

Hence, there is a constant $A > 0$, independent of y and p, such that

$$|\{x \in \mathfrak{O}: Mf(x) > y\}| \leqq A^p y^{-p} (p^2/(p-1))^p.$$

Set $p = y/eA$. Then

$$|\{x \in \mathfrak{D}: Mf(x) > y\}| \leqq \begin{cases} e^2 \exp(-(1/eA)y), & y/eA \geqq 2, \\ 1 \leqq e^2 \exp(-(1/eA)y), & 0 < y/eA \leqq 2. \end{cases}$$

Our result holds with $B_1 = e^2$, $B_2 = 1/eA$.

Proof of Theorem 3. We may assume that $f \geqq 0$ and write $f = \sum_{k=0}^{\infty} f_k$, where

$$f_0(x) = \begin{cases} f(x), & 0 \leqq f(x) \leqq 2, \\ 0, & \text{otherwise}, \end{cases} \qquad f_k = \begin{cases} f(x), & 2^k < f(x) \leqq 2^{k+1}, \\ 0, & \text{otherwise}, \end{cases} \qquad k \geqq 1.$$

Let $\lambda_k(y) = |\{x \in \mathfrak{D}: Mf_k(x) > y\}|$, $k = 0, 1, 2, \cdots$. Since $f_0 \in L^2$, the Basic Result implies that

$$\begin{aligned} \int_{\mathfrak{D}} Mf_0(x)\,dx &= \int_0^\infty \lambda_0(y)\,dy \\ &\leqq 1 + \int_1^\infty C_2^2 y^{-2}\left[\int_{\mathfrak{D}} |f_0(x)|^2\,dx\right] dy \\ &\leqq 1 + 4C_2^2 = C, \end{aligned}$$

where $C > 0$ is independent of f.

Set $\alpha_k = \|f_k\|_p C_p$, $k \geqq 1$. Then

$$\begin{aligned} \int_{\mathfrak{D}} Mf_k(x)\,dx &= \int_0^{\alpha_k} \lambda_k(y)\,dy + \int_{\alpha_k}^\infty \lambda_k(y)\,dy \\ &\leqq \alpha_k + C_p^p \|f_k\|_p^p \int_{\alpha_k}^\infty y^{-p}\,dy \\ &= \alpha_k + \frac{\alpha_k^p \alpha_k^{1-p}}{p-1} = \alpha_k\left(1 + \frac{1}{p-1}\right) \\ &= \frac{p}{p-1} C_p \|f_k\|_p \leqq \frac{Cp^3}{(p-1)^2}\|f_k\|_p, \end{aligned}$$

where $C > 0$ is independent of f.

The argument continues exactly as in Zygmund [14, vol. II, p. 120]. This completes the proof.

Proof of Theorem 4. Let

$$J(f) = \int_{\mathfrak{D}} |f(x)| \{\log^+ |f(x)| \log^+ \log^+ |f(x)| + 1\}\,dx.$$

Since $\int_{\mathfrak{D}} |f(x)| \log^+ |f(x)| \log^+ \log^+ |f(x)|\,dx < \infty$ implies $J(f) < \infty$, and such functions can be approximated in the "J-norm" by functions with everywhere convergent Fourier series, our result will follow from

$$\begin{gathered} |\{x \in \mathfrak{D}: Mf(x) > C_1[J(f)]^{1/5}\}| \leqq C_2[J(f)]^{1/5}, \\ 0 < J(f) < 1/2, \quad \text{for } C_1, C_2 > 0 \text{ independent of } f \text{ and } J(f), \end{gathered} \tag{2.1}$$

by the usual argument. (The density of a "nice" class of functions in the space of functions f such that $J(f) < \infty$ is shown by standard arguments. One such argument is given in the Appendix.)

We may assume $f > 0$ and let $f = f_0 + \sum_{k=4}^{\infty} f_k$, where

$$f_0(x) = \begin{cases} f(x), & 0 \leqq f(x) < 8, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$f_k(x) = \begin{cases} f(x), & x \in F_k = \{x \in \mathfrak{D} : 2^{k-1} \leqq f(x) < 2^k\}, \\ 0, & x \notin F_k, \end{cases} \qquad k \geqq 4.$$

Throughout, we let $C > 0$ be constants independent of f, not necessarily the same in each instance.

Since $f_0 \in L^2$, our Basic Result yields

$$\begin{aligned} |\{x \in \mathfrak{D} : Mf_0(x) > [J(f)]^{1/5}\}| &\leqq C[J(f)]^{-2/5} \int_{\mathfrak{D}} |f_0(x)|^2 \, dx \\ &\leqq C[J(f)]^{-2/5} J(f) \leqq C[J(f)]^{3/5} \leqq C[J(f)]^{1/5}. \end{aligned}$$

Thus; it will suffice to prove our result for $\sum_{k=4}^{\infty} f_k$. We use the following estimate of Sjölin [8]. For each $k \geqq 4$ and $0 < y \leqq 1/2$ choose $p = 1 + (\log y^{-1} 2^k)^{-1}$. Since $C_p \leqq C(p-1)^{-1}$ for p near 1 and $1 < p < 1 + (\log 32)^{-1}$, we obtain that $C_p \leqq C \log (y^{-1} 2^k)$. Since $f_k \in L^p$ for all k, we have

$$\begin{aligned} |\{x \in \mathfrak{D} : Mf_k(x) > y\}| &\leqq C_p^p y^{-p} \int_{\mathfrak{D}} |f_k(x)|^p \, dx \\ &\leqq [C(2^k y^{-1} \log (2^k y^{-1}))]^{1 + (\log 2^k y^{-1})^{-1}} |F_k| \\ &\leqq C 2^k y^{-1} \log (2^k y^{-1}) |F_k| \\ &\leqq C k 2^k |F_k| (\log y^{-1}) y^{-1}, \qquad 0 < y < 1/2, \quad k \geqq 4. \end{aligned} \tag{2.2}$$

To prove (2.1) for $\sum_{k=4}^{\infty} f_k$ we shall need a modified L^1 estimate. This requires the identification of certain exceptional sets P_k where the functions Mf_k are too large. In particular, we let $\rho = [J(f)]^{2/5}$, $P_k = \{x \in \mathfrak{D} : Mf_k(x) > \rho\}$ and $P = \bigcup_{k=4}^{\infty} P_k$.

From (2.2) we obtain:

$$|P_k| \leqq C k 2^k |F_k| \rho^{-1} \log (1/\rho).$$

Thus,

$$\begin{aligned} |P| \leqq \sum_{k=4}^{\infty} |P_k| &\leqq C \left\{ \sum_{k=4}^{\infty} k 2^k \right\} \rho^{-1} \log (1/\rho) \\ &\leqq C J(f) [J(f)]^{-2/5} \log [1/J(f)] \leqq C[J(f)]^{1/5}. \end{aligned}$$

Thus, we may disregard the set P.

Let

$$\lambda_k(y) = |\{x \in \mathfrak{D} : Mf_k(x) > y\}|.$$

$$\begin{aligned}\int_{\mathfrak{D} \sim P_k} Mf_k(x)\,dx &\leqq \int_0^{\rho} \lambda_k(y)\,dy = \int_{\rho k^{-2}}^{\rho} \lambda_k(y)\,dy + \int_0^{\rho k^{-2}} \lambda_k(y)\,dy \\ &\leqq C|F_k| \int_{\rho k^{-2}}^{\rho} \log(2^k y^{-1})(2^k y^{-1})\,dy + \int_0^{\rho k^{-2}} 1\,dy \\ &\leqq C|F_k| 2^k \int_{\rho 2^{-k} k^{-2}}^{\rho 2^{-k}} \log(1/y)\,dy/y + \rho k^{-2} \\ &\leqq C|F_k| 2^k k \log k \log(1/\rho) + \rho k^{-2}.\end{aligned}$$

Hence,

$$\int_{\mathfrak{D} \sim P} M\left(\sum_{k=4}^{\infty} f_k\right)(x)\,dx \leqq \sum_{k=4}^{\infty} \int_{\mathfrak{D} \sim P_k} Mf_k(x)\,dx \leqq C\{\log(1/\rho)J(f) + \rho\}.$$

From this we obtain

$$\begin{aligned}\left|\left\{x \in \mathfrak{D} \sim P : M\left(\sum_{k=4}^{\infty} f_k\right)(x) > [J(f)]^{1/5}\right\}\right| &= \left|\left\{x \in \mathfrak{D} \sim P : M\left(\sum_{k=4}^{\infty} f_k\right)(x) > \rho^{1/2}\right\}\right| \\ &\leqq C\rho^{-1/2}[\log(1/\rho)J(f) + \rho] \\ &= C[\log(1/\rho)\rho^2 + \rho^{1/2}] \\ &\leqq C\rho^{1/2} = C[J(f)]^{1/5}.\end{aligned}$$

This completes the proof of Theorem 4.

3. Notation and preliminary results. We shall list here properties of totally disconnected, locally compact, nondiscrete, complete fields K. The facts noted here can be found in the introduction to [7] or follow immediately from facts listed there. Most of these properties are well known, and proofs can be found in [7], [3] or [13].

Let K be fixed and let dx be a Haar measure on K^+, the additive group of K. There is a natural (non-Archimedian) norm on K such that $d(\alpha x) = |\alpha| dx$, $|x + y| \leqq \max[|x|, |y|]$, and $|x + y| = \max[|x|, |y|]$ if $|x| \neq |y|$.

The ring of integers of K is $\mathfrak{D} = \{x \in K : |x| \leqq 1\}$, the maximal compact subring of K. We assume that dx is normalized so the measure of $\mathfrak{D}$ is 1. ($|\mathfrak{D}| = \int_{\mathfrak{D}} dx = 1$). $\mathfrak{P} = \{x \in K : |x| < 1\}$ is the unique maximal ideal in $\mathfrak{D}$. $\mathfrak{P}$ is principal and $\mathfrak{D}/\mathfrak{P} \cong GF(q)$, where $q = p^r$, p prime, $r \geqq 1$.

We fix $\mathfrak{p}$ a generator of $\mathfrak{P}$. Then $|\mathfrak{p}| = q^{-1}$, and for all $x \in K$, either $|x| = 0$ (if $x = 0$) or if $x \neq 0, |x| = q^k$ for some integer k. Hence if $x \neq 0, x = \mathfrak{p}^k x'$, where $k \in \mathbb{Z}$ and $x' \in \mathfrak{D}^* = \mathfrak{D} \sim \mathfrak{P}$ are unique. ($\mathfrak{D}^*$ is the ring of units in K^*, the multiplicative group of K.)

For each integer k set $\mathfrak{P}^k = \{x \in K : |x| \leqq q^{-k}\}$. The members of the collection $\{\mathfrak{P}^k\}_{k \in \mathbb{Z}}$ are called the fractional ideals of K. For all k they are subgroups and form a natural neighborhood base at the zero element of K. For $k \geqq 0$ they are subrings of K. Cosets of the $\mathfrak{P}^k$ in K are called spheres and are denoted by ω. If $\omega = x + \mathfrak{P}^k$, then $|\omega| = q^{-k}$ and we say that ω has center x and radius q^{-k}. Thus the measure and radius of a sphere are equal. We shall deal principally with spheres $\omega \subset \mathfrak{O}$ and in special circumstances with certain spheres in $\mathfrak{P}^{-1}$. It is important to note that if ω, $\bar{\omega}$ are any two spheres then either $\omega \cap \bar{\omega} = \emptyset$, $\omega \subset \bar{\omega}$ or $\bar{\omega} \subset \omega$. It follows that any point of ω is its center.

Let χ be a fixed nontrivial, continuous, unitary character on K^+. Then χ is constant (and equal to 1) on some $\mathfrak{P}^k$ and nonconstant on $\mathfrak{P}^{k-1}$. Then every other continuous unitary character (henceforth, simply "character") on K is of the form $\chi_u(x) = \chi(ux)$. Normalizing, we may assume χ is equal to 1 on $\mathfrak{O}$ and is nonconstant on $\mathfrak{P}^{-1}$. The map $u \to \chi_u$ from K^+ onto $\hat{K}^+$ is a topological isomorphism, and we identify $\hat{K}^+$ with K^+ under this isomorphism.

There is a simple picture which permits a reduction so that harmonic analysis on $\mathfrak{O}$ can be studied in terms of harmonic analysis on K. Namely, take functions on $\mathfrak{O}$ and extend them to K by setting them equal to zero on $K \sim \mathfrak{O}$. The Fourier transform of such a function (supported on $\mathfrak{O}$) is constant on cosets of $\mathfrak{O}$ in K, and these constants are the Fourier coefficients of f as a function on $\mathfrak{O}$ with respect to a complete set of characters on $\mathfrak{O}$. That is, there is a natural correspondence between characters on $\mathfrak{O}$ and cosets of $\mathfrak{O}$ in K^+. This correspondence is established by showing that every character on $\mathfrak{O}$ is the restriction to $\mathfrak{O}$ of a character on K, and that two characters χ_u, χ_v have the same restriction to $\mathfrak{O}$ if and only if $u - v \in \mathfrak{O}$; that is, if and only if they are in the same coset of $\mathfrak{O}$ in K^+.

We now select a sequence of coset representatives $\{i(n)\}_{n=0}^{\infty}$ of $\mathfrak{O}$ in K^+ such that if χ_n is the restriction of $\chi_{i(n)}$ to $\mathfrak{O}$, then $\{\chi_n\}_{n=0}^{\infty}$ will be a complete set of characters on $\mathfrak{O}$.

Note that $GF(q) = GF(p^r)$ can be written as an r-dimensional vector space $GF(p)$ with basis $\{\xi_0, \xi_1, \cdots, \xi_{r-1}\}$, where $\xi_0 = 1$. Let $\{\beta_0, \beta_1, \cdots, \beta_{r-1}\}$ be coset representatives of $\mathfrak{P}$ in $\mathfrak{O}$ such that β_k corresponds to ξ_k, $k = 0, 1, \cdots, r-1$ (from the isomorphism $GF(q) \cong \mathfrak{O}/\mathfrak{P}$). For $0 \leqq n < q$, n can be uniquely represented as $n = a_0 + a_1 p + a_2 p^2 + \cdots + a_{r-1}p^{r-1}$, $0 \leqq a_k < p$. We set $i(n) = \mathfrak{p}^{-1}(a_0\beta_0 + a_1\beta_1 + \cdots + a_{r-1}\beta_{r-1})$, where $\mathfrak{p}$ is the generator of $\mathfrak{P}$ that was fixed above. For $n \geqq 0$ write $n = c_0 + c_1 q + \cdots + c_k q^k$, $0 \leqq c_\nu < q$, and set $i(n) = i(c_0) + \mathfrak{p}^{-1}i(c_1) + \cdots + \mathfrak{p}^{-k}i(c_k)$. It follows that $i(n) = 0$ if and only if $n = 0$ and $|i(n)| = q^k$ if and only if $q^{k-1} \leqq n < q^k$, $k = 1, 2, \cdots$.

We now set $\chi_n(x) = \chi_{i(n)}(x) = \chi(i(n)x)$ for $x \in \mathfrak{O}$. It follows that $\chi_0(x) \equiv 1$ and if $q^{k-1} \leqq n < q^k$, then χ_n is constant on spheres of radius q^{-k}, $k \geqq 1$.

(3.1) *If $n = lq^\nu + s$, $0 \leqq s < q^\nu$, l a nonnegative integer, then*

$$\chi_n = \chi_{\ell q^\nu}\chi_s.$$

(3.2) *If $0 \leqq s < q^\nu$, then χ_s is constant on spheres of radius less than or equal to $q^{-\nu}$, $\nu = 0, 1, 2, \cdots$.*

For ω any sphere in $\mathfrak{O}$ of radius $q^{-\nu}$ (so $\omega = x + \mathfrak{P}^\nu$) and n a nonnegative integer we define $n[\omega]$ to be the largest integer in $nq^{-\nu}$. Note that $n = n[\omega]|\omega|^{-1} + s$, $0 \leqq s < |\omega|^{-1}$.

In view of (3.1), (3.2) we have

$$\left|\frac{1}{|\omega|}\int_\omega \chi_n(t)\overline{\chi_m(t)}\,dt\right| = \begin{cases} 1 & \text{if } n[\omega] = m[\omega], \\ 0 & \text{if } n[\omega] \neq m[\omega], \end{cases} \tag{3.3}$$

and

(3.4) $\{|\omega|^{1/2}\chi_{n|\omega|^{-1}}\}_{n=0}^{\infty}$ *is a complete orthonormal set on* ω, *and if* ω *contains the origin, it is a complete set of characters on* ω.

If $f \in L^1(\omega, dx)$, $\omega \subset \mathfrak{D}$, we define the "Fourier coefficients" of f on ω by

$$c_n(\omega; f) = \frac{1}{|\omega|} \int_\omega f(t)\bar{\chi}_{n|\omega|^{-1}}(t)\,dt, \qquad n = 0, 1, 2, \cdots.$$

Plancherel's formula on ω gives

$$\sum_{n=0}^{\infty} |c_n(\omega; f)|^2 |\omega| = \int_\omega |f(x)|^2\,dx. \tag{3.5}$$

If $f \in L^1 = L^1(\mathfrak{D})$, the nth partial sum of the Fourier coefficients of f over $\mathfrak{D}$ can be written as $S_n f(x) = \int_{\mathfrak{D}} f(t)D_n(x - t)\,dt$, where the Dirichlet kernel $D_n(t) = \sum_{k=0}^{n-1} \chi_k(t)$, $n \geqq 1$, $D_0(t) \equiv 0$. $D_n^*(t)$, the modified Dirichlet kernel, is defined by $D_n^* = \bar{\chi}_n D_n$.

Note. In the classical treatment of Fourier series, ordinary partial sums are studied in terms of modified partial sums, and these are reduced to the study of the conjugate transform. The same program is carried out here, but it is necessary to introduce a distinct "conjugate transform" for each n.

For $\omega \subset \mathfrak{D}$, $\omega = x + \mathfrak{P}^k$ define $\omega^* = x + \mathfrak{P}^{k-1}$. Thus $\mathfrak{D}^* = \mathfrak{P}^{-1}$. We extend f to $\mathfrak{P}^{-1}$ by setting $f(x) = 0$ for $x \in \mathfrak{P}^{-1} \sim \mathfrak{D}$, and extend the characters χ_n similarly. Note that for each ω^* there are exactly q spheres $\bar{\omega}$ with $\bar{\omega}^* = \omega^*$, and that $|\omega^*| = q|\omega|$.

For each $\omega \subset \mathfrak{D}$ there is a unique sequence of spheres $\{\omega_j\}_{j=0}^{J}$ such that $\mathfrak{D} = \omega_0$, $\omega = \omega_J$, and $\omega_j = \omega_{j-1}^*$, $j = 0, 1, \cdots, J - 1$. This fact allows inductive definitions on ω which are based on knowledge of spheres $\bar{\omega}$ where $\bar{\omega} \supset \omega^*$. Note that for the sequence $\{\omega_j\}$, $|\omega_j| = q^{-j}$, $|\omega_j^*| = q^{-j+1}$.

Suppose $\omega^* \subset \mathfrak{P}^{-1}$. We define

$$S_n^* f(x; \omega^*) = \int_{\omega^*} f(t)\bar{\chi}_n(t)D_n^*(x - t)\,dt, \qquad n \geqq 0, \quad x \in \omega^*.$$

$$|S_n^* f(x; \omega^*)| = \left|\int_{\omega^*} f(t)D_n(x - t)\,dt\right|, \qquad n \geqq 0, \quad x \in \omega^*. \tag{3.6}$$

(3.7) *If* ω *is any sphere and* $x \notin \omega$, *then* $D_n^*(x - t)$ *is constant as* t *varies over* ω.

Proof. This immediately reduces to $D_n^*(x + y) = D_n^*(x)$ if $|y| < |x|$, which is contained in Taibleson [11, Theorem 4].

Define $C_n(\omega^*; f) = \max\{|c_n(\bar{\omega}; f)| : \bar{\omega}^* = \omega^*\}$.

(3.8) *If* $n[\omega] = m[\omega]$, *then*

$$|S_n^* f(x; \omega^*)| \leqq |S_m^* f(x; \omega^*)| + qC_{n[\omega]}(\omega^*; f).$$

Proof.

$$\begin{aligned}
&\left|\int_{\omega^*} f(t)D_n(x-t)\,dt - \int_{\omega^*} f(t)D_m(x-t)\,dt\right| \\
&\quad \leqq \sum_{\bar\omega \ni \bar\omega^* = \omega^*} \left|\int_{\bar\omega} f(t)[D_n(x-t) - D_m(x-t)]\,dt\right| \\
&\quad \leqq \sum_{\bar\omega \ni \bar\omega^* = \omega^*} \sum_{s=0}^{|\omega|^{-1}-2} \left|\int_{\bar\omega} f(t)\chi_{n[\omega]|\omega|^{-1}+s}(x-t)\,dt\right| \\
&\quad = \sum_{\bar\omega \ni \bar\omega^* = \omega^*} \sum_{s=0}^{|\omega|^{-1}-2} |\bar\omega|\,|c_{n[\omega]}(\bar\omega; f)| \\
&\quad \leqq q(|\omega|^{-1} - 1)|\bar\omega| C_{n[\omega]}(\omega^*; f) \leqq q C_{n[\omega]}(\omega^*; f).
\end{aligned}$$

The result follows immediately from (3.6).

Remark. On occasion, ω^* is given explicitly, but ω is not, while we still wish to define $n[\omega]$. In this case, choose any $\bar\omega$ such that $\bar\omega^* = \omega^*$ and define $n[\omega] = n[\bar\omega]$.

The following theorem on the maximal modified partial sums on a sphere ω^* is essential to our Basic Result.

Suppose $g \in L^\infty(\omega^*, dx)$ and $\|g\|_{\infty,\omega^*}$ is the essential supremum of g on ω^*. For $x \in \omega^*$ define

(3.9)
$$T_n^* g(x; \omega^*) = \sup_{x \in \bar\omega^* \subset \omega^*} \left|\int_{\omega^* \sim \bar\omega^*} g(t)D_n^*(x-t)\,dt\right|.$$

Let $y > 0$. Then there is a constant $A > 0$, independent of g, ω^* and y such that

$$|\{x \in \omega^* : T_n^* g(x; \omega) > y\}| < e^{1/2} \exp\{-Ay/\|g\|_{\infty,\omega^*}\}|\omega^*|.$$

Proof. See Taibleson [11, Corollary 4] for a proof.

4. Proof of the Basic Result.

4.1. Reduction to the Basic Lemma.

(4.1) BASIC LEMMA. *Fix y, p and N with $y > 0$, $1 < p < \infty$, and N a positive integer. Let f be a function supported on $\mathfrak{D}$ which is a special function if $p \neq 2$ but is any function in L^2 if $p = 2$.*

(4.2) *Let $L = L(p) = [2p^2/(p-1)] + 1$, where $[\cdot]$ is the greatest integer function.*

Then there are a set $E = E(y, p, N, f) \subset \mathfrak{D}$ and a constant $C > 0$ independent of y, p, N, f such that

$$|E| \leqq C^p y^{-p} \int_{\mathfrak{D}} |f(x)|^p\,dx,$$

and $x \in \mathfrak{D} \sim E$, $0 \leqq n < q^N$, implies

$$|S_{n+1} f(x)| \leqq CLy.$$

Suppose we establish (4.1), the Basic Lemma. Since the estimate of $|E|$ does not depend on N, (4.1) implies the Basic Result, (1.1), with $C_p = CL(p)$. From (4.2) we see that (1.1) would follow.

4.2. Development of assertions. The main idea of the proof of (4.1) is to reduce $S_{n+1}f(x)$ to an integral over a sphere ω^* with $|\omega^*| = 2^{-N+1}$. Then $0 \leqq n < 2^N$ implies $n[\omega] = 0$ and (3.8) gives a good estimate of the integral $S_n^* f(x; \omega^*)$.

The reduction is carried out in a finite number of steps by a careful construction of partitions of certain spheres ω^* into subspheres, the partitions depending on certain integers $n|\omega|$ and the size of $C_{n[\omega]}(\omega^*; f)$. Together with each partition we need an estimate of the size of certain differences, $|S_n^* f(x; \omega^*) - S_n^* f(x; \tilde{\omega}^*)|$, $\tilde{\omega}^* \subset \omega^*$. These estimates also depend on the size of $C_{n[\omega]}(\omega^*; f)$.

To illustrate the main idea of the proof without becoming involved in several side constructions we shall give a proof of (4.1) (in § 4.3) that depends on various assertions. These assertions will be starred, and in § 4.4 we return to these assertions and complete the proof by "removing the stars."

Several parts of the proof depend on the size of various coefficients $C_n(\omega^*; f)$. To control the size of the coefficients we define a set $S^* \subset \mathfrak{D}^*$ such that

$$|S^*| \leqq q y^{-p} \int_{\mathfrak{D}} |f(x)|^p \, dx, \tag{4.3*}$$

and

$$\omega^* \not\subset S^* \Rightarrow C_n(\omega^*; f) < y \quad \textit{for all } n = 0, 1, 2, \cdots. \tag{4.4*}$$

For each positive integer k we shall define a collection G_k^* of pairs $(n|\omega|, \omega^*)$ which satisfy

$$\begin{aligned}&(n|\omega|, \omega^*) \in G_k^* \Rightarrow n|\omega| \textit{ is a nonnegative integer}, \ \omega^* \not\subset S^*, \ \omega^* \subset \mathfrak{D}^*, \\ &|\omega^*| > q^{-N+1}, \ \textit{and } C_{n[\omega]}(\omega^*; f) < q^{-k+1} y.\end{aligned} \tag{4.5*}$$

For each pair $(n|\omega|, \omega^*) \in G_k^*$ we define a partition $\Omega = \Omega(n|\omega|, \omega^*, k)$ of ω^* into a finite union of mutually disjoint spheres $\tilde{\omega}^*$.

A sphere $\tilde{\omega}^* \subset \omega^*$ is an element of Ω if

$$C_{n[\tilde{\omega}]}(\tilde{\omega}^*; f) < q^{-k+1} y \qquad \text{for all } \tilde{\omega}^* \text{ such that } \bar{\omega}^* \subsetneqq \tilde{\omega}^* \subset \omega^*, \tag{4.6}$$

and

$$|\bar{\omega}^*| = q^{-N+1} \tag{4.7}$$

or

$$|\bar{\omega}^*| > q^{-N+1} \quad \text{and} \quad C_{n[\bar{\omega}]}(\bar{\omega}^*; f) \geqq q^{-k+1} y.$$

Note that (4.5) implies that each of the q spheres $\bar{\omega}^*$ such that $(\bar{\omega}^*)^* = \omega^*$ satisfies (4.6) and that $|\bar{\omega}^*| \geqq q^{-N+1}$. It follows that Ω is well-defined and $\bar{\omega}^* \in \Omega \Rightarrow \bar{\omega}^* \subsetneqq \omega^*$.

The partition $\Omega = \Omega(n|\omega|, \omega^*, k)$ defined above when $(n|\omega|, \omega^*) \in G_k^*$ will be used to define a subset $\mathscr{U}^* = \mathscr{U}^*(n|\omega|, \omega^*, k)$ of ω^* such that

$$|\mathscr{U}^*| \leqq q^{-5kL} |\omega^*|, \tag{4.8*}$$

and:

(4.9*) *If $x \in \omega^* \sim \mathscr{U}^*$, $x \in \tilde{\omega}^* \subset \omega^*$, and $\omega^* \sim \tilde{\omega}^*$ is a union of spheres in Ω, then there is a fixed constant $C_0 >$ such that*

$$|S_n^* f(x;\omega^*) - S_n^* f(x;\tilde{\omega}^*)| < C_0 L k q^{-k+1} y.$$

To use the estimate (4.9) we need to avoid points x which are in the set

$$\bigcup_{k=1}^{\infty} \bigcup_{(n|\omega|,\omega^*)\in G_k^*} \mathscr{U}^*(n|\omega|,\omega^*,k) = U^*.$$

To estimate the measure of this set we shall show that

$$(4.10^*) \qquad \sum_{(n|\omega|,\omega^*)\in G_k^*} |\omega^*| \leqq q^{5kL-k} y^{-p} \int_{\mathfrak{O}} |f(x)|^p\,dx, \qquad k = 1,2,\cdots.$$

Combining (4.8) and (4.10) we obtain

$$(4.11) \qquad \begin{aligned} |U^*| &\leqq \sum_{k=1}^{\infty} q^{-5kL} \sum_{(n|\omega|,\omega^*)\in G_k^*} |\omega^*| \\ &\leqq \left(\sum_{k=1}^{\infty} q^{-k}\right) y^{-p} \int_{\mathfrak{O}} |f(x)|^p\,dx \leqq y^{-p} \int_{\mathfrak{O}} |f(x)|^p\,dx. \end{aligned}$$

Note. It is essential that the estimate in (4.11) (and hence in (4.10)) be independent of the integer N. This clearly restricts the number of pairs which belong to G_k^*. In particular, G_k^* will not contain all pairs $(n|\omega|,\omega^*)$ which satisfy (4.5).

In the proof we shall need to partition certain spheres ω^* with respect to certain integers $n[\omega]$ even though $(n|\omega|,\omega^*)$ may not be one of our selected pairs (i.e., be in G_k^* for any k). This is possible because of the following two assertions.

(4.12*) *If $\omega^* \not\subset S^*$, $\omega^* \subset \mathfrak{O}^*$, $|\omega^*| > q^{-N+1}$, and $C_{n[\omega]}(\omega^*;f) \geqq q^{-k}y$, then there exists $(\bar{n},\bar{\omega}^*,\bar{k})$ such that $\bar{n}[\omega] = n[\omega]$, $\bar{\omega}^* \supset \omega^*$, $1 \leqq \bar{k} \leqq k$, $(\bar{n}|\bar{\omega}|, \bar{\omega}^*) \in G_{\bar{k}}^*$, and $C_{\bar{n}[\tilde{\omega}]}(\tilde{\omega}^*;f) < q^{-\bar{k}+1}$ for all $\tilde{\omega}^*$ such that $\mathfrak{O}^* \supset \tilde{\omega}^* \supset \omega^*$.*

In the special case when $\omega^* = \mathfrak{O}^*$ we have

$$(4.13^*) \qquad q^{-k}y \leqq C_n(\mathfrak{O}^*;f) < q^{-k+1}y, \quad \mathfrak{O}^* \neq S^* \Rightarrow (n,\mathfrak{O}^*) \in G_k^*.$$

We now show how (4.12) is used.

(4.14) *Suppose $n[\omega]$, ω^*, and k are as in (4.12) and $(\bar{n},\bar{\omega}^*,\bar{k})$ are chosen as in that result. Let $\bar{\Omega} = \Omega(\bar{n}|\bar{\omega}|,\bar{\omega}^*,\bar{k})$, and $\omega^*(x)$ denote the sphere in $\bar{\Omega}$ that contains x, where x is any fixed point in ω^*. Then:*

(a) $\omega^*(x) \subsetneqq \omega^*$,

(b) $0 \leqq n < q^N \Rightarrow 0 \leqq \bar{n} < q^N$,

(c) $x \in \omega^* \sim \mathscr{U}^*(\bar{n}|\bar{\omega}|,\bar{\omega}^*,\bar{k})$ *implies*

$$|S_n^* f(x;\omega^*)| < q^{-\bar{k}+2}y + 2C_0 L\bar{k}q^{-\bar{k}+1}y + |S_{\bar{n}}^* f(x;\omega^*(x))|.$$

Proof. Since $(\bar{n}|\bar{\omega}|,\bar{\omega}^*) \in G_{\bar{k}}^*$, $\bar{\Omega}$ is defined.

There are two possibilities since $\omega^*(x) \cap \omega^* \neq \varnothing$. Namely, $\omega^*(x) \subsetneqq \omega^*$ or $\omega^* \subset \omega^*(x)$. But if $\omega^* \subset \omega^*(x)$, then (4.7) and the last condition of (4.12) gives a contradiction.

If $0 \leqq n < q^N$, notice that $\bar{n}[\omega] = n[\omega]$ and $|\omega| \geqq q^{-N+1}$. Thus $|\omega|^{-1} = q^{N-\ell}$, $\ell \geqq 1$, $n[\omega] \leqq q^{\ell} - 1$ and so

$$\bar{n} < \bar{n}[\omega]|\omega|^{-1} + |\omega|^{-1} < (q^{\ell} - 1)q^{N-\ell} + q^{N-\ell} = q^N.$$

The estimate of $S_n^* f(x; \omega^*)$ is obtained in two parts.

First, since $\bar{n}[\omega] = n[\omega]$, (3.8) and the last condition of (4.12) imply that

$$\begin{aligned} |S_n^* f(x; \omega^*)| &\leqq q C_{n[\omega]}(\omega^*; f) + |S_{\bar{n}} f(x; \omega^*)| \\ &\leqq q^{-\bar{k}+2} y + |S_{\bar{n}} f(x; \omega^*)|. \end{aligned}$$

Let Ω be a partition of a sphere ω^* into a finite union of mutually disjoint spheres; let $\bar{\omega}^*$ be an element of the partition and $\tilde{\omega}^*$ be a sphere such that $\bar{\omega}^* \subset \tilde{\omega}^* \subset \omega^*$. Then $\omega^* \sim \tilde{\omega}^*$ is a union of elements of Ω. This follows since each sphere in Ω is either disjoint from $\tilde{\omega}^*$ or is contained in it, since otherwise $\tilde{\omega}^*$ would be properly contained in a sphere of the partition which is impossible given the fact that $\bar{\omega}^* \subset \tilde{\omega}^*$.

Apply this observation to $\omega^*(x) \subset \omega^* \subset \bar{\omega}^*$, and we have that $\bar{\omega}^* \sim \omega^*$ is a union of spheres in $\bar{\Omega}$. It follows from (4.9) that if $x \in \omega^* \sim \mathscr{U}^*(\bar{n}|\bar{\omega}|, \bar{\omega}^*, \bar{k})$,

$$\begin{aligned} |S_{\bar{n}}^* f(x; \omega^*) - S_{\bar{n}}^* f(x; \omega^*(x))| &\leqq |S_{\bar{n}}^* f(x; \bar{\omega}^*) - S_{\bar{n}}^* f(x; \omega^*)| \\ &\quad + |S_{\bar{n}}^* f(x; \bar{\omega}^*) - S_{\bar{n}}^* f(x; \omega^*(x))| \\ &< 2C_0 L\bar{k} q^{-\bar{k}+1} y. \end{aligned}$$

If we combine the estimates, (c) follows and the proof of (4.14) is complete.

4.3. Proof of the Basic Lemma. We can now assemble a proof of (4.1) based on the assertions above.

The exceptional set E is $S^* \cup U^*$. From (4.3) and (4.11) we have $|E| \leqq C^p y^{-p} \int_{\mathfrak{D}} |f(x)|^p \, dx$, where $C = \sup_{1<p<\infty} (1+q)^{1/p} = 1 + q$, which meets the requirements of (4.1).

Fix $x \in \mathfrak{D} \sim E$, $0 \leqq n < q^N$, and consider $S_{n+1} f(x)$. We may assume that $c_n(\mathfrak{D}; f) \neq 0$. (Otherwise, $S_{n+1} f(x) = S_n f(x)$, and so on. Eventually some $c_m(\mathfrak{D}; f) \neq 0$, $1 < m < n$, and we may use $S_{m+1} f = S_{n+1} f$ or $S_{m+1} f \equiv 0$, $0 \leqq m \leqq n$.)

From (4.4), $C_n(\mathfrak{D}^*; f) < y$ (else, $\mathfrak{D}^* = E$) and so there exists an integer $k_0 \geqq 1$ such that

$$q^{-k_0} y \leqq C_n(\mathfrak{D}^*; f) = |c_n(\mathfrak{D}; f)| < q^{-k_0+1} y.$$

Statement (4.13) implies $(n, \mathfrak{D}^*) \in G_{k_0}^*$, so the partition $\Omega_0 = \Omega(n, \mathfrak{D}^*, k_0)$ is defined.

Let ω_1^* denote the sphere in Ω_0 that contains x. From (4.9) we have

$$\begin{aligned} |S_{n+1} f(x)| &\leqq |c_n(\mathfrak{D}; f)| + |S_n^* f(x; \mathfrak{D}^*)| \\ &< q^{-k_0+1} y + C_0 L k_0 q^{-k_0+1} y + |S_n^* f(x; \omega_1^*)|. \end{aligned} \tag{0}$$

Note that $\omega_1 \subsetneqq \mathfrak{D}^*$. If $\omega_1^* = q^{-N+1}$, we stop the construction. If $|\omega_1^*| > q^{-N+1}$, we continue with a typical step.

According to (4.7), $|\omega_1^*| > q^{-N+1}$ implies $C_{n[\omega_1]}(\omega_1^*; f) \geqq q^{-k_1} y$ for some k_1, $1 \leqq k_1 < k_0$. Apply (4.12) to obtain $(\bar{n}_1, \bar{\omega}_1^*, \bar{k}_1)$. Let ω_2^* denote the sphere in

$\Omega(\bar{n}_1|\bar{\omega}_1|, \bar{\omega}_1^*, \bar{k}_1)$ which contains x. Then (4.14) implies

$$\begin{aligned} &|S_n^* f(x;\omega_1^*)| \leqq q^{-\bar{k}_1+2}y + 2C_0L\bar{k}_1 q^{-\bar{k}_1+1}y + |S_{\bar{n}_1}^* f(x;\omega_2^*)|, \\ &\omega_2^* \subsetneqq \omega_1^*, \quad 0 \leqq \bar{n}_1 < q^N, \quad 1 \leqq \bar{k}_1 < k_0. \end{aligned} \tag{1}$$

If $|\omega_2^*| = q^{-N+1}$, we stop. If $|\omega_2^*| > q^{-N+1}$, we repeat the step above $(J-1)$-times $(J \leqq N)$ to obtain

$$\begin{aligned} &|S_{\bar{n}_{j-1}}^* f(x;\omega_j^*)| \leqq q^{-\bar{k}_j+2}y + 2C_0L\bar{k}_j y + |S_{\bar{n}_j}^* f(x;\omega_{j+1}^*)|, \\ &\omega_{j+1}^* \subsetneqq \omega_j^*, \quad 0 < \bar{n}_j < q^N, \quad 1 \leqq \bar{k}_j < \bar{k}_{j-1}; \\ &j = 2, 3, \cdots, J, \quad |\omega_{J+1}^*| = q^{-N+1}, \quad x \in \omega_{J+1}^*. \end{aligned} \tag{j}$$

Then $\bar{n}_J[\omega_{J+1}] = 0$ so (3.8), (4.4), and the fact that $S_0^* f \equiv 0$ imply

$$|S_{\bar{n}_J}^* f(x;\omega_{J+1}^*)| \leqq qC_0(\omega_{J+1}^*; f) < qy. \tag{$J+1$}$$

Combining the estimates (0), (1), (2), $\cdots$, (J), ($J+1$) we obtain

$$|S_{n+1} f(x)| \leqq \left\{ \sum_{k=1}^{\infty} q^{-k+2} + 2C_0L \sum_{k=1}^{\infty} kq^{-k+1} + q \right\} y.$$

This completes the proof of (4.1) except for the verification of the starred assertions, (4.3), (4.4), (4.5), (4.8), (4.9), (4.10), (4.12) and (4.13).

4.4. Proof of assertions.

Proof of (4.3) *and* (4.4). Let S be the union of all spheres $\omega \subset \mathfrak{D}$ which satisfy

$$\int_\omega |f(x)|^p \, dx \geqq y^p |\omega|. \tag{4.15}$$

Using the fact that if ω_1 and ω_2 are two spheres, then either they are disjoint, or one contains the other we see that S can be written as a countable union of disjoint spheres which satisfy (4.15). Thus

$$|S| \leqq y^{-p} \int_{\mathfrak{D}} |f(x)|^p \, dx. \tag{4.16}$$

$$\omega \not\subset S \Rightarrow |c_n(\omega; f)| \leqq \frac{1}{|\omega|} \int_\omega |f(x)| \, dx \leqq \left[\frac{1}{|\omega|} \int_\omega |f(x)|^p \, dx \right]^{1/p} < y, \tag{4.17}$$

as follows from (4.15).

Let $S^* = \bigcup_{\omega \in S} \omega^*$. From (4.16) we have

$$|S^*| \leqq q|S| \leqq qy^{-p} \int_{\mathfrak{D}} |f(x)|^p \, dx. \tag{4.3}$$

If $\omega^* \not\subset S^*$, then $\bar{\omega} \not\subset S$ for each $\bar{\omega}$ such that $\bar{\omega}^* = \omega^*$. Hence, from (4.17),

$$\omega^* \not\subset S^* \Rightarrow C_n(\omega^*; f) < y \quad \textit{for all } n = 0, 1, 2, \cdots. \tag{4.4}$$

This proves (4.3) and (4.4).

Proof of (4.5), (4.12) *and* (4.13). The selection of pairs for the collection G_k^* is related to the construction of certain polynomials $p_k(x;\omega)$.

Fix the positive integer k. Define $p_k(x;\mathfrak{D}^*) \equiv 0$. The definition of $p_k(x;\omega)$ is inductive. We fix $\omega \subset \mathfrak{D}$ such that $|\omega| \geqq q^{-N+1}$, and assume $p_k(x;\omega^*)$ is defined. (Recall that any ω is reached by a unique chain $\{\omega_j\}_{j=0}^J$ such that $\omega_0 = \mathfrak{D}^*$, $\omega_J = \omega$, and $\omega_j = \omega_{j-1}^*$, $j = 1, 2, \cdots, J$.)

Let $G_k(\omega) = \{(n,\omega): |c_n(\omega; f - p_k(\cdot;\omega^*))| \geqq q^{-k}y\}$, and let

$$p_k(x;\omega) = p_k(x;\omega^*) + \sum_{(n,\omega)\in G_k(\omega)} c_n(\omega; f - p_k(\cdot;\omega^*))\chi_{n|\omega|^{-1}}(x).$$

We note that for $\omega \subset \mathfrak{D}$,

(4.18) $$(n,\omega) \in G_k(\omega) \Rightarrow |c_n(\omega; f - p_k(\cdot;\omega^*))| \geqq q^{-k}y,$$

and

(4.19) $$|c_m(\omega; f - p_k(\cdot;\omega))| < q^{-k}y \quad \text{for all } (m,\omega).$$

We may write

$$p_k(x;\omega) = \sum_{\substack{\omega' \\ \omega \subset \omega' \subset \mathfrak{D}}} \sum_{(n'|\omega'|,\omega')\in G_k(\omega')} c_{n'|\omega'|}(\omega'; f - p_k(\cdot;(\omega')^*))\chi_{n'}(x).$$

From this representation we see:

(4.20) *If $p_k(x;\omega)$ contains a term $c\chi_{n'}$, then there exists ω' such that $\omega \subset \omega' \subset \mathfrak{D}$ and $(n'|\omega'|, \omega') \in G_k(\omega')$.*

Let

$$G_k^* = \{(n|\omega|,\omega^*): |\omega| \geqq q^{-N+1}, (n|\omega|,\omega) \in G_k(\omega), \omega^* \not\subset S^*, C_{n|\omega|}(\omega^*; f) < q^{-k+1}y\}.$$

(4.5) *The conditions of* (4.5*) *are now met.*

(4.13) *The conditions of* (4.13*) *are also met.*

We now establish a lemma needed to prove (4.12).

(4.21) *If $|c_m(\omega; f)| \geqq q^{-k}y$, then there exist n', ω' such that $\omega \subset \omega' \subset \mathfrak{D}$, $n'[\omega] = m$, and $(n'|\omega'|, \omega') \in G_k(\omega')$.*

Proof. This follows from (4.20) if $p_k(x;\omega)$ contains a term $c\chi_{n'}$ with $n'[\omega] = m$. If $p_k(x;\omega)$ contains no such term, then $c_m(\omega; p_k(\cdot;\omega)) = 0$ and so $|c_m(\omega; f - p_k(\cdot;\omega))| = |c_m(\omega; f)| \geqq q^{-k}y$, which contradicts (4.19).

(4.12) *If $\omega^* \not\subset S^*$, $\omega^* \subset \mathfrak{D}^*$, $|\omega^*| > q^{-N+1}$, and $C_{n[\omega]}(\omega^*; f) \geqq q^{-k}y$, then there exists $(\bar{n}, \bar{\omega}^*, \bar{k})$ such that $\bar{n}[\omega] = n[\omega]$, $\omega^* \subset \bar{\omega}^*$, $1 \leqq \bar{k} \leqq k$, $(\bar{n}|\bar{\omega}|, \bar{\omega}^*) \in G_{\bar{k}}^*$ and $C_{\bar{n}[\tilde{\omega}]}(\tilde{\omega}^*; f) < q^{-\bar{k}+1}y$ for all $\tilde{\omega}^*$ with $\omega^* \subset \tilde{\omega}^* \subset \mathfrak{D}^*$.*

Proof. Let Σ denote the collection of triples (n', ω', k') such that:

(i) $1 \leqq k' \leqq k$,
(ii) $\omega^* \subset (\omega')^* \subset \mathfrak{D}^*$,
(iii) $n'[\omega] = n[\omega]$,
(iv) $(n'|\omega'|, \omega') \in G_k(\omega')$.

From (4.21) we see that there exists $(n', \omega', k) \in \Sigma$ so that Σ is not empty. Let $(\bar{n}, \bar{\omega}, \bar{k})$ be an element of Σ with $\bar{k}$ minimal. It only remains to check the last condition of (4.12), for together with $\omega^* \not\subset S^*$, $|\omega^*| > q^{-N+1}$ and the conditions of Σ we obtain that $(\bar{n}|\varpi|, \varpi^*) \in G_{\bar{k}}^*$.

Suppose there exists $\tilde{\omega}^* \supset \omega^*$ with $C_{\bar{n}[\tilde{\omega}]}(\tilde{\omega}^*; f) \geqq q^{-\bar{k}+1}y$. Since $\tilde{\omega}^* \not\subset S^*$ this coefficient is less than y, and so $(\bar{k} - 1) \geqq 1$. We apply (4.21) with $\bar{n}[\tilde{\omega}]$, $\tilde{\omega}$, $(\bar{k} - 1)$, and obtain n', ω' such that $\tilde{\omega} \subset \omega' \subset \mathfrak{D}$, $n'[\tilde{\omega}] = \bar{n}[\tilde{\omega}]$, and $(n'|\omega'|, \omega') \in G_{\bar{k}-1}(\omega')$.

We have $1 \leqq \bar{k} - 1 < \bar{k} \leqq k$. Note also that $\tilde{\omega} \subset \omega'$ and $\tilde{\omega}^* \supset \omega^*$ implies $(\omega')^* \supset \omega^*$. Further $n'[\tilde{\omega}] = \bar{n}[\tilde{\omega}], |\tilde{\omega}| \geqq |\omega|$ implies $n'[\omega] = \bar{n}[\omega]$. Since $\bar{n}[\omega] = n[\omega]$ we have that $n'[\omega] = n[\omega]$. It follows that $(n', \omega', \bar{k} - 1) \in \Sigma$ since $(n'|\omega'|, \omega') \in G_{\bar{k}-1}(\psi')$.

This contradicts the minimality of $\bar{k}$. The proof of (4.12) is complete.

Proof of (4.10). We need an estimate of

$$\sum_{j=0}^{N-1} \sum_{\substack{(n,\omega)\in G_k(\omega) \\ |\omega| = q^{-j}}} |\omega|.$$

When we multiply this by q we get the required estimate. Set $a_n(\omega) = c_n(\omega; f - p_k(\cdot\,; \omega^*))$. From Plancherel's formula, (3.5), we obtain

$$\begin{aligned}
&\sum_{|\omega| = q^{-N+1}} \int_\omega |f(x) - p_k(x; \omega)|^2 \, dx \\
&= \sum_{|\omega| = q^{-N+1}} \int_\omega |f(x) - p_k(x; \omega^*) - \sum_{(n,\omega)\in G_k(\omega)} a_n(\omega)\chi_{nqN-1}(x)|^2 \, dx \\
&= \sum_{|\omega| = q^{-N+1}} \int_\omega |f(x) - p_k(x; \omega^*)|^2 \, dx - \sum_{\substack{(n,\omega)\in G_k(\omega) \\ |\omega| = q^{-N+1}}} |a_n(\omega)|^2 |\omega| \\
&= \sum_{|\omega| = q^{-N+2}} \int_\omega |f(x) - p_k(x; \omega)|^2 \, dx - \sum_{\substack{(n,\omega)\in G_k(\omega) \\ |\omega| = q^{-N+1}}} |a_n(\omega)|^2 |\omega|.
\end{aligned}$$

The first term on the right is similar to the term on the left. We repeat the argument $N - 1$ times to obtain

$$\begin{aligned}
0 &\leqq \sum_{|\omega| = q^{-N+1}} \int_\omega |f(x) - p_k(x; \omega)|^2 \, dx \\
&= \int_{\mathfrak{D}} |f(x)|^2 \, dx - \sum_{j=0}^{N-1} \sum_{\substack{(n,\omega)\in G_k(\omega) \\ |\omega| = q^{-j}}} |a_n(\omega)|^2 |\omega|.
\end{aligned}$$

Hence,

$$\sum_{j=0}^{N-1} \sum_{\substack{(n,\omega)\in G_k(\omega) \\ |\omega| = q^{-j}}} |a_n(\omega)|^2 |\omega| \leqq \int_{\mathfrak{D}} |f(x)|^2 \, dx.$$

From (4.18), $|a_n(\omega)| \geqq q^{-k}y$. Hence,

$$\sum_{j=0}^{N-1} \sum_{\substack{(n,\omega)\in G_k(\omega) \\ |\omega| = q^{-j}}} |\omega| < q^{2k} y^{-2} \int_{\mathfrak{D}} |f(x)|^2 \, dx. \tag{4.22}$$

We note that $L(2) = 9$, so that for $p = 2$ we require an estimate with the factor $q^{5 \cdot k \cdot 9 \cdot 9 - k} = q^{44k}$. From (4.22) we obtain the factor q^{2k+1}. Since $2k + 1 \leqq 44k$ for $k \geqq 1$ the estimate (4.10) for $p = 2$ is established.

The proof for L^p, $p \neq 2$, f a special function, requires an additional observation.

(4.23) *If $G_k(\omega)$ contains a pair (n, ω) with $\omega \not\subset S$, then $y^{-2} \leqq q^{kL} y^{-p}$, where $L = L(p)$ is defined in* (4.2).

Proof. If $(n, \omega) \in G_k(\omega)$, there is a pair (m, ω') with $\omega' \supset \omega$ and $|c_m(\omega'; f)| \geqq q^{-k} y$. To see this, suppose $|c_m(\omega'; f)| < q^{-k} y$ for all (m, ω') such that $\omega' \supsetneqq \omega$. Then $p_k(x; \omega^*) \equiv 0$. Then $(n, \omega) \in G_k(\omega)$ implies $|c_n(\omega; f)| = |c_n(\omega; f - p_k(\cdot\,; \omega^*)| \geqq q^{-k} y$.

Fix such a pair (m, ω'). In the case $1 < p < 2$, and f is a special function, then (4.15) and $\omega' \not\subset S$ imply

$$q^{-k} y \leqq |c_m(\omega'; f)| \leqq \frac{1}{|\omega'|} \int_{\omega'} |f(x)|\, dx$$

$$\leqq 2^{p-1} \frac{1}{|\omega'|} \int_{\omega'} |f(x)|^p\, dx \leqq 2^{p-1} y^p \leqq q y^p.$$

This yields $y^{1-p} < q^{k+1}$. That $y^{p-2} < q^{kL}$ follows easily from the fact that $(k + 1)(2 - p)/(p - 1) \leqq 2p^2 k/(p - 1)$ for $k = 1, 2, 3, \cdots$.

In the case $p > 2$ and f is a special function, we have

$$q^{-k} y \leqq |c_m(\omega'; f)| \leqq \frac{1}{|\omega'|} \int_{\omega'} |f(x)|\, dx \leqq 1.$$

This yields $y \leqq q^k$, and then $y^{p-2} \leqq q^{kL}$ follows from the estimate $p - 2 \leqq L(p)$ if $p > 2$. This completes the proof of (4.23).

We proceed with the proof of (4.10). If G_k^* is empty, our estimate is trivially true. If G_k^* is not empty, then for some ω, some $(n, \omega) \in G_k(\omega)$ with $\omega \not\subset S$. Using (4.22) and (4.23) we have

$$\sum_{(n,\omega^*) \in G_k^*} |\omega^*| \leqq q^{2k+1} y^{-2} \int_{\mathfrak{D}} |f(x)|^2\, dx$$

$$\leqq q^{2k+1+kL} y^{-p} \int_{\mathfrak{D}} |f(x)|^2\, dx.$$

Recall that $p \neq 2$ and that f is a special function. Thus, $\int_{\mathfrak{D}} |f(x)|^2\, dx \leqq 2^p \int_{\mathfrak{D}} |f(x)|^p\, dx$. Since $q \geqq 2$, $k \geqq 1$ and $L \geqq 2p$, $p > 1$, we obtain

$$2^p q^{2k+1+kL} \leqq q^{5kL-k},$$

which gives

$$\sum_{(n,\omega^*) \in G_k^*} |\omega^*| \leqq q^{5kL-k} y^{-p} \int_{\mathfrak{D}} |f(x)|^p\, dx. \tag{4.10}$$

This completes the proof of (4.10).

Proof of (4.8) *and* (4.9). For each $(n|\omega|, \omega^*) \in G_k^*$ we construct the partition $\Omega = \Omega(n|\omega|, \omega^*, k)$ of ω^*. We define g on ω^* by

$$g(t) = \frac{1}{|\bar{\omega}|} \int_{\bar{\omega}} f(z)\bar{\chi}_n(z)\,dz, \qquad t \in \bar{\omega}, \quad \bar{\omega} \in \Omega. \tag{4.24}$$

For g we have the estimate

$$|g(t)| = |c_{n[\bar{\omega}]}(\bar{\omega}; f)| \leqq C_{n[\bar{\omega}]}(\bar{\omega}^*; f).$$

From (4.6) we have that

$$\|g\|_{\infty,\omega^*} < q^{-k+1}y \tag{4.25}$$

which follows since $\bar{\omega}^*$ properly contains $\bar{\omega} \in \Omega$, and $\bar{\omega}^* \subset \omega^*$.

The function g is used to obtain the following estimate.

(4.26) *Suppose* $x \in \omega^*$, $\tilde{\omega}^* \subset \omega^*$ *is such that* $\omega^* \sim \tilde{\omega}^*$ *is a union of mutually disjoint spheres in* Ω, $x \in \tilde{\omega}^*$. *Then*

$$|S_n^* f(x; \omega^*) - S_n^* f(x; \tilde{\omega}^*)| \leqq T_n^* g(x),$$

where the operator T_n^* *is defined in* (3.9).

$$\begin{aligned}|S_n^* f(x; \omega^*) - S_n^* f(x; \tilde{\omega}^*)| &= \left| \int_{\omega^* \sim \tilde{\omega}^*} f(t)\bar{\chi}_n(t) D_n^*(x - t)\,dt \right| \\ &= \left| \int_{\omega^* \sim \tilde{\omega}^*} g(t) D_n^*(x - t)\,dt \right. \\ &\quad \left. + \int_{\omega^* \sim \omega^*} [f(t)\bar{\chi}_n(t) - g(t)] D_n^*(x - t)\,dt \right|.\end{aligned}$$

$\omega^* \sim \tilde{\omega}^*$ is a union of disjoint spheres $\{\omega'\}$ such that $x \notin \omega'$, and each $\omega' \in \Omega$. From (3.7) we see that $D_n^*(x - t)$ is constant on each ω', and from the definition of g we see that

$$\int_{\omega'} [f(t)\chi_n(t) - g(t)]\,dt = 0.$$

Thus the second integral on the right is zero and the inequality follows.

Let $A > 0$ be the constant of (3.9) (which depends only on K), and let C_0 be a positive number such that $AC_0 \geqq (\frac{1}{2} + 5\log q)$. For each $(n|\omega|, \omega^*) \in G_k^*$ let

$$\mathscr{U}^* = \mathscr{U}^*(n|\omega|, \omega^*, k) = \{x \in \omega^* : T_n^* g(x) > C_0 Lkq^{-k+1}y\}.$$

From (3.9) and (4.25) we have

$$\begin{aligned}|\mathscr{U}^*| &\leqq e^{1/2} \exp\{-AC_0 Lkq^{-k+1}y/\|g\|_{\infty,\omega^*}\}\,|\omega^*| \\ &\leqq e^{1/2} \exp\{-AC_0 Lk\}|\omega^*| \leqq q^{-5kL}|\omega^*|.\end{aligned} \tag{4.8}$$

(4.9) *Statement* (4.26) *and the definition of* $\mathscr{U}^*$ *clearly imply* (4.9).

This completes the proof of (4.9) and (4.8).

We have now established all of the "starred" assertions and the proof is complete.

Appendix. Orlicz spaces. Let K be a local field as in § 3, $\mathfrak{O}$ the ring of integers in K, and suppose q is the integer such that $\mathfrak{O}/\mathfrak{P} \cong GF(q)$ (where $\mathfrak{P}$ is the maximal ideal of $\mathfrak{O}$) and $\{\chi_v\}_{v=0}^{\infty}$ are the characters on $\mathfrak{O}$. A function on $\mathfrak{O}$ of the form $\sum_{v=0}^{n} c_v \chi_v$ is called a polynomial.

Clearly if g is a polynomial, then $S_\mu g = g$ for all $\mu \geqq n + 1$. Our claim, in the proof of Theorem 4, that the "nice" functions are dense in the class of functions such that $J(f) < \infty$ is now shown by establishing the following theorem.

THEOREM A. *Let Φ be a nonnegative-valued, convex, nondecreasing function on $[0, \infty)$ such that $\Phi(0) = 0$. Suppose f is a measurable function on $\mathfrak{O}$ such that $\int_{\mathfrak{O}} \Phi(|f(x)|)\, dx < \infty$. Then*

$$\int_{\mathfrak{O}} \Phi(\tfrac{1}{4}|S_{q^n} f(x) - f(x)|)\, dx \to 0 \quad \text{as } n \to \infty .$$

Proof. The proof follows from the argument in Zygmund [14, vol. I, (5.14), p. 146] as soon as we show that $\int_{\mathfrak{O}} D_{q^n}(x)\, dx = 1$ and $D_{q^n}(x) \geqq 0$ for all $x \in \mathfrak{O}$.

That the integral is 1 follows from the fact that $\{\chi_v\}_{v=0}^{\infty}$ is an orthonormal sequence on $\mathfrak{O}$.

To see that $D_{q^n}(x) \geqq 0$ we extend D_{q^n} to a function on K by setting it equal to zero outside of $\mathfrak{O}$. From the arguments in Sally and Taibleson [7, § 2] we see that $D_{q^n} = \sum_{v=0}^{q^n-1} \chi_{i(v)} \Phi_0$, where Φ_v is the characteristic function of $\mathfrak{P}^v$. We compute the Fourier transform $(\chi_{i(v)}\Phi_0)\hat{}$ and it is the characteristic function of the sphere $i(v) + \mathfrak{O}$. From the choice of the $\{i(v)\}$ it follows that $(D_{q^n})\hat{} = \Phi_{-n}$ and so $D_{q^n} = q^n \Phi_n \geqq 0$.

COROLLARY. *If f is a measurable function on $\mathfrak{O}$ such that $J(f) < \infty$, then $J([S_{q^n} f - f]) \to 0$ as $n \to \infty$.*

Proof. Let $\Phi(s) = s + s \log^+ s \log^+ s \log^+ s, s > 0$, and zero otherwise. The theorem implies $J(\frac{1}{4}(S_{q^n} f - f)) \to 0$ as $n \to \infty$. But Φ satisfies the condition: there exists a $C > 0$ such that $\Phi(2x) \leqq C\Phi(x)$ for all $x \geqq 0$. This shows that $J((S_{q^n} f - f)) \to 0$ as $n \to \infty$.

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